DIFFERENTIAL POSETS AND DOWN-UP ALGEBRAS

Georgia Benkart

Abstract. Down-up algebras originated in the study of differential posets. In this paper we discuss their combinatorial origins, representations, and structure. Down-up algebras exhibit many of the important features of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ including a Poincaré-Birkhoff-Witt type basis and a well-behaved representation theory. They have many interesting connections with Weyl algebras, quantum groups, and Witten's deformations of $U(\mathfrak{sl}_2)$.

Differential posets.

In [St2], Stanley introduced a class of partially ordered sets, which he termed *differential posets*. Many of the remarkable enumerative and combinatorial properties of these posets involve counting saturated chains $y_1 \prec y_2 \prec \cdots \prec y_k$ or Hasse walks $y_1, y_2, \ldots, y_k$, (where either y_{i+1} covers y_i or y_i covers y_{i+1}). Essential in the computations are two operators, d (down) and u (up), which are defined on the complex vector space $\mathbf{C}P$ having basis the elements of the poset P. If $y \in P$, then $d(y)$ is the sum of all elements that y covers and $u(y)$ is the sum of all elements that cover y. For many posets the down and up operators give well-defined linear transformations of $\mathbf{C}P$. Precursors of the operators d and u appeared in [St1] and [P], where they were used to show posets are Sperner or rank unimodal.

The characterizing property of an r-differential poset is that the down and up operators satisfy $du - ud = rI$ for some positive integer r (see [St2, Thm. 2.2]), where I is the identity transformation on $\mathbf{C}P$. Thus, the poset affords a representation of the Weyl algebra, (the associative algebra with generators y, x subject to the relation $yx - xy = 1$), via the mapping $y \mapsto d/r$, and $x \mapsto u$. Since the Weyl algebra also can be realized as differential operators $y \mapsto d/dx$ and $x \mapsto x$ (multiplication by x) on $\mathbf{C}[x]$, Stanley referred to the posets satisfying $du - ud = rI$ as *r-differential* or simply *differential* when $r = 1$. Fomin [F] studied essentially the same class of posets for $r = 1$, calling them Y-graphs". This terminology comes from the fact that Young's lattice Y of all partitions of all nonnegative integers is the prototypical example.

A partition μ of a nonnegative integer m can be regarded as a descending sequence $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ of parts whose sum $|\mu| = \sum_i \mu_i$ equals m. If $\nu = (\nu_1 \geq \nu_2 \geq \ldots)$ is a second partition, then $\mu \leq \nu$ when $\mu_i \leq \nu_i$ for all i.

1991 *Mathematics Subject Classification.* Primary 16S15, 16S30, 17B35, 17B10 Secondary 81R50, 06A07.

The author gratefully acknowledges support from National Science Foundation Grant #DMS–9622447.

The partition ν covers μ (written here as $\mu \prec \nu$) if $\mu < \nu$ and $|\nu| = 1 + |\mu|$. Thus, $\mu \prec \nu$ if the partition μ is obtained from ν by subtracting 1 from exactly one of the parts of ν, and $d(\nu)$ is the sum of all such μ. Analogously, $u(\nu)$ is the sum of all partitions π obtained from ν by adding 1 to one part of ν. Young's lattice Y is a 1-differential poset, and Y^r is r-differential ([St2, Cor. 1.4]).

The down and up operators on Young's lattice have a representation theoretic significance. The simple modules of the symmetric group S_n are indexed by the partitions ν of n. Upon restriction to S_{n-1}, the representation labelled by ν decomposes into a direct sum of simple S_{n-1}-modules indexed by the partitions $\mu \prec \nu$, so it is given by $d(\nu)$. When the simple module labelled by ν is induced to a representation of S_{n+1}, it decomposes into a sum of simple S_{n+1}-modules indexed by the partitions π of $n+1$ such that $\nu \prec \pi$, which is just $u(\nu)$.

In his study [T1] of uniform posets, Terwilliger considered finite ranked posets P whose down and up operators satisfy the following relation

$$d_i d_{i+1} u_i = \alpha_i d_i u_{i-1} d_i + \beta_i u_{i-2} d_{i-1} d_i + \gamma_i d_i,$$

where d_i and u_i denote the restriction of d and u to the elements of rank i. (There is an analogous second relation,

$$d_{i+1} u_i u_{i-1} = \alpha_i u_{i-1} d_i u_{i-1} + \beta_i u_{i-1} u_{i-2} d_{i-1} + \gamma_i u_{i-1},$$

which holds automatically in this case because d_{i+1} and u_i are adjoint operators relative to a certain bilinear form.) In many examples the constants in these relations do not depend on the rank i. In particular, a poset whose down and up operators satisfy

$$d^2 u = q(q+1)dud - q^3 ud^2 + rd$$
$$du^2 = q(q+1)udu - q^3 u^2 d + ru$$

where q and r are fixed complex numbers is said to be (q,r)-*differential.* Many interesting examples of (q,r)-differential posets in [T1] arise from considering certain subspaces of a vector space over the field $GF(q)$ of q elements:

(1) Assume W is an n-dimensional vector space over $GF(q)$ and consider the set of pairs $P = \{(U,f) \mid U$ is a subspace of W and f is an alternating bilinear form on $U\}$ with the ordering: $(U,f) \le (V,g)$ if U is a subspace of V and $g|_U = f$. Then P is a (q,r)-differential poset with $r = -q^n(q+1)$.

(2) In example (1), replace an alternating bilinear form" with a quadratic form". The resulting poset P is $(q, -q^{n+1}(q+1))$-differential.

(3) In this example assume W is an n-dimensional space over $GF(q^2)$ and the bilinear forms are Hermitian. The poset P is $(q^2, -q^{2n+1}(q^2+1))$-differential in this case.

Down-up algebras.

To better understand the algebra generated by the down and up operators of a poset and its action on the poset, we introduced the notion of a down-up algebra in our joint work with Roby (see [BR]). Although the initial motivation for our investigations came from posets, we made no assumptions about the existence of posets whose down and up operators satisfy our relations. However, when such a poset exists, it affords a representation of the down-up algebra, so our primary focus in [BR] was on determining explicit information about the representations of down-up algebras.

Definition 1. *Let α, β, γ be fixed but arbitrary complex numbers. The unital associative algebra $A(\alpha, \beta, \gamma)$ over $\mathbf{C}$ with generators d, u and defining relations*

(R1) $d^2u = \alpha dud + \beta ud^2 + \gamma d$,

(R2) $du^2 = \alpha udu + \beta u^2 d + \gamma u$,

is a down-up algebra.

It is easy to see that when $\gamma \neq 0$ the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$ by the map, $d \mapsto d'$, $u \mapsto \gamma u'$. Therefore, it would suffice to treat just two cases $\gamma = 0, 1$, but to avoid dividing considerations into these two cases, we retain the notation γ.

Examples of down-up algebras.

Example (i). If B is the associative algebra generated by the down and up operators d, u of a (q, r)-differential poset, then relations (R1) and (R2) hold with $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$. Thus, B is a homomorphic image of the algebra $A(\alpha, \beta, \gamma)$ with these parameters, and the action of B on the poset gives a representation of $A(\alpha, \beta, \gamma)$.

Example (ii). The relation $du - ud = rI$ of an r-differential poset can be multiplied on the left by d and on the right by d and the resulting equations can be added to get the relation $d^2u - ud^2 = 2rd$ of a $(-1, 2r)$-differential poset. Thus, the Weyl algebra is a homomorphic image (by the ideal generated by $du - ud - r1$) of the algebra $A(0, 1, 2r)$. Similarly, the q-Weyl algebra is a homomorphic image of the algebra $A(0, q^2, (q+1))$ by the ideal generated by $du - qud - 1$. The skew polynomial algebra $\mathbf{C}_q[d, u]$, or quantum plane (see [M]), is the associative algebra with generators d, u which satisfy the relation $du = qud$. Therefore, $\mathbf{C}_q[d, u]$ is a homomorphic image (by the ideal generated by $du - qud$) of the algebra $A(2q, -q^2, 0)$.

Example (iii). Consider the poset $\mathcal{L}(2, 2) = \{\underline{a} = (a_1, a_2) \mid 2 \geq a_1 \geq a_2$ and $a_1, a_2 \in \mathbf{Z}_{\geq 0}\}$ with the order relation $\underline{a} \leq \underline{b}$ if $a_i \leq b_i$ for $i = 1, 2$. This is just the set of partitions which fit into a 2×2 box. By direct calculation it is easy to verify that the down and up operators on this poset satisfy $d^2u = dud - ud^2 + d$ and $du^2 = udu - u^2d + u$, so the algebra they generate is a homomorphic image of the down-up algebra $A(1, -1, 1)$.

Example (iv). Suppose $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbf{C}$ with basis $x, y, [x, y]$ such that $[x[x, y]] = \gamma x$ and $[[x, y], y] = \gamma y$. In the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ where $[x, y] = xy - yx$, these relations become

$$x^2y - 2xyx + yx^2 = \gamma x$$
$$xy^2 - 2yxy + y^2x = \gamma y.$$

Thus, $U(\mathfrak{g})$ is a homomorphic image of the down-up algebra $A(2, -1, \gamma)$ via the mapping $\phi : A(2, -1, \gamma) \to U(\mathfrak{g})$ with $\phi : d \mapsto x$, $\phi : u \mapsto y$. The mapping $\psi : \mathfrak{g} \to A(2, -1, \gamma)$ with $\psi : x \mapsto d$, $\psi : y \mapsto u$, and $\psi : [x, y] \mapsto du - ud$ extends, by the universal property of $U(\mathfrak{g})$, to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2, -1, \gamma)$ which is the inverse of ϕ. Consequently, $U(\mathfrak{g})$ is isomorphic to $A(2, -1, \gamma)$.

The Lie algebra $\mathfrak{sl}_2$ of 2×2 complex matrices of trace zero has a standard basis $e = E_{1,2}, f = E_{2,1}$, and $h = E_{1,1} - E_{2,2}$ of matrix units, which satisfies $[e, f] =$

h, $[h,e] = 2e$, and $[h,f] = -2f$. From this we see that $U(\mathfrak{sl}_2) \cong A(2,-1,-2)$. The Heisenberg Lie algebra $\mathfrak{H}$ has a basis x, y, z where $[x,y] = z$, and $[z, \mathfrak{H}] = 0$, so $U(\mathfrak{H}) \cong A(2,-1,0)$.

Example (v). The 2×2 complex matrices $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ with supertrace $y_1 - y_4 = 0$ is the special linear Lie superalgebra $L = \mathfrak{sl}(1,1) = L_{\overline{0}} \oplus L_{\overline{1}}$ under the supercommutator $[x,y] = xy - (-1)^{ab} yx$ for $x \in L_{\overline{a}}$, $y \in L_{\overline{b}}$. It has a presentation by generators e, f (which belong to $L_{\overline{1}}$ and can be identified with the matrix units $e = E_{1,2}$, $f = E_{2,1}$) and relations $[e,[e,f]] = 0$, $[[e,f],f] = 0$, $[e,e] = 0$, $[f,f] = 0$. The universal enveloping algebra $U(\mathfrak{sl}(1,1))$ of $\mathfrak{sl}(1,1)$ has generators e, f and relations $e^2 f - fe^2 = 0$, $ef^2 - f^2 e = 0$, $e^2 = 0$, $f^2 = 0$. Thus, $U(\mathfrak{sl}(1,1))$ is a homomorphic image of the down-up algebra $A(0,1,0)$ by the ideal generated by the elements e^2 and f^2, which are central in $A(0,1,0)$.

Example (vi). The orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2) = L_{\overline{0}} \oplus L_{\overline{1}}$ has generators $x, y \in L_{\overline{1}}$ which satisfy

$$xy + yx = t \in L_{\overline{0}} \qquad tx - xt = x \qquad yt - ty = y.$$

By combining these relations, we see that its universal enveloping algebra $U(\mathfrak{osp}(1,2))$ is a homomorphic image of $A(0,1,1)$.

Example (vii). Consider the field $\mathbf{C}(q)$ of rational functions in the indeterminate q over the complex numbers, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra (quantum group) of a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ corresponding to the Cartan matrix $\mathfrak{A} = (a_{i,j})_{i,j=1}^n$. There are relatively prime integers ℓ_i so that the matrix $(\ell_i a_{i,j})$ is symmetric. Let

$$q_i = q^{\ell_i}, \qquad \text{and} \qquad [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$$

for all $m \in \mathbf{Z}_{\geq 0}$. When $m \geq 1$, let $[m]_i! = \prod_{j=1}^m [j]_i$. Set $[0]_i! = 1$ and define

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i![m-n]_i!}.$$

Then $U = U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbf{C}(q)$ with generators E_i, F_i, K_i, K_i^{-1} $(i = 1, \ldots, n)$ subject to the relations

(Q1) $K_i K_i^{-1} = K_i^{-1} K_i, \qquad K_i K_j = K_j K_i$

(Q2) $K_i E_j K_i^{-1} = q_i^{a_{i,j}} E_j \qquad K_i F_j K_i^{-1} = q_i^{-a_{i,j}} F_j$

(Q3) $E_i F_j - F_j E_i = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

(Q4) $\displaystyle\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1 - a_{i,j} \\ k \end{bmatrix}_i E_i^{1-a_{i,j}-k} E_j E_i^k = 0 \qquad \text{for} \qquad i \neq j$

(Q5) $\displaystyle\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1 - a_{i,j} \\ k \end{bmatrix}_i F_i^{1-a_{i,j}-k} F_j F_i^k = 0 \qquad \text{for} \qquad i \neq j.$

Suppose $a_{i,j} = -1 = a_{j,i}$ for some $i \neq j$, and consider the subalgebra $U_{i,j}$ generated by E_i, E_j. In this special case, the quantum Serre relation (Q4) reduces to

$$E_i^2 E_j - [2]_i E_i E_j E_i + E_j E_i^2 = 0 \qquad \text{and}$$
$$E_j^2 E_i - [2]_j E_j E_i E_j + E_i E_j^2 = 0.$$

Since $-\ell_i = \ell_i a_{i,j} = \ell_j a_{j,i} = -\ell_j$, the coefficients $[2]_i$ and $[2]_j$ are equal. The algebra $U_{i,j}$ (with q specialized to a complex number which is not a root of unity) is isomorphic to $A([2]_i, -1, 0)$ by the mapping $E_i \mapsto d$, $E_j \mapsto u$. The same result is true if the corresponding F's are used in place of the E's. In particular, when $\mathfrak{g} = \mathfrak{sl}_3$ (3×3 matrices of trace 0), the algebra $U_{i,j}$ is just the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by the E's.

Example (viii). To provide an explanation of the existence of quantum groups, Witten ([W1], [W2]) introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$. Witten's deformation is a unital associative algebra over a field $\mathbf{K}$ (which is algebraically closed of characteristic zero and which could be assumed to be $\mathbf{C}$) and depends on a 7-tuple $\underline{\xi} = (\xi_1, \ldots, \xi_7)$ of elements of $\mathbf{K}$. It has a presentation by generators x, y, z and defining relations

$$xz - \xi_1 zx = \xi_2 x \tag{2}$$
$$zy - \xi_3 yz = \xi_4 y \tag{3}$$
$$yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z. \tag{4}$$

We denote the resulting algebra by $\mathfrak{W}(\underline{\xi})$. In applications of these deformation algebras, the parameters depend on the coupling constant of the particular physical theory, and Witten [W2] gives an evaluation of them in the special case of the three-dimensional Chern-Simons gauge theory.

Let us assume $\xi_6 = 0$ and $\xi_7 \neq 0$. Then substituting expression (4) into (2) and (3) and rearranging shows that

$$-\xi_5 x^2 y + (1 + \xi_1\xi_5) xyx - \xi_1 yx^2 = \xi_2\xi_7 x$$
$$-\xi_5 xy^2 + (1 + \xi_3\xi_5) yxy - \xi_3 y^2 x = \xi_4\xi_7 y.$$

In particular, when $\xi_5 \neq 0$, $\xi_1 = \xi_3$, and $\xi_2 = \xi_4$ we obtain

$$x^2 y = \frac{1 + \xi_1\xi_5}{\xi_5} xyx - \frac{\xi_1}{\xi_5} yx^2 - \frac{\xi_2\xi_7}{\xi_5} x$$
$$xy^2 = \frac{1 + \xi_1\xi_5}{\xi_5} yxy - \frac{\xi_1}{\xi_5} y^2 x - \frac{\xi_2\xi_7}{\xi_5} y.$$

From this it is easy to see that a Witten deformation algebra $\mathfrak{W}(\underline{\xi})$ with $\xi_6 = 0$, $\xi_5\xi_7 \neq 0$, $\xi_1 = \xi_3$, and $\xi_2 = \xi_4$ is a homomorphic image of the down-up algebra $A(\alpha, \beta, \gamma)$ with

$$\alpha = \frac{1 + \xi_1\xi_5}{\xi_5}, \qquad \beta = -\frac{\xi_1}{\xi_5}, \qquad \gamma = -\frac{\xi_2\xi_7}{\xi_5}. \tag{5}$$

In fact, in [B, Thm. 2.6] we proved the following

Proposition 6. *A Witten deformation algebra* $\mathfrak{W}(\underline{\xi})$ *with*

$$\xi_6 = 0, \ \ \xi_5\xi_7 \neq 0, \ \xi_1 = \xi_3, \ \ \textit{and} \ \xi_2 = \xi_4 \tag{7}$$

is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ *with* α, β, γ *given by (5). Conversely, any down-up algebra* $A(\alpha, \beta, \gamma)$ *with not both* α *and* β *equal to 0 is isomorphic to a Witten deformation algebra* $\mathfrak{W}(\underline{\xi})$ *whose parameters satisfy (7).*

A deformation algebra $\mathfrak{W}(\underline{\xi})$ has a filtration, and Le Bruyn ([L1], [L2]) investigated the algebras $\mathfrak{W}(\underline{\xi})$ whose associated graded algebras are Auslander regular. They determine a 3-parameter family of deformation algebras which are called *conformal* $\mathfrak{sl}_2$ *algebras* and whose defining relations are

$$xz - azx = x, \qquad zy - ayz = y, \qquad yx - cxy = bz^2 + z \tag{8}$$

When $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with defining relations given by (8) is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1 + ac), \beta = -ac^{-1}$ and $\gamma = -c^{-1}$. If $b = c = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}$, $\beta = 0$ and $\gamma = -a^{-1}$.

In a recent paper [K1], Kulkarni has shown that under certain assumptions on the parameters a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2$ algebra or to a double skew polynomial extension. Kulkarni studies the simple modules of the conformal sl_2 algebras and of the skew polynomial algebras. Critical to the investigations in [K1] is the observation that the conformal $\mathfrak{sl}_2$ algebra of (8) can be realized as a *hyperbolic ring.* Kulkarni then applies results of Rosenberg [R] on noncommutative algebraic geometry to describe the left ideals in the left spectrum of the algebra and to determine the maximal left ideals for the conformal sl_2 algebras.

Example (ix). The quadratic Askey-Wilson algebras studied in [GLZ] can be regarded as having generators a, b and defining relations which depend on fixed parameters $(\alpha, \gamma, \delta, \epsilon, \zeta, \eta, \mu, \nu)$ according to:

$$\begin{aligned} a^2b &= \alpha aba - ba^2 + \zeta(ab + ba) + \eta a^2 + \gamma a + \delta b + \mu 1 \\ ab^2 &= \alpha bab - b^2a + \eta(ab + ba) + \zeta b^2 + \gamma b + \epsilon a + \nu 1. \end{aligned}$$

It is apparent that when $\delta = \epsilon = \zeta = \eta = \mu = \nu = 0$, this algebra is just the down-up algebra $A(\alpha, -1, \gamma)$. Askey-Wilson algebras are related to the Leonard systems introduced in [T2] as abstract algebraic generalizations of q-Racah polynomials and of families of orthogonal polynomials that include the quantum q-Krawtchouk, Racah, Hahn, dual Hahn, and Krawtchouk polynomials.

Highest weight modules.

Down-up algebras have a rich representation theory (see [BR, Sec. 2]). In particular, they have highest weight modules and weight modules which mimic those of $\mathfrak{sl}_2$.

A module V for $A = A(\alpha, \beta, \gamma)$ is said to be a *highest weight module of weight* λ if V has a vector y_0 such that $d \cdot y_0 = 0$, $(du) \cdot y_0 = \lambda y_0$, and $V = Ay_0$. The vector y_0 is a *maximal vector* or *highest weight vector* of V.

Proposition 9. *(See [BR, Sec. 2]) Set $\lambda_{-1} = 0$ and let $\lambda_0 = \lambda \in \mathbf{C}$ be arbitrary. For $n \geq 1$, define λ_n inductively by the recurrence relation,*

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma. \tag{10}$$

The $\mathbf{C}$-vector space $V(\lambda)$ with basis $\{v_n \mid n = 0, 1, 2, \dots\}$ and with $A(\alpha, \beta, \gamma)$-action given by

$$\begin{aligned} d \cdot v_n &= \lambda_{n-1}v_{n-1}, \quad n \geq 1, \quad \textit{and} \quad d \cdot v_0 = 0 \\ u \cdot v_n &= v_{n+1}. \end{aligned} \tag{11}$$

is a highest weight module for $A(\alpha, \beta, \gamma)$. Every $A(\alpha, \beta, \gamma)$-module of highest weight λ is a homomorphic image of $V(\lambda)$. The module $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for any n.

Because it shares the same universal property and many of the same features as Verma modules for finite-dimensional semisimple complex Lie algebras, the module $V(\lambda)$ is said to be a *Verma module* for $A(\alpha, \beta, \gamma)$.

Weight modules.

If we multiply the relation $d^2u - \alpha dud - \beta ud^2 = \gamma d$ on the left by u and the relation $du^2 - \alpha udu - \beta u^2d = \gamma u$ on the right by d and subtract the second from the first, the resulting equation is

$$0 = ud^2u - du^2d \qquad \text{or} \qquad (du)(ud) = (ud)(du).$$

Therefore, the elements du and ud commute in $A = A(\alpha, \beta, \gamma)$. For any basis element $v_n \in V(\lambda)$, we have $du \cdot v_n = \lambda_n v_n$ and $ud \cdot v_n = \lambda_{n-1}v_n$. Using that with $n = 0$ and $\lambda \neq 0$, it is easy to see that du and ud are linearly independent. Let $\mathfrak{h} = \mathbf{C}du \oplus \mathbf{C}ud$.

We say an A-module V is a *weight module* if $V = \bigoplus_{\nu \in \mathfrak{h}^*} V_\nu$, where $V_\nu = \{v \in V \mid h \cdot v = \nu(h)v \text{ for all } h \in \mathfrak{h}\}$, and the sum is over elements in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Any submodule of a weight module is a weight module. If $V_\nu \neq (0)$, then ν is a *weight* and V_ν is the corresponding *weight space*. Each weight ν is determined by the pair (ν', ν'') of complex numbers, $\nu' = \nu(du)$ and $\nu'' = \nu(ud)$. In particular, highest weight modules are weight modules in this sense. The basis vector v_n of $V(\lambda)$ is a weight vector whose weight is given by the pair $(\lambda_n, \lambda_{n-1})$. Finding these weights explicitly involves solving the linear recurrence relation in (10), which can be done by standard methods as in [Br, Chap.7] for example.

Proposition 12. *Assume $\lambda_{-1} = 0$, $\lambda_0 = \lambda \in \mathbf{C}$, and λ_n for $n \geq 1$ is given by the recurrence relation $\lambda_n - \alpha\lambda_{n-1} - \beta\lambda_{n-2} = \gamma$. Fix $t \in \mathbf{C}$ such that*

$$t^2 = \frac{\alpha^2 + 4\beta}{4}.$$

(i) If $\alpha^2 + 4\beta \neq 0$, then

$$\lambda_n = c_1 r_1^n + c_2 r_2^n + x_n, \qquad \textit{where}$$

$$r_1 = \frac{\alpha}{2} + t, \qquad r_2 = \frac{\alpha}{2} - t,$$
$$x_n = \begin{cases} (1-\alpha-\beta)^{-1}\gamma & \textit{if} \quad \alpha+\beta \neq 1 \\ (2-\alpha)^{-1}\gamma n & \textit{if} \quad \alpha+\beta = 1 \quad \textit{(necessarily} \quad \alpha \neq 2), \end{cases}$$
$$\textit{and} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{r_2 - r_1} \begin{pmatrix} r_2 & -1 \\ -r_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda - x_0 \\ \alpha\lambda + \gamma - x_1 \end{pmatrix}.$$

(ii) If $\alpha^2 + 4\beta = 0$ and $\alpha \neq 0$, then

$$\lambda_n = c_1 s^n + c_2 n s^n + x_n \quad \textit{where}$$

$$s = \frac{\alpha}{2}$$
$$x_n = \begin{cases} (1-\alpha-\beta)^{-1}\gamma & \textit{if} \quad \alpha+\beta \neq 1 \\ 2^{-1}n^2\gamma & \textit{if} \quad \alpha+\beta = 1 \quad \textit{i.e.} \quad \textit{if} \quad \alpha = 2, \ \beta = -1, \end{cases}$$
$$\textit{and} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2\alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda - x_0 \\ \alpha\lambda + \gamma - x_1 \end{pmatrix}.$$

(iii) If $\alpha^2 + 4\beta = 0$ and $\alpha = 0$, then $\beta = 0$ and $\lambda_n = \gamma$ for all $n \geq 1$.

If α, β are real, then it is natural to take $t = \dfrac{\sqrt{\alpha^2 + 4\beta}}{2}$ in the above calculations.

Let us consider several special cases.

Example (a). Recall that the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is isomorphic to the algebra $A(2, -1, -2)$, and the universal enveloping algebra $U(\mathfrak{H})$ of the Heisenberg Lie algebra $\mathfrak{H}$ is isomorphic to $A(2, -1, 0)$. Applying (ii) with $s = \alpha/2 = 1$ and $x_n = n^2\gamma/2$ for any algebra $A(2, -1, \gamma)$, we have that

$$\lambda_n = \lambda + (\lambda + \frac{\gamma}{2})n + \frac{\gamma n^2}{2} = (n+1)(\lambda + \frac{\gamma n}{2}).$$

In the $\mathfrak{sl}_2$-case, it is customary to use the operator $h = du - ud$ rather than du. The eigenvalues of h are $\lambda_n - \lambda_{n-1} = \lambda + n\gamma = \lambda - 2n$, $n = 0, 1, \ldots$. The analogous computation in the Heisenberg Lie algebra shows that the central element $z = du - ud$ has constant eigenvalue $\lambda_n = \lambda$.

Example (b). Recall that the quantum case discussed earlier involves the down-up algebra $A([2]_i, -1, 0)$. In the particular case of $U_q(\mathfrak{sl}_3)$, the subalgebra generated by the E_i's is isomorphic to $A([2], -1, 0)$ where $[2] = \dfrac{q^2 - q^{-2}}{q - q^{-1}}$, and $\lambda_n = [n+1]\lambda = \Big(\dfrac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}\Big)\lambda$ for all $n \geq 0$ in that case.

Example (c). For the algebra $A(1, 1, 0)$, the solutions to the associated linear recurrence $\lambda_n = \lambda_{n-1} + \lambda_{n-2}$, $\lambda_0 = \lambda$, $\lambda_{-1} = 0$, (hence the eigenvalues of du and ud on $V(\lambda)$) are given by the Fibonacci sequence $\lambda_0 = \lambda$, $\lambda_1 = \lambda$, $\lambda_2 = 2\lambda$, $\lambda_3 = 3\lambda$, $\lambda_4 = 5\lambda$, $\ldots$. In this case, the equations in Proposition 12 reduce to $\lambda_n = \lambda\dfrac{\sqrt{5}}{5}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

In [BR] we investigated in detail the weight space and submodule structure of the Verma module $V(\lambda)$. Roots of unity play a critical role in determining the dimension of a weight space. We introduced category $\mathcal{O}$" modules in the spirit of [BGG] and showed the simple objects were highest weight modules, and we explored a more general category $\mathcal{O}'$ of modules for down-up algebras. We briefly summarize some of the main results.

Proposition 13. *([BR, Secs. 2, 4, 5])*

(a) In $V(\lambda)$ each weight space is either one-dimensional or infinite-dimensional. If an infinite-dimensional weight space occurs, there are only finitely many weights.

(b) If each weight space of $V(\lambda)$ is one-dimensional, then the proper submodules of $V(\lambda)$ have the form $N = span_{\mathbf{C}}\{v_j \mid j \geq n+1\}$ for some $n \geq 0$ with $\lambda_n = 0$. Hence they are contained in $M(\lambda) = span_{\mathbf{C}}\{v_j \mid j \geq m+1\}$, where $\lambda_m = 0$ and m is minimal with that property.

(c) If $\gamma = 0 = \lambda$, then $V(\lambda)$ has infinitely many maximal proper submodules, each of the form $N^{(\tau)} = span_{\mathbf{C}}\{v_n - \tau v_{n-1} \mid n = 1, 2, \dots\}$ for some $\tau \in \mathbf{C}$, and infinitely many one-dimensional simple modules, $L(0,\tau) = V(0)/N^{(\tau)}$. In all other cases, $M(\lambda)$ is the unique maximal submodule of $V(\lambda)$, and there is a unique simple highest weight module, $L(\lambda) = V(\lambda)/M(\lambda)$, of weight λ up to isomorphism.

Example. Recall that the poset $\mathcal{L}(2,2)$ affords a representation of the down-up algebra $A(1,-1,1)$. It is easy to see that the down and up operators satisfy $d(\underline{0}) = 0$ and $du(\underline{0}) = \underline{0}$, where $\underline{0} = (0,0)$. Thus, the element $\underline{0} \in \mathcal{L}(2,2)$ generates a highest weight module with $\lambda = 1$. If we solve the corresponding recurrence relation in Proposition 12, we get from (i) that $r_1 = 1/2(1+\sqrt{-3})$, $r_2 = 1/2(1-\sqrt{-3})$, and $\lambda_n = 1 + (r_2^n - r_1^n)/(r_2 - r_1)$. Since $r_1^3 = -1 = r_2^3$, (and hence r_1, r_2 are 6th roots of unity), we see that the sequence $\lambda_0 = \lambda, \lambda_1, \lambda_2, \dots,$ is given by $1,2,2,1,0,0,1,2,2,1,0,0,\dots$. Thus, in the Verma module $V(1)$, the maximal submodule $M(1) = \text{span}_{\mathbf{C}}\{v_j \mid j \geq 5\}$. The irreducible quotient $L(1) = V(1)/M(1)$ is 5-dimensional, and it is spanned modulo $M(1)$ by v_0, v_1, v_2, v_3, v_4. As an $A(1,-1,1)$-module, $\mathcal{L}(2,2)$ decomposes as $L(1) \oplus L(0)$, where we identify the copy of $L(1)$ with the span of the vectors $v_0 = (0,0)$, $v_1 = (1,0)$, $v_2 = (2,0) + (1,1)$, $v_3 = 2 \cdot (2,1)$, $v_4 = 2 \cdot (2,2)$, and $L(0)$ with the span of $(2,0) - (1,1)$.

The structure of down-up algebras.

¿From a ring theoretic viewpoint, down-up algebras exhibit many interesting features. For example, it is apparent from the defining relations that the monomials $u^i(du)^j d^k$, $i,j,k = 0,1,\dots$ in a down-up algebra $A = A(\alpha,\beta,\gamma)$ determine a spanning set. In [BR, Thm. 3.1] we applied the Diamond Lemma (see [Be]) to prove a Poincaré-Birkhoff-Witt type result for down-up algebras. There is one essential ambiguity, $(d^2u)u = d(du^2)$, and the result of resolving the ambiguity in the two possible ways is the same.

Theorem 14. *(Poincaré-Birkhoff-Witt Theorem) Assume $A = A(\alpha,\beta,\gamma)$ is a down-up algebra over $\mathbf{C}$. Then $\{u^i(du)^j d^k \mid i,j,k = 0,1,\dots\}$ is a basis of A.*

The Gelfand-Kirillov dimension is a natural dimension to assign to an algebra A, and in many cases (such as when A is a domain), it provides important structural information. Theorem 14 enables us to compute the GK-dimension of any down-up algebra $A = A(\alpha, \beta, \gamma)$. The spaces $A^{(n)} = \text{span}_{\mathbf{C}}\{u^i(du)^j d^k \mid i+2j+k \leq n\}$ afford a filtration $(0) \subset A^{(0)} \subset A^{(1)} \subset \cdots \subset \cup_n A^{(n)} = A(\alpha, \beta, \gamma)$ of the down-up algebra, and $A^{(m)}A^{(n)} \subseteq A^{(m+n)}$ since the defining relations replace the words d^2u and du^2 by words of the same or lower total degree. The number of monomials $u^i(du)^j d^k$ with $i+2j+k = \ell$ is $(m+1)(m+1)$ if $\ell = 2m$ and is $(m+1)(m+2)$ if $\ell = 2m+1$. Thus, $\dim A^{(n)}$ is a polynomial in n with positive coefficients of degree 3, and the Gelfand-Kirillov dimension is given by

$$\text{GKdim}\big(A(\alpha,\beta,\gamma)\big) = \limsup_{n\to\infty} \log_n(\dim A^{(n)}) = \lim_{n\to\infty} \frac{\ln\Big(\dim A^{(n)}\Big)}{\ln n} = 3.$$

Proposition 15. *([BR, Sec. 3]) If $A(\alpha,\beta,\gamma)$ has infinitely many simple Verma modules $V(\lambda)$, then the intersection of the annihilators of the simple Verma modules is zero.*

It follows immediately that for such a down-up algebra $A(\alpha,\beta,\gamma)$ the Jacobson radical, which is the intersection of the annihilators of all the simple modules, is zero.

When $\beta = 0$, then $d(du - \alpha ud - \gamma) = 0$ so that $A(\alpha,\beta,\gamma)$ has zero divisors for any choice of $\alpha, \gamma \in \mathbf{C}$. Necessary and sufficient conditions for $A(\alpha,\beta,\gamma)$ to be a domain or to be Noetherian have been proven recently using Theorem 14:

Proposition 16. *([KMP], and compare also [K2].) For a down-up algebra $A = A(\alpha,\beta,\gamma)$, the following are equivalent:*

(i) $\beta \neq 0$.

(ii) A is a domain.

(iii) A is right and left Noetherian.

(iv) $\mathbf{C}[du, ud]$ is a polynomial ring in the two variables, (i.e. du and ud are algebraically independent).

A down-up algebra $A = A(\alpha,\beta,\gamma)$ is $\mathbf{Z}$-graded by assigning $\deg(d) = -1$ and $\deg(u) = 1$ and extending to all of A by setting $\deg(ab) = \deg(a) + \deg(b)$. Then $A = \bigoplus_{n\in\mathbf{Z}} A_n$ where $A_n = \{a \in A \mid \deg(a) = n\}$, and A_0 is a commutative subalgebra. It is shown in [BR] that if A has infinitely many simple Verma modules, the center lies in A_0. The center of A has been completely described in recent work ([K2] and [Z]).

References

[B] G. Benkart,, *Down-up algebras and Witten's deformations of the universal enveloping algebra of* $\mathfrak{sl}_2$,, Contemp. Math. Amer. Math. Soc. (to appear).

[BR] G. Benkart and T. Roby, *Down-up algebras*, J. Algebra (to appear).

[Be] G.M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), 178–218.

[BGG] J. Bernstein, I.M. Gelfand, and S.I. Gelfand, *A category of* $\mathfrak{g}$*-modules*, Func. Anal. Appl. **10** (1976), 87–92.

[Br] R.A. Brualdi, *Introductory Combinatorics*, Second Edition, North Holland, New York, 1992.

[F] S.V. Fomin, *Duality of graded graphs*, J. Alg. Comb. **3** (1994), 357–404.

[GLZ] Ya. I. Granovskiĭ, I.M. Lutzenko, and A.S. Zhedanov, *Quadratic algebras and dynamical symmetry of the Schrödinger equation*, Soviet Phys. JETP **72 (2)** (1991), 205-209.

[K1] R. Kulkarni, *Irreducible representations of Witten's deformations of* $U(sl_2)$ (to appear).

[K2] R. Kulkarni, *Down-up algebras and their representations* (to appear).

[KMP] E. Kirkman, I. Musson, and D. Passman, *Noetherian down-up algebras* (to appear).

[L1] L. Le Bruyn, *Two remarks on Witten's quantum enveloping algebra*, Comm. Algebra **22** (1994), 865–876.

[L2] L. Le Bruyn, *Conformal* $\mathfrak{sl}_2$ *enveloping algebras*, Comm. Algebra **23** (1995), 1325–1362.

[M] Yu. Manin, *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier **37** (1987), 191–205.

[P] R.A. Proctor, *Representations of* $sl(2, \mathbf{C})$ *on posets and the Sperner property, SIAM J. Algebraic and Discrete Methods* **3** (1982), 275–280.

[R] A.L. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Kluwer, 1995.

[St1] R.P. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic and Discrete Methods **1** (1980), 168–184.

[St2] R.P. Stanley, *Differential posets*, J. of Amer. Math. Soc. **4** (1988), 919–961.

[T1] P. Terwilliger, *The incidence algebra of a uniform poset*, 193-212; *Coding Theory and Design Theory, Part I*, D. Ray-Chaudhuri, ed., IMA Series, vol. 20, Springer-Verlag, New York, 1990.

[T2] P. Terwilliger, *Leonard Systems, An Algebraic Approach to the q-Racah Polynomials* (to appear).

[W1] E. Witten, *Gauge theories, vertex models, and quantum groups*, Nuclear Phys. B **330** (1990), 285–346.

[W2] E. Witten, *Quantization of Chern-Simons gauge theory with complex gauge group*, Comm. Math. Phys. **137** (1991), 29–66.

[Z] K. Zhao, *Centers of down-up algebras* (to appear).

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706 USA
benkart@math.wisc.edu