Extended Abstract

Iwahori-Hecke algebras of type A, bitraces and symmetric functions

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Abstract. Let S_n be the symmetric group and let H_n be the corresponding Iwahori-Hecke algebra. Let $\gamma_r = (1, 2, \ldots, r)$ be the r-cycle in S_r and for a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ of n let $\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \in S_{\mu_1} \times \cdots \times S_{\mu_\ell} \subseteq S_n$. Let T_{γ_μ} be the standard basis element of the Iwahori-Hecke algebra corresponding to γ_μ. Let L_μ and R_μ be the matrices describing the actions of the element T_{γ_μ} on H_n by left multiplication and right multiplication respectively. We give an explicit formula for $\mathrm{Tr}(L_\mu R_\nu)$ as a weighted sum over the nonnegative integer matrices with row sums μ and column sums ν. This gives an explicit determination of the bitrace of the regular representation of the Iwahori-Hecke algebra of type A. We derive several corollaries of our main theorem and give interpretations of the value $\mathrm{Tr}(L_\mu R_\nu)$ in terms of inner products of symmetric functions, inner products on Iwahori-Hecke algebras, and the Robinson-Schensted-Knuth insertion algorithm.

Résumé. Soit S_n le groupe symétrique et H_n l'algèbre d'Iwahori-Hecke correspondante. Soit $\gamma_r = (1, 2, \ldots, r)$ le r-cycle de S_r et pour toute composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ de n, soit $\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \in S_{\mu_1} \times \cdots S_{\mu_\ell ll} \subseteq S_n$. Soit T_{γ_μ} l'élément de la base standard de l'algèbre d'Iwahori-Hecke qui correspond à γ_μ. Soit L_μ (resp. R_μ) la matrice de l'action de $T_{\gamma_m u}$ sur H_n par la multiplication à gauche (resp. à droite). On donne une formule explicite pour $\mathrm{Tr}(L_\mu R_\nu)$ comme somme pondérée sur les matrices à coefficients entiers positifs dont les sommes par ligne et par colonne sont μ et ν respectivement. Ceci fournit une formule explicite pour la bitrace de la représentation régulière d'une algèbre d'Iwahori-Hecke de type A. On obtient plusieurs corollaires de ce résultat principal et on donne une interprétation de $\mathrm{Tr}(L_\mu R_\nu)$ en termes de produits internes de fonctions symétriques, de produits internes sur des algèbres d'Iwahori-Hecke, et de l'algorithme d'insertion de Robinson-Schensted-Knuth.

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1. The bitrace of the regular representation of $\mathcal{H}_n(q)$

We use the notation $\lambda \models n$ to indicate that λ is a composition of n; that is $\lambda = (\lambda_1, \lambda_2, \ldots)$ where the parts, λ_i, are nonnegative for all i and $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ if λ is a partition of n, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. The length $\ell(\lambda)$ is the number of nonzero parts of λ. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ are compositions such that $\lambda_i \leq \mu_i$ for $1 \leq i \leq \ell$ then we write $\lambda \subseteq \mu$ and denote their difference or skew shape by μ/λ. In general we adopt the notation of [Mac] for partitions and symmetric functions.

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$, and let $q \in \mathbb{C}$ such that $q \neq 0$ and q is not a root of unity. The Iwahori-Hecke algebra $\mathcal{H}_n(q)$ corresponding to S_n is the algebra over $\mathbb{C}$ given by generators $1, T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_iT_j &= T_jT_i, && \text{if } |i-j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \text{for } 1 \leq i \leq n-2, \\ T_i^2 &= (q-1)T_i + q, && \text{for } 1 \leq i \leq n-1. \end{aligned} \tag{1.1}$$

Let $s_i = (i, i+1) \in S_n$ denote the simple transposition that exchanges i and $i+1$. Given a reduced word $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in S_n$, let $T_w = T_{i_1}T_{i_2}\cdots T_{i_k} \in \mathcal{H}_n(q)$. The element T_w is well-defined (independent of choice of the reduced word for w). The elements T_w, $w \in S_n$, form a basis of $\mathcal{H}_n(q)$.

The irreducible representations of $\mathcal{H}_n(q)$ are labeled by the partitions $\lambda \vdash n$, and their traces χ_q^λ are the irreducible characters of $\mathcal{H}_n(q)$. A *character* of $\mathcal{H}_n(q)$ is a linear map $\chi: \mathcal{H}_n(q) \to \mathbb{C}$ which satisfies $\chi(ab) = \chi(ba)$ for all $a, b \in \mathcal{H}_n(q)$. Let $\gamma_r = (1, 2, \ldots, r) \in S_r$ in cycle notation and for a composition $\mu = (\mu_1, \ldots, \mu_\ell) \models n$ define $\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \in S_{\mu_1} \times \cdots \times S_{\mu_\ell}$. Any character of $\mathcal{H}_n(q)$ is completely determined by its values on the elements T_{γ_μ}, $\mu \vdash n$ (see [Ca] and [Ra1]).

The bitrace

Let $x, y \in S_n$ and define

$$\mathrm{btr}(T_x, T_y) = \sum_{z \in S_n} T_xT_zT_y\big|_{T_z}, \tag{1.2}$$

where $T_xT_zT_y|_{T_z}$ denotes the coefficient of the basis element T_z when $T_xT_zT_y$ is expanded in terms of the basis T_w, $w \in S_n$. If $x \in S_n$ let L_x and R_x denote the linear transformations of $\mathcal{H}_n(q)$ induced by the action of T_x on $\mathcal{H}_n(q)$ by left multliplication and by right multiplication, respectively. If $x, y \in S_n$ then L_x and R_y commute and

$$\mathrm{btr}(T_x, T_y) = \mathrm{Tr}(L_xR_y). \tag{1.3}$$

Left and right multiplication make $\mathcal{H}_n(q)$ into a bimodule and, by double centralizer theory, we have

$$\mathcal{H}_n(q) \cong \bigoplus_{\lambda \vdash n} H_\lambda \otimes H^\lambda,$$

as $\mathcal{H}_n(q)$-bimodules, where H_λ is the irreducible left $\mathcal{H}_n(q)$-module labeled by λ and H^λ is the irreducible right $\mathcal{H}_n(q)$-module labeled by λ. Taking traces on both sides of this identity gives

$$\mathrm{btr}(T_x, T_y) = \sum_{\lambda \vdash n} \chi_q^\lambda(T_x)\chi_q^\lambda(T_y). \tag{1.4}$$

This formula is an $\mathcal{H}_n(q)$ analogue of the second orthogonality relation for the irreducible characters of the symmetric group S_n.

Keeping in mind that any character of $\mathcal{H}_n(q)$ is completely determined by its values on the elements T_{γ_μ}, $\mu \vdash n$, we define

$$\mathrm{btr}(\mu, \nu) = \mathrm{btr}(T_{\gamma_\mu}, T_{\gamma_\nu}) \tag{1.5}$$

for any two compositions $\mu, \nu \models n$.

An inner product on $\mathcal{H}_n(q)$

Suppose that q is a prime power and let $\mathbb{F}_q$ be the finite field with q elements. Let B be the subgroup of the general linear group $GL_n(\mathbb{F}_q)$ consisting of upper triangular matrices. Let 1_B^G be the $GL_n(\mathbb{F}_q)$-module which as a vector space is the linear span of the cosets in G/B and where the G-action on cosets is by left multiplication. There is a natural action of $\mathcal{H}_n(q)$ on 1_B^G and

$$\mathcal{H}_n(q) \cong \mathrm{End}_G(1_B^G).$$

Let $w \in S_n$. Then the trace of the action of T_w on 1_B^G is given by the formula

$$\mathrm{tr}(T_w) = \begin{cases} [\![n]\!]!, & \text{if } w \text{ is the identity,} \\ 0, & \text{otherwise,} \end{cases}$$

where $[\![n]\!] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[\![n]\!]! = [\![n]\!][\![n-1]\!] \cdots [\![2]\!][\![1]\!]$. Define a bilinear form on $\mathcal{H}_n(q)$ by

$$\langle a, b \rangle = \frac{1}{[\![n]\!]!}\mathrm{tr}(ab), \quad \text{for } a, b \in \mathcal{H}_n(q).$$

Note that the inner product $\langle a, b\rangle$ is the coefficient of 1 in the product ab. The dual basis to the basis T_w, $w \in S_n$, with respect to the inner product $\langle , \rangle$ is the basis $q^{-\ell(w)}T_{w^{-1}}$, $w \in S_n$.

Very general arguments [CR] 9.17, which work for any semisimple algebra, combined with the computation of the generic degrees in type A ([Ca2] 13.5 or [Hf] 3.4.14) will show that

$$\sum_{w \in S_n} \chi^\lambda(T_w)\chi^\mu(q^{-\ell(w)}T_{w^{-1}}) = \delta_{\lambda\mu} n! \frac{q^{-n(\lambda)}H_\lambda(q)}{h(\lambda)}, \tag{1.6}$$

where

$$h(\lambda) = \prod_{x \in \lambda} h(x), \quad H_\lambda(q) = \prod_{x \in \lambda} \frac{1 - q^{h(x)}}{1 - q}, \quad n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i, \quad \text{and}$$

if $x \in \lambda$ is the box in position (i,j) in λ then $h(x) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length at x. Formula (1.6) is the $\mathcal{H}_n(q)$-analogue of the first orthogonality relation for the irreducible characters of the symmetric group S_n.

For any element $x \in S_n$ define

$$[T_x] = \sum_{w \in S_n} T_w T_x q^{-\ell(w)} T_{w^{-1}}.$$

This is some sort of analogue of a conjugacy class sum in the group algebra of S_n. If $x, y \in S_n$,

$$\begin{aligned}\langle T_x, [T_y]\rangle &= \sum_{w \in S_n} \langle T_x, T_w T_y q^{-\ell(w)} T_{w^{-1}}\rangle = \sum_{w \in S_n} \frac{1}{[\![n]\!]!} q^{-\ell(w)} \mathrm{tr}(T_x T_w T_y T_{w^{-1}}) \\ &= \sum_{w \in S_n} \langle T_x T_w T_y, q^{-\ell(w)} T_{w^{-1}}\rangle = \sum_{w \in S_n} T_x T_w T_y \big|_{T_w},\end{aligned}$$

and thus

$$\langle T_x, [T_y]\rangle = \langle [T_x], T_y\rangle = \mathrm{btr}(T_x, T_y). \tag{1.7}$$

Specializing q to 1

For each $\mu \vdash n$ the character $\chi_q^\lambda(T_{\gamma_\mu})$ is a polynomial in q with integer coefficients and

$$\chi_q^\lambda(T_{\gamma_\mu})\big|_{q=1} = \chi^\lambda(\mu),$$

where $\chi^\lambda(\mu)$ denotes the irreducible character of the symmetric group S_n corresponding to the partition λ evaluated at a permutation of cycle type μ. It follows from (1.4) and the second orthogonality relation for the characters of the symmetric group that

$$\mathrm{btr}(\mu, \nu)\big|_{q=1} = \sum_{\lambda \vdash n} \chi^\lambda(\mu)\chi^\lambda(\nu) = \delta_{\mu\nu} z_\mu, \quad \text{where } z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

if μ is the partition $\mu = (1^{m_1} 2^{m_2} \ldots)$.

Symmetric functions

Let $x_1, x_2, \ldots, x_n$ be commuting variables. Define $q_0(x_1, x_2, \ldots, x_n; q) = 1$ and for $r > 0$, define $q_r(x_1, x_2, \ldots, x_n; q)$ by the generating function

$$\prod_{i=1}^n \frac{1 - x_i z}{1 - q x_i z} = 1 + (q-1) \sum_{r > 0} q_r(x; q) z^r.$$

For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define $q_\mu(x; q) = q_{\mu_1} q_{\mu_2} \cdots q_{\mu_\ell}$. From [Ra1], [VK], [KW] we know that if $\mu \models n$,

$$q_\mu(x; q) = \sum_{\lambda \vdash n} \chi_q^\lambda(T_{\gamma_\mu}) s_\lambda(x), \tag{1.8}$$

where $s_\lambda(x)$ is the Schur function corresponding to λ, see [Mac]. There is a standard inner product on the ring of symmetric functions given by $\langle s_\mu, s_\nu\rangle = \delta_{\mu\nu}$ for all partitions μ,ν. It follows from (1.8) and (1.4) that

$$\mathrm{btr}(\mu,\nu) = \langle q_\mu(x;q), q_\nu(x;q)\rangle. \tag{1.9}$$

2. The main theorem and corollaries

The following theorem is the main result of this paper, its proof is outlined in Section 3.

Theorem 2.1. *Let $\mu,\nu \models n$ and let $\mu=(\mu_1,\ldots,\mu_\ell)$ and $\nu=(\nu_1,\ldots,\nu_m)$. Then*

$$btr(\mu,\nu) = (q-1)^{-\ell(\mu)-\ell(\nu)} \sum_M wt(M),$$

where the sum is over all $\ell\times m$ nonnegative integer matrices with row sums $\mu_1,\ldots,\mu_\ell$, and column sums $\nu_1,\ldots,\nu_m$, and

$$wt(M) = \prod_{x\in\mathcal{P}(M)} (q-1)^2[\![x]\!]_{q^2},$$

where $\mathcal{P}(M)$ is the multiset of nonzero entries x in the matrix M and $[\![x]\!]_{q^2} = 1 + q^2+q^4+\cdots+q^{2(x-1)}$.

Corollary 2.2. *The trace of the regular representation of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$ is given by*

$$Tr(T_{\gamma_\mu}) = (q-1)^{n-\ell(\mu)}\frac{n!}{\mu_1!\mu_2!\cdots\mu_\ell!}, \quad \textit{for all compositions } \mu=(\mu_1,\ldots,\mu_\ell)\models n.$$

For a non-negative integer r, define the symmetric function t_r by the formula

$$\sum_{r\geq 0} t_r(x;q)z^r = \prod_i \frac{(1-qx_iz)^2}{(1-q^2x_iz)(1-x_iz)}, \tag{2.3}$$

and for a composition $\mu=(\mu_1,\mu_2,\ldots,\mu_\ell)$ define $t_\mu(x;q)=t_{\mu_1}t_{\mu_2}\cdots t_{\mu_\ell}$.

Corollary 2.4. *If $\mu,\nu\models n$, then*

$$btr(\mu,\nu) = (q-1)^{-\ell(\mu)-\ell(\nu)}\, t_\mu(x;q)\big|_{m_\nu}$$

where $t_\mu(x;q)|_{m_\nu}$ denotes the coefficient of the monomial symmetric function m_ν in the symmetric function t_μ.

Corollary 2.5. *Let $\mu,\nu \models n$ and let q_μ and t_μ be the symmetric functions defined in (1.8) and (2.3), respectively. Then*

$$\langle q_\mu(x;q), q_\nu(x;q)\rangle = (q-1)^{-\ell(\mu)-\ell(\nu)}\langle t_\mu(x;q), h_\nu(x)\rangle,$$

where $h_\nu(x)$ is the homogeneous symmetric function and $\langle , \rangle$ is the inner product on symmetric functions that makes the Schur functions orthonormal.

Specializations of $\langle q_\mu, q_\nu\rangle$

Define $\tilde{q}_0(x;q,t) = 1$ and, for positive integers r, define symmetric functions $\tilde{q}_r(x;q,t)$ by the formula

$$(q-t)\sum_{r\geq 0}\tilde{q}_r(x;q,t)z^r = \prod_i \frac{(1-tx_iz)}{(1-qx_iz)}. \tag{2.6}$$

For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define $\tilde{q}_\mu(x;q,t) = \tilde{q}_{\mu_1}\tilde{q}_{\mu_2}\cdots\tilde{q}_{\mu_\ell}$. These symmetric functions differ from the symmetric functions $q_\mu(x;q)$ only by a change in normalization. On the other hand they have the advantage that one can specialize either q or t or both as follows:

(a) $\tilde{q}_\mu(x;q,0) = q^{|\mu|-\ell(\mu)}h_\mu(x)$, where h_μ is the homogeneous symmetric function,

(b) $\tilde{q}_\mu(x;0,t) = (-t)^{|\mu|-\ell(\mu)}e_\mu(x)$, where e_μ is the elementary symmetric function,

(c) $\tilde{q}_\mu(x;q,q) = q^{|\mu|-\ell(\mu)}p_\mu(x)$, where p_μ is the power symmetric function.

The combinatorics of the symmetric functions $\tilde{q}_\mu(x;q,t)$ is studied in depth in [RRW]. The appropriate modifications to Theorem 2.1 give

$$\langle \tilde{q}_\mu, \tilde{q}_\nu\rangle = (q-t)^{-\ell(\mu)-\ell(\nu)}\sum_M \widetilde{\mathrm{wt}}(M), \quad \text{where } \widetilde{\mathrm{wt}}(M) = \prod_x (q-t)^2t^{2(x-1)}[\![x]\!]_{q^2t^{-2}},$$

where the sum is over all nonnegative integer matrices M with row sums μ and column sums ν, the product is over all nonzero entries x in the matrix M, and $t^{2(x-1)}[\![x]\!]_{q^2t^{-2}} = t^{2(x-1)} + q^2t^{2(x-2)} + \cdots + q^{2(x-2)}t^2 + q^{2(x-1)}$. By specializing q and t we have new proofs of the following well known formulas ([Mac] I (6.6) (iv), (6.7)(ii), (4.7)):

(2.7a) $\langle e_\mu, e_\nu\rangle$ is the number of nonnegative integer matrices with row sums μ and column sums ν,

(2.7b) $\langle h_\mu, h_\nu\rangle$ is the number of nonnegative integer matrices with row sums μ and column sums ν,

(2.7c) $\langle p_\mu, p_\nu\rangle = \delta_{\mu\nu}z_\mu$, where $z_\mu = 1^{m_1}m_1!2^{m_2}m_2!\cdots$ if μ is the partition $\mu = (1^{m_1}2^{m_2}\ldots)$.

The adjoint of multiplication by $\tilde{q}_r$

If f is a symmetric function define $f^\perp$ to be the adjoint of multiplication by f, with respect to the inner product $\langle , \rangle$, i.e.

$$\langle fg_1, g_2\rangle = \langle g_1, f^\perp g_2\rangle \quad \text{for all symmetric functions } g_1, g_2.$$

In Section 3 we will prove the following recursion rule for the bitrace.

Proposition 2.8. *Let $\mu, \nu \models n$ and $\nu = (\nu_1, \ldots, \nu_\ell)$. Then*

$$btr(\mu,\nu) = \sum_\alpha (q-1)^{s(\alpha,\mu)} btr(\mu/\alpha, \nu') btr(\alpha, (\nu_\ell))$$

where the sum is over all compositions $\alpha \models \nu_\ell$ such that are $\alpha \subseteq \mu$ and $s(\alpha,\mu) = Card(\{k \mid 0 < \alpha_k < \mu_k\})$.

It follows from Theorem 2.1 that if $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a composition of n then

$$\mathrm{btr}(\alpha, (n)) = (q-1)^{\ell(\alpha)-1} \prod_{\alpha_i \neq 0} [\![\alpha_i]\!]_{q^2}.$$

Combining this formula with Proposition 2.8 and 1.9 gives the following corollary, where we have done the necessary modifications to use $\tilde{q}_\mu$ instead of q_μ.

Corollary 2.9. *Let r be a positive integer and let μ be a composition. Let $\tilde{q}_\mu(x;q,t)$ be the symmetric function defined in (2.6) and, if α is a composition contained in μ, let $s(\alpha,\mu)$ be as given in Proposition 2.8. Then*

$$\tilde{q}_r^\perp \tilde{q}_\mu = \sum_{\alpha \models r} f(\alpha,\mu)\tilde{q}_{\mu/\alpha}, \qquad \text{where } f(\alpha,\mu) = (q-t)^{\ell(\alpha)-1+s(\alpha,\mu)} \prod_{\alpha_i \neq 0} t^{2(\alpha_i - 1)} [\![\alpha_i]\!]_{q^2 t^{-2}}.$$

By specializing q and t we get the following results:

(a) $e_r^\perp e_\mu = \sum_{\alpha \models r} e_{\mu/\alpha}$,

(b) $h_r^\perp h_\mu = \sum_{\alpha \models r} h_{\mu/\alpha}$, and

(c) $p_r^\perp p_\mu = z_\mu z_\nu^{-1} p_\nu$, if r is a part of μ and ν is the partition obtained by removing one part of size r from μ.

The result in (c) is well known, see [Mac] I §5 Ex. 3c and the results in (a) and (b) can also be deduced directly from (2.7a) and (2.7b), above.

3. A Recurrence Relation for the Bitrace.

The Roichman formula

The starting point for the proof of our main result is a recent formula of Y. Roichman [Ro] which expresses the irreducible character of the Iwahori-Hecke algebra as a weighted sum over standard tableaux. Let $\mu, \lambda \vdash n$ be partitions of n and let Q be a standard tableau of shape λ. Then the *μ-Roichman weight* of Q is

$$\mathrm{rwt}_q^\mu(Q) = \prod_{\substack{i=1 \\ i\notin B(\mu)}}^{n} f_\mu(i,Q) \quad \text{where } B(\mu) = \{\mu_1+\mu_2+\ldots+\mu_r \mid 1 \le r \le \ell(\mu)\}, \text{ and}$$

$$f_\mu(i,Q) = \begin{cases} -1, & \text{if } i+1 \text{ is southwest of } i \text{ in } Q, \\ 0, & \text{if } i+1 \text{ is northeast of } i \text{ in } Q,\ i+1 \notin B(\mu), \\ & \quad \text{and } i+2 \text{ is southwest of } i+1 \text{ in } Q, \\ q, & \text{otherwise.} \end{cases}$$

In the definition of the Roichman weight our notations for partitions and their Ferrers diagrams are as in [Mac], "northeast" means weakly north and strictly east, and "southwest" means strictly south and weakly west.

Theorem 3.1. [Ro] *If $\lambda \vdash n$ and $\mu \models n$, then*

$$\chi_q^\lambda(T_{\gamma_\mu}) = \sum_Q \mathrm{rwt}_q^\mu(Q)$$

where χ_q^λ is the irreducible character of $\mathcal{H}_n(q)$ indexed by the partition λ and the sum is taken over all standard tableaux Q of shape λ.

An elementary proof of (the type A case) of Roichman's theorem was given in [Ra2]. One of the ideas of [Ra2] was to convert the Roichman weight to a weight on sequences as follows. A sequence $w_1, w_2, \ldots, w_r$ of elements of $\{1,2,\ldots,n\}$ has weight

$$\mathrm{wt}(w_1,w_2,\ldots,w_r) = \begin{cases} 1, & \text{if } r=1 \text{ or the sequence is empty;} \\ (-1)^{t-1}q^{r-t}, & \text{if } w_1 < w_2 < \cdots < w_t > w_{t+1} > \cdots > w_r; \\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a composition of n and $w \in S_n$ is a permutation, define (w,λ) to be the injective λ-tableau obtained by filling in the boxes of λ with $w(1), w(2), \ldots, w(n)$ from left-to-right and top-to-bottom. Define

$$\begin{aligned} \mathrm{wt}_\lambda(w) &= \text{the product of the weights of the rows of } (w,\lambda) \text{ and} \\ \mathrm{wt}^\lambda(w) &= \mathrm{wt}_\lambda(w^{-1}). \end{aligned}$$

For $w \in S_n$, write $w = [w_1, w_2, \dots, w_n]$ if $w(i) = w_i$ for each $1 \le i \le n$. If $\lambda = (4,3,2)$ and $w = [2,7,5,1,9,8,3,4,6]$, then $w^{-1} = [4,1,7,8,3,9,2,6,5]$,

$$(w,\lambda) = \begin{matrix} 2 & 7 & 5 & 1 \\ 9 & 8 & 3 & \\ 4 & 6 & & \end{matrix}\ , \qquad (w^{-1},\lambda) = \begin{matrix} 4 & 1 & 7 & 8 \\ 3 & 9 & 2 & \\ 6 & 5 & & \end{matrix}\ ,$$

$\mathrm{wt}_\lambda(w) = (-q^2)(q^2)(-1) = q^4$, and $\mathrm{wt}^\lambda(w) = 0(-q)q = 0$.

The connection between this definition and the Roichman weight of a tableaux Q is via Robinson-Schensted-Knuth (RSK) column insertion. (The original references for the RSK insertion scheme are [Sz], [Sch] and [Kn]; for an expository treatment see [Sa].) Applying RSK insertion on the sequence w produces a pair (P,Q) of standard tableaux of the same shape $\lambda \vdash n$, where P is the result of insertion and Q is the so-called "recording tableau."

(a) RSK column insertion is a bijection between S_n and the set of all pairs of standard tableaux (P,Q) having the same shape $\lambda \vdash n$.

(b) If applying RSK insertion to $w \in S_n$ produces the pair (P,Q) then applying RSK insertion to w^{-1} produces (Q,P) ([Scü], [Sa]).

(c) We have $\mathrm{rwt}_q^\mu(Q) = \mathrm{wt}_\mu(w)$, where Q is the recording tableau produced by column insertion of the sequence $w = [w_1, \dots, w_n]$ (cf. [Ra2]).

The following lemma uses Roichman's result and RSK insertion to write the bitrace in terms of weights on symmetric group elements. We use this reformulation to prove the recurrence relation for $\mathrm{btr}(\mu,\nu)$.

Lemma 3.2. *If $\mu,\nu \models n$ then* $\quad \mathit{btr}(\mu,\nu) = \sum_{w\in S_n} \mathit{wt}_\mu(w)\mathit{wt}^\nu(w).$

Outline of Proof of Theorem 2.1

Let $\mathcal{C}_n$ denote the set of compositions of n. For $(w,\mu) \in S_n \times \mathcal{C}_n$, let $(\hat{w},\lambda) \in S_{n-m} \times \mathcal{C}_{n-m}$ be the injective λ-tableau obtained by deleting $\{n-m+1,\dots,n\}$ from (w,μ) and left justifying the resulting tableau. Let $(w/\hat{w}, \mu/\lambda)$ be the diagram obtained by deleting $\{1,2,\dots,n-m\}$ from (w,μ). Reading the elements of $((w/\hat{w}),\mu/\lambda)$ from left to right and top to bottom, we can view $w/\hat{w}$ as a permutation in the symmetric group S'_m on $\{n-m+1, n-m+2, \dots, n\}$. We write $(w/\hat{w}) \to ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda))$. As an example, let $m = 6$, $\mu = (4,3,2,2)$, and $w = [2,7,6,1,9,8,3,11,10,4,5] \in S_{11}$. Then the deletion of $\{6,7,8,9,10,11\}$ from

$$(w,\mu) = \begin{matrix} 2 & 7 & 6 & 1 \\ 9 & 8 & 3 & \\ 11 & 10 & & \\ 4 & 5 & & \end{matrix} \qquad \text{is} \qquad ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda)),$$

where

$$(\hat{w},\lambda) = \begin{matrix} 2 & 1 \\ 3 & \\ & \\ 4 & 5 \end{matrix} \qquad \text{and} \qquad (w/\hat{w},\mu/\lambda) = \begin{matrix} & 7 & 6 \\ 9 & 8 & \\ 11 & 10 & \end{matrix}\ .$$

Thus, $\hat{w} = [2,1,3,4,5] \in \mathcal{S}_5$, $\lambda = (2,1,0,2)$, and $w/\hat{w} = [7,6,9,8,11,10] \in \mathcal{S}'_6$.

Lemma 3.3. *Assume that $(w,\mu) \to ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda))$ denotes the deletion of $\{n-m+1,\ldots,n\}$. If $wt_\mu(w) \neq 0$, then*

(a) *In each row of (w,μ), the elements from $\{n-m+1,\ldots,n\}$ appear in a contiguous block;*

(b) *$wt_\lambda(\hat{w}) \neq 0$ (thus the rows of $(\hat{w},\lambda)$ form up-down sequences).*

(c) *$wt_{\mu/\lambda}(w/\hat{w}) \neq 0$ (thus the rows of $(w/\hat{w},\mu/\lambda)$ form up-down sequences).*

(d) *In each row of (w,μ), the elements from $\{n-m+1,\ldots,n\}$ appear either immediately to the left or immediately to the right of the largest element from $\{1,2,\ldots,n-m\}$.*

In Lemma 3.3 (d), an insertion to the left of the largest element is called a *left insertion* and an insertion to the right of the largest element is called a *right insertion*. Each $(w,\mu) \to ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda))$ with $\mathrm{wt}_\mu(\hat{w}) \neq 0$ gives rise to a unique sequence $I = (I_1, I_2, \ldots, I_{\ell(\mu)})$, where for each nonempty row k of μ we have

$$I_k = \begin{cases} \mathrm{T}, & \text{if } \lambda_k = 0 \text{ or } \lambda_k = \mu_k, \\ \mathrm{L}, & \text{if in row } k \text{ a left insertion takes } ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda)) \text{ to } (w,\mu), \\ \mathrm{R}, & \text{if in row } k \text{ a right insertion takes } ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda)) \text{ to } (w,\mu). \end{cases}$$

In our example the insertion sequence is $I = (\mathrm{R},\mathrm{L},\mathrm{T},\mathrm{T})$.

Lemma 3.4. *Let $\mu,\nu \models n$ with $\nu = (\nu_1,\ldots,\nu_\ell)$. Let $\nu' = (\nu_1,\ldots,\nu_{\ell-1})$ and $m = \nu_\ell$. Assume that $wt_\mu(\hat{w}) \neq 0$ and let*

$$(w,\mu) \to ((\hat{w},\lambda),(w/\hat{w},\mu/\lambda),I)$$

denote the deletion of $\{n-m+1,\ldots,n\}$ from (w,μ). Then

(a) *$wt_\mu(w) = (-1)^{R(I)} q^{L(I)} wt_\lambda(\hat{w}) wt_{\mu/\lambda}(w/\hat{w})$,*
where $L(I)$ is the number of Ls in the insertion sequence I and $R(I)$ is the number of Rs in I, and

(b) *$wt^\nu(w) = wt^{\nu'}(\hat{w}) wt^{(m)}(w/\hat{w})$.*

Using Lemmas 3.2 and 3.4 we prove the following recurrence relation for the bitrace $\mathrm{btr}(\mu,\nu)$ by deleting $\{n-\nu_\ell+1,\ldots,n\}$ from each $w \in S_n$.

Proposition 3.5. *Let $\mu,\nu \models n$, $\nu = (\nu_1,\ldots,\nu_\ell)$ and $\nu' = (\nu_1,\ldots,\nu_{\ell-1})$. Then*

$$btr(\mu,\nu) = \sum_{\substack{\lambda \models (n-\nu_\ell) \\ \lambda \subseteq \mu}} (q-1)^{s(\lambda,\mu)} btr(\lambda,\nu') btr(\mu/\lambda,(\nu_\ell))$$

where the sum is over all compositions λ of $n-\nu_\ell$ that are contained in μ and

$$s(\lambda,\mu) = Card(\{k \mid 0 < \lambda_k < \mu_k\}).$$

We then give the following closed formulas for the bitrace in the special case where ν consists of a single part.

Proposition 3.6.

(a) $btr((n),(n)) = [\![n]\!]_{q^2}$.

(b) $btr(\alpha,(n)) = (q-1)^{\ell(\alpha)-1} \prod_{\alpha_i \neq 0} [\![\alpha_i]\!]_{q^2}$, *if* α *is the composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

Theorem 2.1 is then proved using Propositions 3.4 and 3.5.

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