Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa–Intriligator formula.

(Extended abstract)

Anatol N. Kirillov*

Centre de Recherchers Mathematiques, Université de Montréal,

C.P. 6128, Succursale, Centre-Ville, Montréal, Québec, H3C 3J7, Canada

and

Toshiaki Maeno†

Department of Mathematics, Faculty of Science, Kyoto University,

Sakyo-ku, Kyoto 606, Japan

Introduction

The cohomology ring of the flag variety $Fl_n = SL_n/B$ is isomorphic to the quotient ring of the polynomial ring by the ideal generated by symmetric polynomials without constant term. The Schubert cycles give a linear basis of the cohomology ring and they are represented by Schubert polynomials. Our aim is to introduce the notion of quantum double Schubert polynomials, which represent the Schubert cycles in the equivariant quantum cohomology ring, and investigate their properties.

*On leave from *Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191011, Russia*

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Our approach based on the quantum Cauchy identity ([KM], Theorem 3)

$$\sum_{w\in S_n}\widetilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y)=\widetilde{\mathfrak{S}}_{w_0}(x,y):=\prod_{k=1}^{n-1}\Delta_k(y_{n-k}\mid x_1,\ldots,x_k),$$

and on the Lascoux-Schützenberger type formula for the quantum double Schubert polynomials ([KM], Definition 4)

$$\widetilde{\mathfrak{S}}_w(x,y)=\partial^{(y)}_{ww_0}\widetilde{\mathfrak{S}}_{w_0}(x,y).$$

We start from the Jack–Macdonald type definition ([M], Chapter VI) of quantum Schubert polynomials and deduce the properties of quantum and double quantum Schubert polynomials from the quantum Cauchy identity. A proof of quantum Cauchy identity based on geometrical approach due to I. Ciocan-Fontanine [C]. As a corollary of quantum Cauchy identity, we obtain the following simple formula for the quantum Schubert polynomials:

$$\widetilde{\mathfrak{S}}_w(x)=\partial^{(y)}_{ww_0}\widetilde{\mathfrak{S}}_{w_0}(x,y)\mid_{y=0}.$$

Finally, we define the extended coloured Ehresmannöeder for the symmetric group and give a quantum analog of Pieri's rule.

We would like to mention, that in the recent preprint "Quantum Schubert polynomials" by S. Fomin, S. Gelfand and A. Postnikov, [FGP], developed a different approach to the theory of quantum Schubert polynomials, based on the remarkable family of commuting operators X_i ([FGP], (3.2)). Among main results, obtained by S. Fomin, S. Gelfand and A. Postnikov, are definitions, orthogonality, quantum Monk's formula and other properties of quantum Schubert polynomials; definition of quantization map and quantum multiplication.

Besides some overlap with the preprint of S. Fomin, S. Gelfand and A. Postnikov, our works were done independently and based on the different approaches. We obtained, among others, the following new results:

1) definition of big quantum Schubert polynomials, using the residue pairing ([KM], Introduction);
2) quantum Cauchy identity (Theorem B);
3) Lascoux–Schützenberger's type formula for quantum Schubert polynomials (Theorem C);
4) quantum double Schubert polynomials (Theorem–Definition A, and [KM], Section 3);

5) equivariant quantum Pieri's rule (Theorem E, and [KM], Corollary 7);
6) Vafa–Intriligator's type formula for higher genus correlation functions on the flag manifold (Theorem D);
7) residue formula ([KM], Section 8.3).

Schubert polynomials, definition and known results

Let us remind ([LS2], [M1], [F]) some facts on the cohomology ring of the flag variety. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbf{C}^n \otimes \mathcal{O}_{Fl_n}$$

be the universal flag of subbundles on Fl_n. It is well known that the cohomology ring $H^*(Fl_n, \mathbf{Z})$ is generated by $x_i = c_1(E_{n-i+1}/E_{n-i})$, $i = 1, \ldots, n$, and

$$H^*(Fl_n, \mathbf{Z}) = P_n/I_n,$$

where $P_n = \mathbf{Z}[x_1, \ldots, x_n]$ and I_n is an ideal generated by elementary symmetric polynomials $e_1(x), \ldots, e_n(x)$.

We consider the universal sequence of quotient bundles

$$L_n \to \cdots \to L_1,$$

where $L_i = \mathbf{C}^n \otimes \mathcal{O}_{Fl_n}/E_{n-i}$ and fix a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{C}^n.$$

Then we have induced maps

$$f_{pq} : V_p \otimes \mathcal{O}_{Fl_n} \to L_q.$$

For a permutation $w \in S_n$, the Schubert cycle Ω_w is defined as the locus where $\mathrm{rank} f_{pq} \leq r_w(q,p)$, $1 \leq p, q \leq n-1$, and $r_w(q,p) := \sharp\{i \mid i \leq q, w_i \leq p\}$. Schubert cycles give an orthonormal basis of $H^*(Fl_n, \mathbf{Z})$ with respect to the intersection pairing.

In the quotient ring $P_n/I_n = H^*(Fl_n, \mathbf{Z})$, the Schubert cycles are represented by Schubert polynomials. Let us define the divided difference operator ∂_i acting on P_n by

$$(\partial_i f)(x) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$

Any permutation $w \in S_n$ is decomposed into a product of simple transpositions. We choose a reduced decomposition $w = s_{a_1} \cdots s_{a_p}$, where $p = l(w)$ and s_i is the simple transposition $(i, i+1)$. Then the operator ∂_w is given by

$$\partial_w = \partial_{a_1} \cdots \partial_{a_p}.$$

Definition (Lascoux-Schützenberger [LS1]). *For each permutation $w \in S_n$ the Schubert polynomial $\mathfrak{S}_w$ is defined to be*

$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0}(x^\delta),$$

where w_0 is the longest element of S_n and δ is a multi-index $(n-1, \ldots, 1, 0)$.

The intersection pairing on the cohomology ring coincides with a pairing induced by the Grothendieck residue

$$\langle f, g \rangle_{I_n} = \mathrm{Res}_{I_n}(fg), \quad f, g \in P_n,$$

where $P_n := \mathbf{Z}[x_1, \ldots, x_n]$.

We refer to [GH], Chapter 5, for definition and basic properties of Grothendieck's residue.

It is well known that the Schubert polynomials form an orthonormal basis with respect to the pairing $\langle , \rangle_{I_n}$. Conversely, this property characterize the Schubert polynomials. Namely, Schubert polynomials can be obtained as Gram-Schmidt's orthogonalization of the set of ordered lexicographically monomials $\{x^I\}_{I \subset \delta}$, with respect to the scalar product $\langle , \rangle_{I_n}$, cf. [KV]. We will use this property of classical Schubert polynomials in order to define the quantum Schubert polynomials. The study of quantum Schubert polynomials was initiated in [FGP] and, independently, in [KM]. The Grassmanian case was considered earlier in [B], [C] and [W]. We refer to [MS] and [RT] for definition and basic properties of the quantum cohomology.

Main results

Let us explain briefly the definition of quantum and quantum double Schubert polynomials and the main results on them. Follow to A. Givental and B. Kim [GK], and I. Ciocan–Fontanine [C], we define the quantum elementary symmetric polynomials $\widetilde{e}_1, \ldots, \widetilde{e}_n$ by the formula

$$\det \begin{pmatrix} x_1+t & q_1 & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & x_2+t & q_2 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & x_3+t & q_3 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & x_{n-2}+t & q_{n-2} & 0 \\ 0 & \ldots & \ldots & 0 & -1 & x_{n-1}+t & q_{n-1} \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & x_n+t \end{pmatrix}$$

$$= t^n + \widetilde{e}_1 t^{n-1} + \widetilde{e}_2 t^{n-2} + \cdots + \widetilde{e}_n,$$

where $q_1, \ldots, q_{n-1}$ are the independent parameters. The defining ideal $\tilde{I}_n$ of the small quantum cohomology ring is generated by the quantum elementary symmetric polynomials, namely

$$QH^*(Fl_n) = \mathbf{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]/(\widetilde{e}_1, \ldots, \widetilde{e}_n).$$

We have to find the polynomials which represent the Schubert cycles in the quantum cohomology ring. In the quantum cohomology ring, the intersection pairing is identified with the pairing induced by the residue pairing

$$\langle f, g \rangle_Q = \mathrm{Res}_{\tilde{I}_n}(fg), \quad f, g \in \bar{P}_n = \mathbf{Z}[x; q],$$

with values in $\mathbf{Z}[q]$. Based on the analogy with classical case (cf. [KV]), we give the Jack-Macdonald type definition (see [M2], chapter VI) of quantum Schubert polynomials

Definition *Define the quantum Schubert polynomials $\widetilde{\mathfrak{S}}_w$ as Gram-Schmidt's orthogonalization of the set of lexicographically ordered monomials $\{x^I \mid I \subset \delta\}$ with respect to the residue pairing $\langle , \rangle_Q$:*

1) $\langle \widetilde{\mathfrak{S}}_u, \widetilde{\mathfrak{S}}_v \rangle_Q = \langle \mathfrak{S}_u, \mathfrak{S}_v \rangle = \begin{cases} 1, & \text{if } v = w_0 u \\ 0 & \text{otherwise} \end{cases}$

2) $\widetilde{\mathfrak{S}}_w(x) = x^{c(w)} + \sum_{I < c(w)} a_I(q) x^I$, *where* $a_I(q) \in \mathbf{Z}[q_1, \ldots, q_{n-1}]$ *and* $I < c(w)$ *means the lexicographic order.*

Here $c(w)$ is the code of a permutation $w \in S_n$, [M], p.9.

Though we treat the (small) quantum cohomology ring, it turns out that to work with the equivariant quantum cohomology algebra ([GK], [K2]) is more convenient. The main reason is that one can find Lascoux–Schützenberger's type representative for any equivariant quantum cohomology class. In other words, each quantum double Schubert polynomial $\widetilde{\mathfrak{S}}_w(x,y)$ can be obtained from the top one by using the divided difference operators.

Theorem-Definition A *Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ be two sets of variables, and*

$$\widetilde{\mathfrak{S}}_{w_0}(x,y) := \prod_{i=1}^{n-1} \Delta_i(y_{n-i} \mid x_1,\ldots,x_i),$$

where $\Delta_k(t \mid x_1,\ldots,x_k) := \sum_{j=0}^{k} t^{k-j} e_j(x_1,\ldots,x_k \mid q_1,\ldots,q_{k-1})$ is the generating function for the quantum elementary symmetric functions in $x_1,\ldots,x_k$. Then $\widetilde{\mathfrak{S}}_w(x,y) = \partial^{(y)}_{ww_0}\widetilde{\mathfrak{S}}_{w_0}(x,y)$.

One of our main results is the quantum analog of Cauchy's identity for (classical) Schubert polynomials, [M], (5.10).

Theorem B (Quantum Cauchy's identity)

$$\sum_{w\in S_n} \widetilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y) = \widetilde{\mathfrak{S}}_{w_0}(x,y). \tag{1}$$

Corollary *For each permutation $w \in S_n$,*

$$\widetilde{\mathfrak{S}}_w(x,y) = \sum_{u\in S_n,\ l(u)+l(uw^{-1})=l(w)} \widetilde{\mathfrak{S}}_u(x,z)\mathfrak{S}_{uw^{-1}}(y,-z).$$

This theorem is proved in geometric way by using the arguments due to I. Ciocan-Fontanine [C] (see also [K1]); more particularly, we reduce directly a proof of Theorem B to that of the following geometric statement:

Lemma *Let $I \subset \delta = (n-1, n-2, \ldots, 1, 0)$ and $w \in S_n$ be a permutation, then*

$$\langle \widetilde{e}_I(x), \widetilde{\mathfrak{S}}_w(x)\rangle_Q = \langle e_I(x), \mathfrak{S}_w(x)\rangle, \tag{2}$$

where $e_I(x) := \prod_{k=1}^{n-1} e_{i_k}(x_1,\ldots,x_{n-k})$

(resp. $\widetilde{e}_I(x) := \prod_{k=1}^{n-1} \widetilde{e}_{i_k}(x_1,\ldots,x_{n-k} \mid q_1,\ldots,q_{n-k-1})$)

is the elementary polynomial (resp. quantum elementary polynomial).

By product, it follows from our proof that quantum Schubert polynomials $\hat{\mathfrak{S}}_w(x)$ defined geometrically (cf. [C]) coincide with those defined algebraically:

$$\hat{\mathfrak{S}}_w(x) \equiv \tilde{\mathfrak{S}}_{w^{-1}}(x) \pmod{\tilde{I}}.$$

It is interesting to note, that the intersection numbers $\langle e_I(x), \mathfrak{S}_w(x)\rangle$ (which are nonnegative!) are precisely the coefficients of corresponding Schubert polynomial:

$$\mathfrak{S}_w(x) = \sum_{I\subset\delta} \langle e_I(x), \mathfrak{S}_w(x)\rangle x^{\delta - I}.$$

The quantum Cauchy formula (1) plays the important role in our approach to the quantum double Schubert polynomials. As a direct consequence of (1), we obtain the Lascoux–Schützenberger type formula for quantum Schubert polynomials (cf. Theorem-Definition A).

Theorem C *Let $\tilde{\mathfrak{S}}_{w_0}(x,y)$ be as in Theorem-Definition A, then*

$$\tilde{\mathfrak{S}}_w(x) = \partial^{(y)}_{ww_0} \tilde{\mathfrak{S}}_{w_0}(x,y)|_{y=0}.$$

We introduce a quantization map

$$P_n \to \overline{P}_n, \quad f \mapsto \tilde{f}$$

by the rule

$$\tilde{f}(x) = \sum_{w\in S^{(n)}} \partial^{(y)}_w f(y) \tilde{\mathfrak{S}}_w(x,y) \mid_{\overline{P}_n},$$

where for a polynomial $f \in \overline{P}_\infty$, the symbol $f|_{\overline{P}_m}$ means the restriction of f to the ring of polynomials $\overline{P}_m$.

The quantization is a linear map which preserves the pairings, i.e.,

$$\langle \tilde{f}, \tilde{g}\rangle_Q = \langle f, g\rangle, \quad f,g \in P_n.$$

Using the quantum Cauchy formula (1), we prove that the quantum double Schubert polynomials are the quantization of classical ones. Another class of polynomials having a nice quantization is the set of elementary polynomials

$$e_I(x) := \prod_{k=1}^{n-1} e_{i_k}(x_1, \ldots x_{n-k}), \quad I = (i_1, \ldots, i_{n-1}) \subset \delta.$$

It follows from Theorem B that the quantization $\tilde{e}_I(x)$ of elementary polynomial $e_I(x)$ is given by

$$\tilde{e}_I(x) = \prod_{k=1}^{n-1} e_{i_k}(x_1, \ldots, x_{n-k} \mid q_1, \ldots, q_{n-k-1}).$$

Remark To our knowledge, originally, construction of the quantization map, using a remarkable family of commuting operators X_i, appeared in [FGP]; independently, construction of quantization map was introduced in [KM] in a different form, using the Interpolation formula and quantum double Schubert polynomials. The fact that the quantum elementary polynomials $\widetilde{e}_I(x)$ are the quantization of classical ones (in the sense of [FGP]) was obtained in [FGP]. It can be shown that two forms of quantizations mentioned above are equivalent.

Based on several examples, we make a conjecture ("quantum Schur functions ") that quantization of the flagged Schur function (see [M], (3.1), (4.9) and (6.16))

$$s_{\lambda/\mu}(X_1, \ldots, X_n) = \det \left(h_{\lambda_i - \mu_j - i + j}(X_i) \right)_{1 \leq i,j \leq n}$$

is given by

$$\tilde{s}_{\lambda/\mu}(X_1, \ldots, X_n) = \det \left(\tilde{h}_{\lambda_i - \mu_j - i + j}(X_i) \right)_{1 \leq i,j \leq n},$$

where $\tilde{h}_k(X)$ is the quantum complete homogeneous symmetric polynomial of degree k, and $X_1 \subset \cdots \subset X_n$ are the flagged sets of variables (for the definitions of polynomials $\widetilde{h}_k(x)$ see [FGP], Section 7.3, and [KM], Section 5.2).

More generally, using the quantization procedure, we can define the quantum Grothendieck, quantum Macdonald and quantum Key polynomials. The work is in progress and we hope to present our results in the nearest future.

Now let us consider a problem how to quantize the monomials. It seems to be difficult to find an explicit determinantal formula for a quantum monomial $\tilde{x}^I$, i.e., to find a quantum analog of the Billey-Jockusch-Stanley formula for Schubert polynomials in terms of compatible sequences [BJS]. We obtain the following formulae for quantum monomials

$$\tilde{x}^I = \sum_{w \in S_n} \eta(\partial_w x^I) \tilde{\mathfrak{S}}_w(x), \quad I \subset \delta,$$

$$\tilde{\mathfrak{S}}_{w_0}(x,y) = \sum_{I\subset\delta} \tilde{x}^I e_{\delta-I}(y).$$

As an application of our results, we give the higher genus analog of the Vafa–Intriligator type formula for the flag manifold.

Theorem D *Let $\langle P(x_1,\ldots,x_n)\rangle_g$ be the genus g correlation function corresponding to a polynomial P. Then*

$$\begin{aligned}\langle P(x_1,\ldots,x_n)\rangle_g &= \mathrm{Res}_{\tilde{I}}(P\Phi^g)\\ &= \sum_{\widetilde{e}_1=\cdots=\widetilde{e}_n=0} P(x_1,\ldots,x_n)\det\left(\frac{\partial\widetilde{e}_i}{\partial x_j}\right)^{-1}(\Phi(x_1,\ldots,x_n))^g,\end{aligned}$$

where $\Phi(x) := \Phi(x|q) = \langle\widetilde{\mathfrak{S}}_{w_0}(x,y), \widetilde{\mathfrak{S}}_{w_0}(x,y)\rangle^{(y)} = \sum_{w\in S_n}\widetilde{\mathfrak{S}}_w(x)\widetilde{\mathfrak{S}}_{w_0w}(x)$.

We also study a problem how to compute the quantum residues. This is very important for computation of small quantum cohomology ring correlation functions and the Gromov–Witten invariants. For this purpose we introduce the quantum residues generating function

$$\Psi(t) = \langle\prod_{i=1}^{n-1}\frac{t_i}{t_i - x_i}\rangle.$$

Then we have

$$\mathrm{Res}_{\tilde{I}} P(t_1,\ldots,t_{n-1}) = \mathrm{Res}_I\left(P(x_1,\ldots,x_{n-1})\Psi(x)\right).$$

So it is important to determine this generating function. We can give a characterization of this function as the unique solution to some system of differential equations, see [KM], Proposition 16.

Finally, we introduce the extended coloured Ehresmannöeder and give the equivariant quantum Pieri rule, [KM], Corollary 7. The extended coloured Ehresmannöeder for S_n is a set of all Ehresmann-Bruhat paths on S_n (cf. [LS2]). Let us define the Ehresmann–Bruhat path (BE–path, for short) $v \Leftarrow w$ on S_n. First of all, let us remind a definition of the relation $v \to w$, [M1], p.5. Relation $v \to w$ means that

1) $l(w) = l(v) + 1$,

2) $w = v \cdot t$, where t is a transposition.

Secondly, we define (see, also, [FGP]) a relation $v \leftarrow w$. Relation $v \leftarrow w$ means that

1) $w = v \cdot t$, where t is a transposition,

2) $l(w) \geq l(v) + l(v^{-1}w)$. We define (see also [FGP]) a weight of an arrow $v \leftarrow w$, denoted by $wt(v \leftarrow w)$, to be equal to the product $q_i \ldots q_{i+s-1}$, if $t = t_{ij}$ and $2s := l(w) + 1 - l(v)$. We assume that weight of any arrow $v \rightarrow w$ is equal to 1.

Let us say that an arrow $v \leftarrow w$ (resp. $v \rightarrow w$) has a color k if $w = vt_{ij}$ and $1 \leq i \leq k < j \leq n$.

Finally, the Ehresmann–Bruhat path between two (ordered) permutations v and w in S_n (notation $v \Leftarrow w$) is a sequence of permutations $v_0, v_1, \ldots, v_r$ in S_n such that

$$v = v_0 \rightleftharpoons v_1 \rightleftharpoons v_2 \rightleftharpoons \cdots \rightleftharpoons v_r = w, \tag{3}$$

where symbol $v_i \rightleftharpoons v_{i+1}$ means either $v_i \rightarrow v_{i+1}$ or $v_i \leftarrow v_{i+1}$.

We denote the number r in a representation (3) by $l(v \Leftarrow w)$.

Let us define a weight of a BE–path $v \Leftarrow w$ as follows

$$wt(v \Leftarrow w) = \prod_{i=0}^{r-1} wt(v_i \rightleftharpoons v_{i+1}).$$

We will say that a BE–path $v \Leftarrow w$ has a color k, notation $v \overset{k}{\Longleftarrow} w$, if in the representation (3) all arrows $v_i \rightleftharpoons v_{i+1}$ $(i = 0, \ldots, r-1)$ have the same color k.

Theorem E (Quantum Pieri's rule). *Let us consider the Grassmanian permutation* $[b, d] = (1, 2, \ldots, b-d-1, b, b-d, b-d+1 \ldots, b-1, b+1, \ldots, n)$, *for* $2 \leq b \leq n$, $1 \leq d \leq b$. *Then*

$$\widetilde{\mathfrak{S}}_{[b,d]} \cdot \widetilde{\mathfrak{S}}_v \equiv \sum_w wt(v \overset{b}{\Longleftarrow} w) \widetilde{\mathfrak{S}}_w \pmod{\widetilde{I}_n},$$

where the sum runs over all BE–paths $v \overset{b}{\Longleftarrow} w$, *such that*

1) $l(v \Leftarrow w) = d$;

2) if $v_l = v_{l+1}(i_l j_l)$ $(l = 0, \ldots, d-1)$, *then all* i_l *are different.*

Remarks *i)* Generalization of Theorem E for quantum double Schubert polynomials (equivariant quantum Pieri's rule) is given in [KM], Section 9.

ii) A proof of important particular case of Theorem E for $d = 1$ (quantum Monk's formula) was first published in [FGP].

iii) To our knowledge, in the classical case $q = 0$, the Pieri rule for Schubert polynomials was first stated in [LS1], (2.2). Our formulation of Theorem E is very close to that given in [BB]. The difference is: we use the paths in the extended coloured Ehresmannöeder instead of the paths in the ordinary Ehresmann–Bruhat order (classical case). Very transparent proof of Monk's formula one can find in the I. Macdonald book [M1], (4.15). It is the proof that was generalized in [FGP] to the case of quantum Schubert polynomials. Recently, F. Sotile [S] gave a proof of the Pieri rule based on geometrical approach.

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