Using the software EBENMASS for the symmetric groups and symmetric functions

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HARDWARE REQUIREMENT: 486 PC or above

Summary

In this paper we give the theoretical background and the new algorithms developed particularly for this software. First of all we give a brief survey of the existing methods of calculating the characters of symmetric groups then present a new formula for the calculation of ordinary and spin characters using the properties of symmetric functions. This leads to a more efficient method of calculating the Kronecker products of ordinary and spin representations and the plethysm of S and Q functions.
This software can calculate the ordinary χ_ρ^λ and the spin ζ_ρ^λ characters of the irreducible representation labelled by the partition λ associated with the class structure ρ. It can also calculate the dimension of the representation, number of elements in a class and the characters of a given ordinary or spin representation. It also generates the entire character table for a given integer n. Using these characters it performs operations like Kronecker products of the spin and ordinar irreducible representations and the plethysm of both the S and Q functions associated with the ordinary and spin irreducible representations respectively. For larger outputs it has the facility of saving it on a log file either as a standard output or in LaTeX format. On-line help is available to make it user friendly. The PC version of this software is in PASCAL whereas the UNIX version is in C^{++}.
This software is available free with the expectation that its use will be appropriately acknowledged.

1 Introduction

Characters of the symmetric groups have been of interest to both the mathematicians and the physicists for their wide applications. Because of the complexity of their computation the explicit use of the characters in physics has been very limited. The combinatorial properties of the symmetric functions had been exploited instead. This method is proved to be too cumbersome for the calculation of S-function plethysm. A parallel theory of Schur's Q-functions associated with the spin representations appears to be even more cumbersome. For example the computation of Kronecker products of the spin irreducible representations or the Schur's Q-functions as given in [12] is painfully slow. Where as an explicit use of characters is not only faster but makes it possible to calculate the Kronecker products of larger group representations. Here we will give a brief description of the theory of symmetric functions in connection with the characters of the symmetric groups and derive the new algorithm. For details we refer to [7].

2 Ring of symmetric functions

Let $x_1, ..., x_n$ be independent indeterminates. The symmetric group S_n acts on the polynomial ring $\mathcal{Z}[x_1, ..., x_n]$ by permuting the x's, and we shall write [7],

$$\Lambda_n = \mathcal{Z}[x_1, ..., x_n]^{S_n},$$

for the subring of symmetric polynomials in $x_1, ..., x_n$.

Λ_n is a *graded* ring

$$\Lambda_n = \bigoplus_{r \geq 0} \Lambda_n^r,$$

where Λ_n^r is the additive group of symmetric polynomials of degree r in $x_1, ..., x_n$. Let

$$\Lambda^r = \lim_{\overleftarrow{n}} \Lambda_n^r,$$

for each $r \geq 0$, then

$$\Lambda = \bigoplus_{r \geq 0} \Lambda^r.$$

The graded ring Λ is the ring of symmetric functions. If $\mathcal{Q}$ is any commutative ring, we will write

$$\Lambda_{\mathcal{Q}} = \Lambda \otimes_{\mathcal{Z}} \mathcal{Q},$$

for the ring of symmetric functions with coefficients in $\mathcal{Q}$.

There are various $\mathcal{Z}$-bases of the ring Λ, and they all are indexed by partitions. Let $\lambda = (1^{m_1}, 2^{m_2} \cdots)$ be a partition, where m_1 is number of parts equal to 1, m_2 is number of parts equal to 2, and so on. It defines a monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$. The monomial symmetric function m_λ is the sum of all distinct monomials obtainable from x^λ by permutation of the x.

When $\lambda = (1^n)$ we have

$$m_{(1^n)} = e_n = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n},$$

the n-th elementary symmetric function.

For $\lambda = (n)$ we have

$$m_{(n)} = p_n = \sum_i x_i^n,$$

the n-th power sum symmetric function.

Let $l(\lambda)$ be the length of the partition λ and form the determinant

$$D_\lambda = \det \left(x_i^{\lambda_j + l - j} \right)_{1 \leq i,j \leq l(\lambda)},$$

and the Vandermonde determinant

$$D_\circ = \prod_{i<j}(x_i - x_j),$$

then the quotient

$$s_\lambda(x_1, ..., x_l) = D_\lambda / D_\circ,$$

is a homogeneous symmetric polynomial of degree $\mid \lambda \mid$ in $x_1, ..., x_l$ called S-functions. An S-function indexed by a partition λ is denoted by s_λ. S-functions indexed by disordered partitions are called *non-standard* S-functions and must be modified to produce either a signed standard partition or a null result by application of the modification rules derived directly from the determinantal definition of the S-functions [6, 12].

A scalar product $\langle \, , \, \rangle$ is defined on Λ as follows:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$

where

$$z_\lambda = \prod_i i^{m_i} m_i!.$$

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ be two sequences of indeterminates over Λ_t, where t is another indeterminate and define [7],

$$\begin{aligned} P(\mathrm{x,y};t) &= \prod_{i,j} \frac{(1 - t x_i y_j)}{(1 - x_i y_j)}, \\ &= \sum_\lambda b_\lambda(t) P_\lambda(\mathbf{x};t) P_\lambda(\mathbf{y};t), \\ &= \sum_\lambda P_\lambda(\mathbf{x};t) Q_\lambda(\mathbf{y};t), \\ &= \sum_\lambda q_\lambda(\mathbf{x};t) m_\lambda(\mathbf{y};t), \end{aligned}$$

as the generating function of *Hall-Littlewood symmetric functions* $P_\lambda(t)$. $Q_\lambda(\mathbf{y};t)$ is defined as

$$Q_\lambda(\mathbf{y};t) = b_\lambda(t) P_\lambda(\mathbf{y};t),$$

where

$$b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t), \qquad \phi_n(t) = \prod_{j \geq 1}^{n} (1 - t^j).$$

A scalar product $\langle \, , \, \rangle_{(t)}$ over Λ_t is defined as follows:

$$\langle p_\lambda, p_\mu \rangle_{(t)} = \delta_{\lambda\mu} z_\lambda(t),$$

where

$$z_\lambda(t) = z_\lambda \prod_i (1 - t^{\lambda_i})^{-1}.$$

For $t = 0$ we have S-functions where as for $t = -1$ we obtain Schur's Q-functions. Also for $t = 0$ the generalised homogeneous symmetric functions $q_\lambda(t)$ become h_λ. For $t = -1$ we get

$$Q_{(\lambda)} = \prod_{i<j} \frac{(1 - R_{ij})}{(1 + R_{ij})} q_{(\lambda)} \tag{1}$$

where

$$R_{ij} q_{(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_j, \cdots, \lambda_{l(\lambda)})} = q_{(\lambda_1, \cdots, \lambda_i + 1, \cdots \lambda_j - 1, \cdots, \lambda_{l(\lambda)})}$$

and

$$q_{(\lambda)} = q_{(\lambda_1)} q_{(\lambda_2)} \cdots q_{(\lambda_{l(\lambda)})}.$$

3 Ordinary characters of the symmetric groups

S-functions can be expressed in the basis of power sum symmetric functions as follows.

$$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho \tag{2}$$

where χ_ρ^λ is the character of the representation λ and class ρ.

There have been three methods of calculating characters. Here we will give a brief review and then move on to the new algorithm that is much more simple and elegant.

3.1 Polynomial expansion

This method is derived from the properties of the symmetric functions and requires the following four steps [7].

1. To calculate the characters of the irreducible representation χ^λ, expand the S-function s_λ in terms of power sum symmetric functions using Eq. 2.

2. Expand s_λ in terms of homogeneous symmetric functions

$$h_\lambda \equiv h_{\lambda_1}, h_{\lambda_2}, \cdots, h_{\lambda_l},$$

where l is the length of the partition λ.

$$s_\lambda = \prod_{1 \le i < j \le l} (1 - R_{ij}) h_\lambda \tag{3}$$

where R_{ij} is Young raising operator defined as

$$R_{ij}(h_{\lambda_1}, \cdots, h_{\lambda_i}, \cdots, h_{\lambda_j}, \cdots, h_{\lambda_l}) = h_{\lambda_1}, \cdots, h_{(\lambda_i+1)}, \cdots, h_{(\lambda_j-1)} \cdots, h_{\lambda_l}$$

remembering that $h_0 = 1$ and h_n is zero for $n < 0$.

3. Expand each h_{λ_k} in terms of power sum symmetric functions as

$$h_n = \sum_\rho z_\rho^{-1} p_\rho,$$

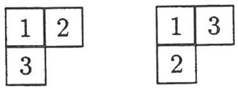
where ρ is a partition of n.

4. Now comparing the coefficients of the power sum symmetric functions p_ρ, one can calculate the characters χ_ρ^λ.

There are obvious disadvantages of this method. It requires three huge expansions that grow fast with increasing length of the partitions. Secondly, it doesn't facilitate the calculation of individual characters as required for the plethysm of S-functions, shown later.

3.2 Young permutation operators

Associated with every partition $\lambda \equiv \lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)}$, there exist a Young diagram Y^λ of λ_1 boxes in first row, λ_2 boxes in second row and so on. If λ is a partition of n then a standard tableaux is obtained by filling in the boxes of Y^λ such that $Y_{i,j}^\lambda < Y_{i+1,j}^\lambda$ and $Y_{i,j}^\lambda < Y_{i,j+1}^\lambda$ where $Y_{i,j}^\lambda$ is the box in ith row and jth column. As an example, following are the possible standard Young tableau for $\lambda = (2, 1)$.



DEFINITION 1 *A Young row operator is defined as*

$$Y_R^\lambda = \sum p_R$$

where sum is taken over all the permutations of the objects in the rows of the tableaux.

DEFINITION 2 *A Young column operator is defined as*

$$Y_C^\lambda = \sum \pm p_C$$

where sum is taken over all the column permutations with + or − signs for even or odd permutations respectively.

DEFINITION 3 *A Young permutation operator is defined as*

$$Y^\lambda = Y_R^\lambda Y_C^\lambda.$$

As an example if we consider the first tableaux of $\lambda \equiv (2,1)$ then

$$Y_R^{(2,1)} = e + (12)$$

where $e \equiv (1)(2)(3)$ is the identity element and (12) is the permutation of objects 1 and 2.

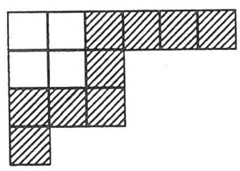
This method has same disadvantages as the previous one. It requires the expansion of the whole representation and the number of permutations grow too large for higher values of n and even a powerful computer can't handle it.

3.3 Staircase method

The recursive staircase method for calculating the characters of the irreducible representations of the symmetric group exploits the combinatorial properties of Young diagrams [2].

DEFINITION 4 *The* staircase *of a Young diagram consists of all the boxes in a continuous outer ribbon going from the upper right to the lower left.*

As an example, consider $\lambda = (6, 3^2, 1)$

$\tau =$ 

The shaded boxes are the staircase of the graph.

Let χ_ρ^λ be the character of the class ρ of S_n in the irreducible representation χ^λ. Suppose that the class (ρ) contains a cycle of length ρ_i. Then we have

$$\chi_\rho^\lambda = \sum_{\bar{\lambda}} \pm \chi_{\bar{\rho}}^{\bar{\lambda}} \tag{4}$$

where the sum is over all legitimate Young graphs of $\bar{\lambda}$ with $n - \rho_i$ boxes. The (+) sign above is when the subtracted staircase segment lies on an odd number of rows. We can strip off as many cycles until we reach very simple groups such as S_1 or S_2. In order to complete the calculations, we must assume that the characters of S_0 are all equal to +1.

Though this method has the advantage of calculating a specific character with out expanding the whole representation, it lacks the computational simplicity. Since it requires the Young diagrams and the hooklenths, it takes up a lot of space in a computer as 2-dimensional array and hence limits the capacity. We have to admit this is far better than the other two.

4 The new algorithm for ordinary characters

In this section we derive a new method for the calculation of the ordinary characters of the symmetric groups and will give an example, reflecting its power and usefulness.

DEFINITION 5 *In the basis of power sum symmetric functions we define the adjoint multiplication by p_n, $D(p_n): \Lambda \to \Lambda$, as [7]*

$$\langle D(p_n)p_\mu, p_\nu\rangle = \langle p_\mu, p_n p_\nu\rangle$$

which is zero if $\mu \neq \nu \cup (n)$, and is equal to z_μ if $\mu = \nu \cup (n)$.

It follows that $D(p_n) = n\frac{\partial}{\partial p_n}$ acting on symmetric functions expressed in terms of power sum symmetric functions and for a partition $\lambda \equiv (\lambda_1, \cdots, \lambda_l)$ we have $D(p_\lambda) = D(p_{\lambda_1})D(p_{\lambda_2})\cdots D(p_{\lambda_l})$. This leads to the following property of these operators.

$$D(p_n)f_\lambda = \sum_i f_{\lambda_1,\lambda_2,\cdots,\lambda_i-n,\cdots} \tag{5}$$

where f_λ is an element of Λ indexed by the partition λ. The above list is obtained by subtracting n from each part of λ. The resulting list of symmetric functions are modified according to the rules related to them. For example, S-functions are modified by the following rules given in [5].

1. $$s_{(\cdots,\lambda_i,\lambda_{i+1},\cdots)} = -s_{(\cdots,\lambda_{i+1}-1,\lambda_i+1,\cdots)},$$

   whenever $|\lambda_{i+1}| > |\lambda_i|$, remembering that $s_{(\mu,0)} = s_{(\mu)}$.

2. S-functions with $\lambda_{i+1} = \lambda_i + 1$ are null.

3. Any S-function indexed by a partition in non decreasing order with last part a negative number is null and $s_0 = 1$.

As an example,

$$D(p_3)s_{(6,3,2,1)} = s_{(3,3,2,1)} + s_{(6,0,2,1)} + s_{(6,3,-1,1)} + s_{(6,3,2,-2)}.$$

According to rule 1, $s_{(6,3,-1,1)} = -s_{(6,3)}$ and $s_{(6,0,2,1)} = -s_{(6,1,1,1)}$. By rule 3, $s_{(6,3,2,-2)} = 0$ Thus

$$D(p_3)s_{(6,3,2,1)} = s_{(3,3,2,1)} - s_{(6,2,1)} - s_{(6,3)}.$$

THEOREM 1 *The character χ^λ_π is the integer $D(p_\pi)s_\lambda$.*

Proof

We apply the operator $D(p_\pi)$ on both sides of (1).

$$D(p_\pi)s_\lambda = D(p_\pi)\sum_\rho z_\rho^{-1}\chi^\lambda_\rho p_\rho$$

then a repetitive use of (5) gives an integer k on the left side of the above equation where as on the right hand side, all the terms will be zero except for $\rho \equiv \pi$. In that case we get

$$k = z_\pi^{-1}\chi^\lambda_\pi z_\pi.$$

Thus

$$k = D(p_\pi)s_\lambda = \chi^\lambda_\pi.$$

As an example,

$$\begin{aligned}
\chi_{43}^{421} &= D(p_{43})s_{421} \\
&= D(p_4)D(p_3)s_{421} \\
&= D(p_4)\{s_{42-2} + s_{4-11} + s_{121}\} \\
&= D(p_4)(-s_4) \\
&= -s_0 \\
&= -1
\end{aligned}$$

Using this algorithm one can produce the character tables for very large groups.

5 Kronecker product of ordinary representations

Now that we can calculate the characters that efficiently, we can use the following formula for the calculation of the inner products.

$$\chi^\mu \chi^\nu = \sum_\lambda f_{\mu\nu}^\lambda \chi^\lambda,$$

where $f_{\mu\nu}^\lambda$ is calculated using the charcters.

$$f_{\mu\nu}^\lambda = \frac{1}{g} \sum_\rho h_\rho \chi_\rho^\mu \chi_\rho^\nu \chi_\rho^\lambda$$

where the sum is taken over all the classes, g is the order of the group and h_ρ is the number of elements in the class ρ.

6 Plethysm of S-functions

It is well known that the character of a polynomial representation of a linear group $GL(n, C)$ is a symmetric polynomial $\rho(x_1, x_2, \cdots, x_n)$ in n variables. If μ is the character of a representation of $GL(n, C)$ and ν is the character of a representation of $GL(m, C)$ then their *composition* or *plethysm* will be the character of a representation of $GL(nm, C)$. Plethysm as a composition or substitution of polynomial functions was introduced by Littlewood [5]. Since then it has found many applications in physics [10, 16]. The S-function method as is very tedious and inefficient. Recently, Thibon has inferred a formula from Murnaghan [9] using the differential operators associated with the power sum symmetric functions [15]. This method requires the characters of symmetric groups. The algorithm developed in this paper is very useful in carrying out this method of the plethysm of S-functions.

THEOREM 2 *Let s_μ, $s_\nu \in \Lambda_Q$, then*

$$D(p_n)(s_\mu \otimes s_\nu) = \sum_{m|n} ((D(p_m)s_\mu) \otimes s_\nu)\, p_m(D(p_{n/m})s_\nu), \tag{6}$$

where $(s_\mu \otimes s_\nu)$ is the plethysm of the S-functions s_μ and s_ν.

As an example, let $f_{(2)(2)}^{(2,2)}$ be the number of times the cycle structure (22) appears in the expansion of $(s_2 \otimes s_2)$ then $f_{(2)(2)}^{(2,2)}$ is obtained by the application of $D(p_{(2,2)})$ on $(s_2 \otimes s_2)$ as follows.

$$\begin{aligned}
D(p_{(2,2)})(s_2 \otimes s_2) &= D(p_2)\{D(p_2)(s_2 \otimes s_2)\} \\
&= D(p_2)\{s_1 \otimes s_2 p_1(D(p_2)s_2) + s_0 \otimes s_1 p_2(D(p_1)s_2)\}
\end{aligned}$$

$$\begin{aligned} &= D(p_2)\{s_2 + p_2\} \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

Thus $f^{(2,2)}_{(2)(2)} = 3$. Similary $f^{(3,1)}_{(2)(2)} = 0$, $f^{(2,1,1)}_{(2)(2)} = 1$, $f^{(1^4)}_{(2)(2)} = 3$ and $f^{(4)}_{(2)(2)} = 1$. Now the power sum symmetric functions can be converted back using the follwing relation [7].

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda$$

where χ^λ_ρ has the usual meaning. Thus

$$(s_2 \otimes s_2) = s_{(4)} + s_{(2,2)}.$$

Our algorithm provides an easy way of calculated the individual characters as they are needed in the expansion of S-function plethysm.

7 Spin characters of the symmetric groups

Spin characters arise from Schur's study of the linear fractional substitution representation group Γ_n of the symmetric group S_n. For details we refer to [3, 8]. A detailed method for the calculation of spin characters is given in [1, 8]. Morris has given a polynomial expansion method that we will briefly describe here.

Let $\zeta^{(\lambda)}_{(\rho)}$ denote the spin character associated with the irreceducible representation λ and class ρ.

THEOREM 3 (Theorem 4, *[8]*)
The basic spin character of the class $(\rho) = (1^{m_1}, 3^{m_3}, 5^{m_5}, \cdots)$, *that is,* ζ^n_ρ *of* Γ_n *is* $2^{(l(\rho)-1-\epsilon)/2}$, *where* $\epsilon = 0$ *or* 1 *according as* n *is odd or even. Further, the basic spin character of class* (n) *when* n *is even, is* $i^{n/2}\sqrt{(n/2)}$, *where* $i = \sqrt{-1}$. *The basic spin character of all other classes is zero.*

In order to calculate the remaining spin characters of Γ_n, Schur's Q-functions are used.

THEOREM 4 (Theorem 5, *[8]*)
For each partition $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)})$ *of* n *into* $l(\lambda)$ *distinct parts, the simple spin character* $\zeta^{(\lambda)}_{(\rho)}$ *of the positive class* (ρ) *of* Γ_n *are given by*

$$Q_\lambda = \sum_{\rho \in OP_n} 2^{(l(\lambda)+l(\rho)+\epsilon)/2} z_\rho^{-1} \zeta^{(\lambda)}_{(\rho)} p_\rho \tag{7}$$

where ϵ *is 0 if* $l(\lambda) + l(\rho)$ *is even and 1 otherwise and* OP_n *is the set of odd part partitions of weight* n. *If* $\epsilon = 0$, $\zeta^{(\lambda)}_{(\rho)}$ *is a double spin character, and if* $\epsilon = 1$, $\zeta^{(\lambda)}_{(\rho)}$ *is an associate spin character. When* $\epsilon = 1$, *the negative class* λ *has a nonzero spin character, given by*

$$\zeta^{(\lambda)}_{(\lambda)} = i^{(n-l(\lambda)+1)/2}\sqrt{\frac{\lambda_1 \times \lambda_2 \times \cdots \times \lambda_{l(\lambda)}}{2}}.$$

In his method, Eq. 1 is used to expand Q_λ in terms of q_μ's and then these $q_{(\mu)}$'s are expanded in terms of power sum symmetric functions using the relation

$$q_r = \sum_{\rho \in OP_r} z_\rho^{-1} 2^{l(\rho)} p_\rho. \tag{8}$$

Then comparing the coefficients of the power sum symmetric functions from Eq. 7, the characters $\zeta^{(\lambda)}_{(\rho)}$ are calculated. This method requires huge expansions and the whole representations has to be exapnded even if a single character is needed.

EXAMPLE 1 *Consider* Γ_4. *The basic spin characters,* $\zeta^{(4)}_{(\rho)}$ *can be calculated using Theorem 3. The basic spin characters of positive classes,* (1^4) *and* $(3,1)$ *are given by*

$$\zeta^{(4)}_{(1^4)} = 2^{4-1-1)/2} = 2,$$

and

$$\zeta^{(4)}_{(3,1)} = 2^{2-1-1)/2} = 1.$$

Since n *is even and* $\epsilon = 1$, *we also have the basic spin character of the class* (4), *that is,*

$$\zeta^{(4)}_{(4)} = i^{4/2}\sqrt{(4/2)} = -\sqrt{2}.$$

The basic spin character of all other classes is zero.

For the characters $\zeta^{(3,1)}_{(\rho)}$ *we need to expand* $Q_{(3,1)}$ *in terms of* $q_{(\mu)}$*s using Eq. 1, that is,*

$$\begin{aligned} Q_{(3,1)} &= (1 - 2R_{12})q_{(3,1)} \\ &= q_{(3,1)} - 2q_{(4)} \end{aligned} \tag{9}$$

Now using Eq. 8 we can exapnd the $q_{(\mu)}$*s in terms of power sum symmetric functions, that is,*

$$\begin{aligned} q_{(3)} &= \tfrac{2}{3}p_{(3)} + \tfrac{4}{3}p_{(1,1,1)} \\ q_{(1)} &= 2p_{(1)} \\ q_{(3,1)} = q_{(3)}q_{(1)} &= \tfrac{4}{3}p_{(3)} + \tfrac{8}{3}p_{(1,1,1)} \\ q_{(4)} &= \tfrac{4}{3}p_{(3,1)} + \tfrac{2}{3}p_{(1,1,1,1)} \end{aligned}$$

Substituting the above to Eq. 9 we get

$$Q_{(3,1)} = -\frac{4}{3}p_{(3,1)} + \frac{4}{3}p_{(1,1,1,1)} \tag{10}$$

Now we expand $Q_{(3,1)}$ *in terms of power sum symmetric functions and spin characters, using Eq. 7, that is,*

$$Q_{(3,1)} = \frac{4}{3}\zeta^{(3,1)}_{(3,1)}p_{(3,1)} + \frac{1}{3}\zeta^{(3,1)}_{(1,1,1,1)}p_{(1,1,1,1)} \tag{11}$$

Finally comparing the coefficients of power sum symmetric function in the equations 10 and 11 we calculate the following characters.

$$\zeta^{(3,1)}_{(3,1)} = -1 \quad \text{and} \quad \zeta^{(3,1)}_{(1,1,1,1)} = 4.$$

8 The new algorithm for spin characters

In this section we develop a very efficient and simple method of calculating the spin characters. Other main advantage of this method is that a particular character associated with a particular class can be calculated without expanding the whole representation. This feature enables us to calculate the Kronecker products of the irreducible representations of Γ_n for very large values of n.

In the basis of power sum symmetric functions we have the adjoint multiplication by p_n, $D(p_n) : \Lambda_t \to \Lambda_t$, defined as [7]

$$\langle D(p_n)p_\mu, p_\nu\rangle_t = \langle p_\mu, p_n p_\nu\rangle_t$$

which is zero if $\mu \neq \nu \cup (n)$, and is equal to $z_\mu(t)$ if $\mu = \nu \cup (n)$. It follows that $D(p_n) = \frac{n}{1-t^n}\frac{\partial}{\partial p_n}$ acting on symmetric functions expressed in terms of power sum symmetric functions and for a partition $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_l)$ we have $D(p_\lambda) = D(p_{\lambda_1})D(p_{\lambda_2})\cdots D(p_{\lambda_l})$. This leads to the following property of these operators.

$$D(p_n)f_\lambda = \sum_i f_{\lambda_1,\lambda_2,\cdots,\lambda_i-n,\cdots} \tag{12}$$

where f_λ is an element of Λ_t indexed by the partition λ. The above list is obtained by subtracting n from each part of λ. The resulting list of symmetric functions are modified according to the rules related to them. For example, Schur's Q-functions are modified by the following rules given in [12].

1.

$$Q_{(\cdots,\lambda_i,\lambda_{i+1},\cdots)} = -Q_{(\cdots,\lambda_{i+1},\lambda_i,\cdots)},$$

whenever $|\lambda_{i+1}| > |\lambda_i|$, remembering that $Q_{(\mu,0)} = Q_{(\mu)}$.

2. Q-functions with consecutive repeated parts are null.

3.

$$Q_{(\lambda_1\cdots,-\lambda_p,\lambda_p,\cdots,\lambda_l)} = (-1)^{\lambda_p} 2Q_{(\lambda_1,\cdots,\lambda_l)}.$$

4. Any Q-function indexed by a partition in decreasing order with last part a negative number is null and $Q_0 = 1$.

EXAMPLE 2

$$D(p_3)Q_{(6,3,2,1)} = Q_{(3,3,2,1)} + Q_{(6,0,2,1)} + Q_{(6,3,-1,1)} + Q_{(6,3,2,-2)}.$$

According to rule 2, $Q_{(3,3,2,1)} = 0$ *and the rule 4 suggests* $Q_{(6,3,2,-1)} = 0$. *Rule 3 will lead to* $Q_{(6,3,-1,1)} = -2Q_{(6,3)}$ *and repeatative use of rule 1 gives* $Q_{(6,0,2,1)} = Q_{(6,2,1)}$. *Thus*

$$D(p_3)Q_{(6,3,2,1)} = Q_{(6,2,1)} - 2Q_{(6,3)}.$$

Using these properties we give following theorem.

THEOREM 5 *The simple spin character* $\zeta_{(\mu)}^{(\lambda)}$ *of positive classes of* Γ_n *is the integer* $2^{(l(\mu)-l(\lambda)-\epsilon)/2}D(p_\mu)Q_\lambda$ *, where* $|\lambda| = |\mu| = n$.

Proof

We apply the operator $D(p_\mu)$ on both sides of Eq.7:

$$D(p_\mu)Q_\lambda = D(p_\mu) \sum_{\rho\in OP_n} 2^{(l(\lambda)+l(\rho)+\epsilon)/2} z_\rho^{-1} \zeta_\rho^\lambda p_\rho.$$

Every term on right hand side of the above equation would be zero except for $\mu = \rho$ and the left hand side is an integer. On simplification we get

$$m = 2^{(l(\lambda)+l(\mu)+\epsilon)/2} z_\mu^{-1} \zeta_\mu^\lambda z_\mu 2^{-l(\mu)}$$

where m is an integer. Thus

$$\zeta_\mu^\lambda = 2^{(l(\mu)-l(\lambda)-\epsilon)/2} m = 2^{(l(\mu)-l(\lambda)-\epsilon)/2} D(p_\mu)Q_\lambda.$$

EXAMPLE 3 *Using the above theorem and the Q-function modification rules we calculate the follwing character in a few simple steps.*

$$\begin{aligned}
\zeta_{(7,5,3)}^{(9,6)} &= 2^{(3-2-1)/2} D(p_{(7,5,3)})Q_{(9,6)} \\
&= D(p_7)D(p_5)D(p_3)Q_{(9,6)} \\
&= D(p_7)D(p_5)\{Q_{(9,3)} + Q_{(6,6)}\} \\
&= D(p_7)\{Q_{(9,-2)} + Q_{(4,3)}\} \\
&= \{Q_{(4,-4)} + Q_{(-3,3)}\} \\
&= (-1)^3 2Q_0 \\
&= -2
\end{aligned}$$

9 Kronecker products of spin irreducible representations

The Kronecker, or inner product corresponds to the tensor product of the representations of Γ_n. For details, please see [13, 14]. If ζ^μ and ζ^ν are the spin irreducible representations corresponding to the distinct part partitions μ and ν then their Kronecker product is given as

$$\zeta^\mu \zeta^\nu = \epsilon_\mu \epsilon_\nu \sum_\lambda f^\lambda_{\mu\nu} \chi^\lambda, \tag{13}$$

where $\epsilon_\mu = 2$ if $(n - l(\mu))$ is odd and 1 otherwise, λ is a partition of n and χ^λ is an ordinary irreducible representations of Γ_n. An indirect method of calculating the coefficients $f^\lambda_{\mu\nu}$ is given in [13]. That method exploits the combinatorial properties of the shifted Young tablaux and is rather complicated. Since we have developed a very efficient algorithm for the spin characters, it is more reasonable to use these characters for the Kronecker products. This method is simple and efficient. The following formula gives the coefficients $f^\lambda_{\mu\nu}$ in terms of the characters.

$$f^\lambda_{\mu\nu} = \frac{1}{g} \sum_\rho h_\rho \zeta^\mu_\rho \zeta^\nu_\rho \chi^\lambda_\rho \tag{14}$$

where the sum is taken over all the classes, g is the order of the group and h_ρ is the number of elements in the class ρ.

10 Plethysm of Q-functions

Analogous to the Eq.6 of S-function plethysm we can derive the following for Q-functions.

THEOREM 6 *Let Q_μ, $Q_\nu \in \Lambda_Q$, then*

$$D(p_n)(Q_\mu \otimes Q_\nu) = \sum_{m|n} ((D(p_m)Q_\mu) \otimes Q_\nu)\, p_m(D(p_{n/m})Q_\nu), \tag{15}$$

where $(Q_\mu \otimes Q_\nu)$ is the plethysm of the Q-functions Q_μ and Q_ν and m is an odd integer.

The restriction that m be an odd integer makes it computationally more efficient than the plethysm of S-functions

11 Important features of the software

This software can calculate the ordinary χ^λ_ρ and the spin ζ^λ_ρ characters of the irreducible representation labelled by the partition λ associated with the class structure ρ. It can also calculate the dimension of the representation, number of elements in a class and the characters of a given ordinary or spin representation. It also generates the entire character table for a given integer n. Using these characters it performs operations like Kronecker products of the spin and ordinar irreducible representations and the plethysm of both the S and $Q-$functions associated with the ordinary and spin irreducible representations respectively.

For larger outputs it has the facility of saving it on a log file either as a standard output or in LaTeX format. A built-in demo and on-line help is available to make it user friendly. The line command format makes it easy to follow through. Once the command is entered the software will ask for the required inputs and options one at a time.

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