From C_n to $n!$: permutations avoiding $S_j(j+1)(j+2)$

E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani *

Abstract. A permutation π is said to be τ–avoiding if it does not contain any subsequence having all the same pairwise comparisons as τ. This paper concerns the characterization and enumeration of permutations which avoid a set F of subsequences increasing in number and in length at the same time. Let F^j be the set of subsequences of the form "$\sigma(j+1)(j+2)$", σ being a permutation on $\{1, ..., j\}$. For $j = 1$ the only subsequence in F^1 is 123 and the 123–avoiding permutations are enumerated by the Catalan numbers; for $j = 2$ the subsequences in F^2 are 1234, 2134 and the $(1234, 2134)$–avoiding permutations are enumerated by the Schröder numbers; for each other value of j greater than 2 the subsequences in F^j are $j!$ and their length is $(j+2)$, the permutations avoiding these $j!$ subsequences are enumerated by a number sequence $\{a_n\}_{n\geq 1}$ such that $C_n \leq a_n \leq n!$, C_n being the n–th Catalan number. For $j = \infty$ permutations are obtained as permutations avoiding an infinite set of infinite length subsequences. For each j we determine the generating function of permutations avoiding the subsequences in F^j, according to their length, the number of their left minima and the number of their non inversions.

Résumé. Une permutation π est dite à motif exclus τ si elle ne contient aucune sous séquence dont les éléments sont dans la même relation d'ordre que ceux de τ. Cet article considère la caractérisation et l'énumeration des permutations qui ne contienent pas un nombre croissant de motifs de longueur également croissante. Soit F^j l'ensemble des sous séquences de la forme "$\sigma(j+1)(j+2)$", σ étant une permutation sur $\{1, ..., j\}$. Pour $j = 1$ la seule sous séquence de F^1 est 123 et les permutations à motifs 123 exclus sont comptees par les nombres de Catalan; pour $j = 2$ les sous séquences de F^2 sont 1234, 2134 et les permutations á motifs $(1234, 2134)$ exclus sont comptées par les nombres de Schröder; pour chaque valeur $j > 2$ le nombre des sous séquences de F^j est $j!$, chaqune de longueur $(j+2)$, et les permutations avec ces $j!$ motifs exclus sont comptées par $\{a_n\}_{n\geq 1}$ tel que $C_n \leq a_n \leq n!$, C_n étant le n–ième nombre de Catalan. Pour $j = \infty$ on obtient les permutations comme les permutations avec un nombre infini de motifs exclus de longueur infinie. Pour chaque j on détermine la fonction génératrice des permutations avec les motifs exclus dans F^j, selon leur longueur, le nombre de leur minima à gauche et le nombre de leur non inversions.

1 Introduction

The study of permutations represents an interesting and relevant discipline in Mathematics dating back a long time. Euler in [14] begun to analyze permutation statistics related to the study of parameters different from their length. MacMahon in [23] further developped this large field but meaningful progresses have been made in the last thirty years.
More recently incoming problems from Computer Science made the concept of permutations with

*Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, 50134 Firenze, Italy, e-mail:pire@ingfi1.ing.unifi.it

forbidden subsequences developped. They arise in sorting problems [8, 20, 25, 28, 29], in the analysis of regularities in words [3, 21], in particular instances of pattern matching algorithms optimization [6]; just to mention some examples. Anyway the enumeration of permutations with specific forbidden subsequences has its applications in areas like Algebraic Geometry and Combinatorics. The 2143–avoiding permutations, called vexilliary permutations, are relevant to the theory of Schubert polynomials [22]. In Combinatorics permutations with forbidden subsequences play an important role as they present bijections with a great number of nontrivial combinatorial objects [11, 12, 13, 16, 18, 19] and moreover their enumeration gives classical number sequences. The former remark brings to translate the study of hard combinatorial structures on the corresponding forbidden subsequence permutations, the latter permits to interpretate some properties of number sequences involved in.

The n–th Catalan number is the common value of permutations with a single forbidden subsequence of length three [20]. More precisely Knuth shows that 312–avoiding permutations are the one stack sortable permutations. Simion et al. in [26] turn their attention in enumerating permutations avoiding two and three subsequences of length three.
As far as forbidden subsequences of length four are concerned enumeration results involve the subsequence 1234 [15], 1342 [5] and all the ones behaving identically [1, 27, 28], while for 1324–avoiding permutations the only result is proved by Bóna in [4] and it concerns their number lower bound. Permutations avoiding some couples of subsequences of length four give the Schröder numbers [18]; results concerning permutations avoiding more than one forbidden subsequence of length four there exist; we refer to [18] for an exhaustive survey on the results about permutations with forbidden subsequences.
For what it concerns permutations avoiding a single subsequence of length greater than four the most important result solves the problem of one increasing subsequence of any length giving an asymptotic value of the number of permutations avoiding the subsequence $(1...(k+1))$ [24]. In [9] Chow and West study $(123, (k...1(k+1)))$–avoiding permutations; their generating functions can be expressed as a quotient of modified Chebyshev polynomials and they give rise to number sequences lying between the well–known Fibonacci and Catalan numbers. In [2] the authors study $(321, ((k+2)\bar{1}(k+3)2...(k+1)))$–avoiding permutations where the latter forbidden subsequence means that the subsequence $(k+1)(k+2)1...(k)$ [that is $(k+2)(k+3)2...(k+1)$ restricted on $\{1, ..., (k+2)\}$] is allowed only in the case it is of type $(k+2)1(k+3)2...(k+1)$. Their generating functions give rise to sequences of numbers lying between the well–known Motzkin and Catalan numbers involving the left–Motzkin factors numbers as a particular case. Even if a lot of work has been carried out on permutations with forbidden subsequences, many interesting problems remain still open: they involve both the simple enumeration of permutations avoiding a given set of subsequences and the study of other parameters on permutations, that is statistics which can play an important role in the structures represented bijectively by the permutations.

We have just mentioned the results obtained in the enumeration of permutations with a certain forbidden subsequence of increasing length. In this paper we are interested in the enumeration of permutations with an increasing number of forbidden subsequences which increase in length on their turn; this means that the cardinality of the set of forbidden subsequences is growing as the length of the forbidden subsequences themselves. Section 2 of this paper contains the basic definitions on permutations with forbidden subsequences we refer to later in the work. In Section 3, we describe the tools used to obtain the enumerative results, that is succession rules and generating trees. The formers are rules stating the growing behavior of an object with a fixed parameter value, the latters are schematic representations of the formers. In Section 4, we characterize the permutations we are studying in terms of succession rules. We translate the construction, represented by the generating tree, into formulae obtaining a set of functional equations. Its solution gives the generating function of the permutations according to their length, the number of their left minima, the number of their non inversions and the number of their active sites. We are able to determine the generating function according to the length of the permutations, the number of their left minima and the number of their non inversions.

2 Notations and definitions

A permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ on $[n] = \{1, 2, \dots, n\}$ is a bijection between $[n]$ and $[n]$. Let S_n be the set of permutations on $[n]$.

A permutation $\pi \in S_n$ *contains a subsequence of type* $\tau \in S_k$ iff a sequence of indices $1 \le i_1 < i_2 < \dots < i_k \le n$ exists such that $\pi(i_1)\pi(i_2)\dots\pi(i_k)$ has all the same pairwise comparisons as τ. We denote the set of permutations of S_n not containing subsequences of type τ by $S_n(\tau)$.

Example 2.1 The permutation 6145732 belongs to $S_7(2413)$ because all its subsequences of length 4 are not of type 2413. This permutation does not belong to $S_7(3142)$ because there exist subsequences of type 3142, that is $\pi(1)\pi(2)\pi(5)\pi(6) = 6173$, $\pi(1)\pi(2)\pi(5)\pi(7) = 6172$.

If we have the set $\tau_1 \in S_{k_1}, \dots, \tau_p \in S_{k_p}$ of permutations, we denote the set $S_n(\tau_1) \cap \dots \cap S_n(\tau_p)$ by $S_n(\tau_1, .., \tau_p)$. We call the family $F = \{\tau_1, .., \tau_p\}$ a *family of forbidden subsequences*, the set $S_n(F)$ a *family of permutations with forbidden subsequences*.

Let $\pi \in S_n$, we denote the position lying on the left of $\pi(1)$ by s_0, the position lying between $\pi(i)$ and $\pi(i+1)$, $1 \le i \le n-1$, by s_i and the position lying on the right of $\pi(n)$ by s_n. $s_0, s_1, ..., s_{n-1}, s_n$ are the *site* of π.

Definition 2.1 *Let $F = \{\tau_1, .., \tau_p\}$, a site s_i, $0 \le i \le n$, of a permutation $\pi \in S_n(F)$ is* active *if the insertion of $(n+1)$ into s_i gives a permutation belonging to the set $S_{n+1}(F)$; otherwise it is said to be* inactive.

Definition 2.2 *Let $\pi \in S_n$. The pair (i, j), $i < j$, is a* non inversion *if $\pi(i) < \pi(j)$. An element $\pi(i)$ is a* left minimum *if $\pi(i) < \pi(j)$, $\forall j \in [1, i-1]$.*

Example 2.2 The permutation $\pi = 6145732$ has 9 non inversions: $(1,5)(2,3)(2,4)(2,5)(2,6)$ $(2,7)(3,4)(3,5)(4,5)$ and 2 left minima: $\pi(1) = 6$ and $\pi(2) = 1$.

3 Succession rules and generating trees

In this Section we briefly describe the tools used to deduce our enumerative results, that is succession rules and generating trees; they were introduced in [10] for the study of Baxter permutations and furtherly applied to the study of permutations with forbidden subsequences by West [9, 30, 31].

Definition 3.1 *A* generating tree *is a rooted, labelled tree having the property that the labels of the set of children of each node* v *can be determined from the label of* v *itself. Thus, any particular generating tree can be specified by a recursive definition consisting in:*

1, the basis. *the label of the root,*

2, the inductive step. *a set of succession rules that yields a multiset of labelled children depending solely on the label of the parent.*

A succession rule contains at least the information about the number of children. Let τ be a forbidden subsequence, following the idea developed in [10], the generating tree for τ–avoiding permutations is a rooted tree such that the nodes on level n are exactly the elements of $S_n(\tau)$; the children of a permutation $\pi = \pi(1)....\pi(n)$ are all the τ free permutations obtained by inserting $(n+1)$ into π. Labels must be assigned to the nodes and they record the number of children of a given node.

Example 3.1 Catalan tree and 123–avoiding permutations

$$\begin{cases} basis: \quad (2) \\ inductive \;\; step: \quad (k) \rightarrow (k+1)(2).....(k) \end{cases} \tag{1}$$

The permutation of length one has two active sites (*basis in rule (1)*). Let $\pi = \pi(1)...\pi(n) \in S_n(123)$; and k, $2 \leq k \leq n$, be the minimum index in π such that $i_1 < k$ exists and $\pi(i_1) < \pi(k)$; then the active sites of π are $s_0, ..., s_{k-1}$. The insertion of $(n+1)$ into each other site on the right of s_{k-1} gives the subsequence $\pi(i_1)\pi(k)(n+1)$ that is forbidden. This means that the active sites of π are all the ones lying between the elements of π constituting the longest initial decreasing subsequence. If π has k active sites then its longest initial decreasing subsequence has length $(k-1)$. The permutation obtained by inserting $(n+1)$ into s_0 give a new permutation with $(k+1)$ active sites; the permutation obtained by inserting $(n+1)$ into s_i, $1 \leq i \leq k-1$, gives $(i+1)$ active sites, (*inductive step in rule (1)*). The generating tree representing 123–avoiding permutations can be obtained by developping rule (1) and by labelling each permutation with the right label (k).

Example 3.2 Schröder tree and $(1234, 2134)$–avoiding permutations

$$\begin{cases} basis: \quad (2) \\ inductive \;\; step: \quad (k) \rightarrow (k+1)(k+1)(3).....(k) \end{cases} \tag{2}$$

The permutation of length one has two active sites (*basis in rule (2)*). Let $\pi = \pi(1)...\pi(n) \in S_n(1234, 2134)$ and k, $3 \leq k \leq n$, be the minimum index in π such that there exist $i_1 < i_2 < k$ for which $\pi(i_1)\pi(i_2)\pi(k)$ is of type 123, or 213; then the active sites of π are $s_0, ..., s_{k-1}$. The insertion of $(n+1)$ into each other site $s_k, ..., s_n$ gives at least one of the forbidden subsequences $1234, 2134$. Let π be a permutation with k active sites; the permutations obtained by inserting $(n+1)$ into s_0 and s_1 have $(k+1)$ active sites; the permutation obtained by inserting $(n+1)$ into s_i, $2 \leq i \leq k-1$, has $(i+1)$ active sites; each other site gives at least one of the two forbidden subsequences because $(n+1)$ has at least two smaller elements on its left (*inductive step in rule (2)*). The generating tree related to rule (2) and representing $(1234, 2134)$–avoiding permutations is showed in fig. 1.

Notice that the succession rule (2) is easily obtained from rule (1) by replacing (2) with $(k+1)$ in the *inductive step*.

4 Permutations with an increasing number of forbidden subsequences increasing in length

In this Section we study the class of permutations $S^j = \bigcup_{n \geq 1} S_n(F^j)$, where F^j is a set of subsequences such that $|F^j| = j!$ and any $\tau \in F^j$ has the form $\tau = \sigma(j+1)(j+2)$, σ being a permutation

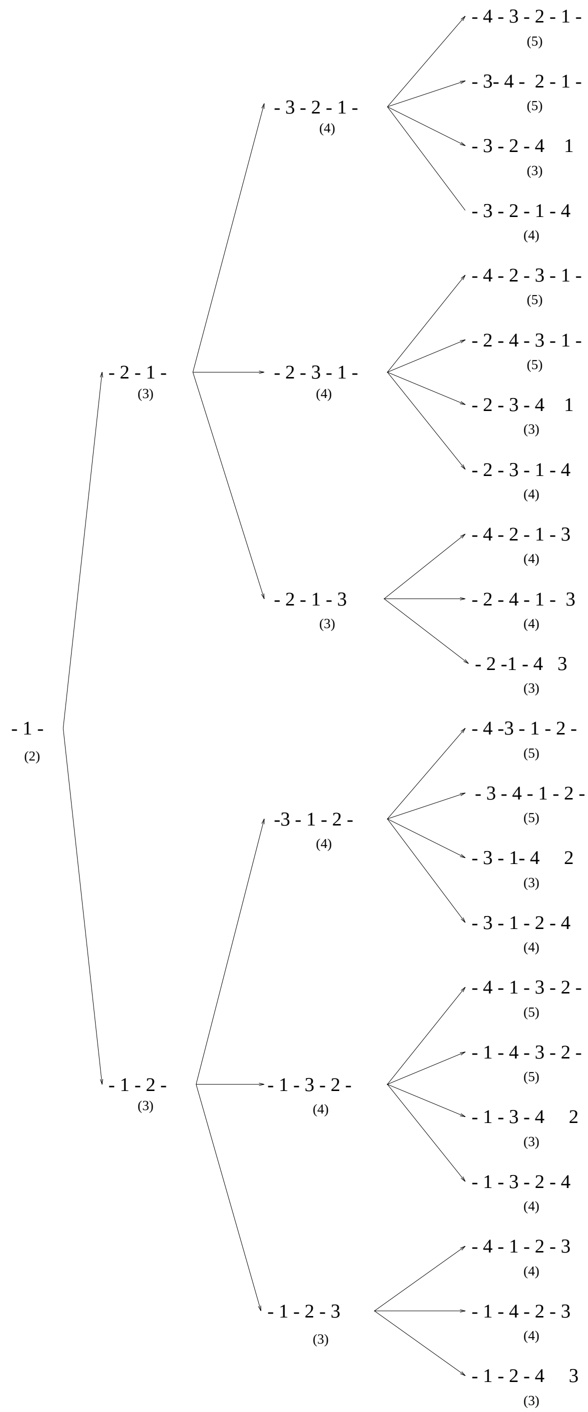

Figure 1: The generating tree for $(1234, 2134)$–avoiding permutations.

belonging to the symmetric group S_j. These permutations are enumerated by numbers lying between the Catalan numbers and the factorial (see fig. 2). We use some combinatorics to describe the structure of their generating tree. The constructive method it represents can be translated into a set of functional equations verified by the generating function of the class of permutations S^j. Some computations allow us to determine this generating function according to the length of the permutations, the number of their left minima and the number of their non inversions.

Index	Family of Permutations	Number of Forbidden Subsequences	Numbers
j=1	S^1_n (123)= C_n	1	1 2 5 14 42 . . .
			+1
j=2	S^2_n (1234,2143)	2	1 2 6 22 90 . . .
			+2
j=3	S^3_n (12345,13245,21345,23145,31245,32145)	6	1 2 6 24 114 . . .
			+3!
j=4	S^4_n ({1234}!56)	12	1 2 6 24 120 . . .
⋮	⋮	⋮	⋮
			+(n-2)!
j= ∞	$S^\infty_n = S_n$	0	1 2 6 24 120 . . . n! . . .

Figure 2: First numbers of the sequences counting the permutations in S^j.

4.1 S^j permutations

The set F^j contains subsequences having length $(j+2)$ such that the two largest elements, that is $(j+1)$, $(j+2)$, are the $(j+1)$–th and the $(j+2)$–th element of each sequence, while the other j elements $\{1,...,j\}$ can be in any order. This means that a site s_i, $0 \le i \le n$, in a permutation $\pi \in S_n(F^j)$, is active if and only if a sequence of indices $i_1,...,i_j,i_{j+1}$ such that $i_1 < ... < i_j < i_{j+1} \le i$ and $max_{1\le t\le j}\{\pi(i_t)\} < \pi(i_{j+1})$ does not exist. It follows that the active sites lie on the left of the minimum index i_{j+1} that gives such a sequence; this means that the $(j+1)$ leftmost sites of a permutation are always active, if they exist.

Let $\pi \in S_n(F^j)$, $j \ge 1$, be a permutation with k active sites: $\{s_0,...,s_{k-1}\}$, so there is a sequence of indices $i_1,...,i_{j+1}=k$ such that $i_1 < ... < i_j < k$ and $max_{1\le t\le j}\{\pi(i_t)\} < \pi(k)$. By inserting $(n+1)$ into s_i, $0 \le i \le j-1$, we obtain a new permutation, say π^i, with $(k+1)$ active sites as the new element plays no role for the existence of the sequence of indices $i_1,...,i_j,i_{j+1}$ and the minimum index that gives such a sequence is $(k+1)$ as the $(k+1)$–th element of π^i is equal to $\pi(k)$. By inserting $(n+1)$ into s_i, $j \le i \le k-1$, we obtain a new permutation π^i with $(i+1)$ active sites. As a matter of fact the sequence of indices $1,...,j,(i+1)$ verifies the required properties and moreover $(i+1)$ is the minimum index giving rise to such a sequence (see fig. 3). The above arguments prove the following Proposition:

Proposition 4.1 *Let $\pi \in S_n(F^j)$, $j \ge 1$, be a permutation with k active sites: $\{s_0,...,s_{k-1}\}$ and π^i be the permutation obtained from π by inserting $(n+1)$ into the active site s_i, then the number of active sites of π^i is $(k+1)$ for $0 \le i \le j-1$ and $(i+1)$ for $j \le i \le k-1$.*

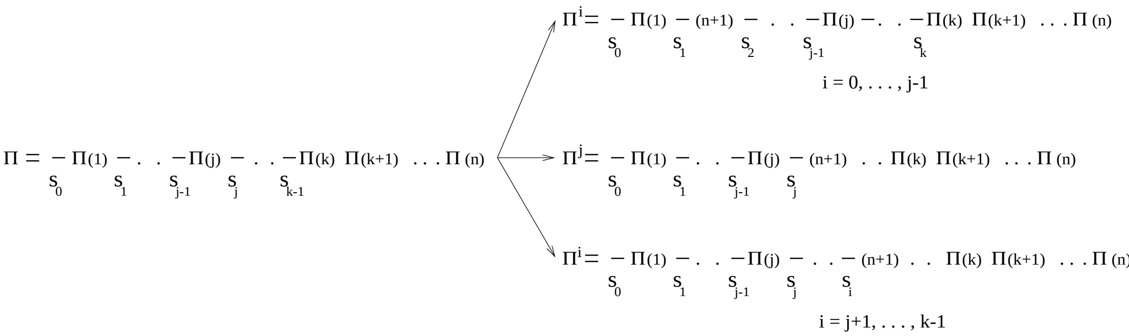

Figure 3: The active sites in the children of π.

The succession rule for their generating tree follows immediatly:

$$\begin{cases} basis: \quad (2) & \\ inductive \;\; step: \quad (k) \to (k+1)^k, & k < j \\ inductive \;\; step: \quad (k) \to (k+1)^j (j+1).....(k), & k \geq j. \end{cases} \tag{3}$$

In a "Catalan permutation" π a site s_i, $0 \leq i \leq n$, is active if and only if there do not exist $i_i < i_2 \leq i$ such that $\pi(i_1) < \pi(i_2)$; in a "Schröder permutation" π a site s_i, $0 \leq i \leq n$, is active if and only if there do not exist $i_1 < i_2 < i_3 \leq i$ such that $max\{\pi(i_1), \pi(i_2)\} < \pi(i_3)$. By comparing these conditions with the definition of active site for a permutations $\pi \in S^j$; we realise that the role of the parameter j is to increase the length of the index sequence $i_1, ..., i_{j+1}$ such that $i_1 < ... < i_j < i_{j+1}$ and each $\pi(i_t)$, $t = 1, ..., j$, is smaller than $\pi(i_{j+1})$. Notice that no constraint is required on the order of the set $\{\pi(i_1), ..., \pi(i_j)\}$. Once the permutations in S^j are characterized we are interested in the results concerning their enumeration.

4.2 The generating function

For each j, we are interested in the generating function for the permutations in S^j according to their length, the number of their left minima and the number of their non inversions. Let $\pi \in S^j$; we denote the length of π by $n(\pi)$, the number of its left minima by $m(\pi)$, the number of its non inversions by $ni(\pi)$ and the number of its active sites by $a(\pi)$. The generating function of S^j according to the above mentioned parameters is the following:

$$S^j(x, y, q, s) = \sum_{\pi \in S^j} x^{n(\pi)} y^{m(\pi)} q^{ni(\pi)} s^{a(\pi)}.$$

From (3) we have that a permutation $\pi \in S^j$ is the father of $a(\pi)$ permutations, namely $\pi^0, ..., \pi^{a(\pi)-1}$ in S^j, obtained by inserting a new element into each of its active site $\left\{s_0, ..., s_{a(\pi)-1}\right\}$. If we look at the parameters changes in each permutation π^i, $0 \leq i \leq a(\pi) - 1$, we obtain:

a) if $0 \leq i \leq min\{j, a(\pi)\} - 1$ then:

$$n(\pi^i) = n(\pi) + 1, \quad \begin{cases} m(\pi^0) = m(\pi) + 1, \\ m(\pi^i) = m(\pi), \end{cases} \quad ni(\pi^i) = ni(\pi) + i, \quad a(\pi^i) = a(\pi) + 1;$$

b) if $min\{j, a(\pi)\} \leq i \leq a(\pi) - 1$ then:

$$n(\pi^i) = n(\pi) + 1, \quad m(\pi^i) = m(\pi), \quad ni(\pi^i) = ni(\pi) + i, \quad a(\pi^i) = i + 1.$$

The set S^j can be partitioned into j subsets: $S^j_2, \ldots.., S^j_j, S^j_{>j}$, where $S^j_k = \{\pi \in S^j : a(\pi) = k)\}$ $2 \le k \le j$, and $S^j_{>j} = \{\pi \in S^j : a(\pi) > j)\}$. In terms of generating functions we obtain the decomposition:

$$S^j(x,y,q,s) = \sum_{k=2}^{j} S^j_k(x,y,q,1)s^k + S^j_{>j}(x,y,q,s).$$

Let S^j_1 be the set representing the permutation of length 0. Notice that each permutation π such that $a(\pi) \le j-1$ gives $a(\pi)$ permutations with $(a(\pi)+1)$ active sites, so S^j_k is given by S^j_{k-1}, $2 \le k \le j$, and the parameter changes are described in **a)**. This means that a permutation of S^j having exactly $k \le j$ active sites is just any permutation of length $(k-1)$. Moreover, each permutation π such that $a(\pi) \ge j$ gives $a(\pi)$ permutations with at least $(j+1)$ active sites and the parameter changes are described in **a)** for the j leftmost active sites and in **b)** for the remaining ones.
Let $[i]_q = \frac{1-q^i}{1-q}$ be the standard notation for the q–analogue of i and $F(x,y,q,s) := F(s)$; if we translate the above arguments into formulae we obtain:

Proposition 4.2 *The generating function $S^j(s)$ of permutations in S^j is:*

$$S^j(s) = \sum_{i=2}^{j} S^j_i(s) + S^j_{>j}(s),$$

where:

$$\begin{cases} S^j_1(s) & = s \\ S^j_k(s) & = x^{k-1}s^k y \prod_{i=1}^{k-2} (y + q[i]_q), \ 2 \le k \le j \\ S^j_{>j}(s) & = \frac{xs(y+q[j-1]_q)}{1-xs(y+q[j-1]_q)} S^j_j(s) + \frac{xs}{(1-sq)(1-xs(y+q[j-1]_q))} \left[s^j q^j S^j_{>j}(1) - S^j_{>j}(sq) \right]. \end{cases}$$

The third equation of Proposition 4.2 can be solved by using the Lemma of Bousquet-Mélou [7]. The expression we obtain for $S^j_{>j}(s)$ gets simplify if we put $s=1$ as $S^j_{>j}(1) := S^j_{>j}(x,y,q)$ satisfies:

Proposition 4.3 *The generating function $S^j_{>j}(x,y,q)$ is given by:*

$$S^j_{>j}(x,y,q) = \frac{J_{1,j}(x,y,q)}{J_{0,j}(x,y,q)}$$

where:

$$J_{1,j}(x,y,q) = \sum_{n \ge 0} \frac{(-1)^n x^{n+1} q^{jn+\binom{n+1}{2}}}{(q)_n (x(y+q[j-1]_q))_{n+1}} (y + q[j-1]_q) S^j_j(1),$$

$$J_{0,j}(x,y,q) = 1 + \sum_{n \ge 0} \frac{(-1)^{n+1} x^{n+1} q^{j(n+1)+\binom{n+1}{2}}}{(q)_{n+1} (x(y+q[j-1]_q))_{n+1}};$$

with $(a)_n = (a,q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right)$.

From Proposition 4.3 it follows:

Theorem 4.4 *The generating function $S^j(x,y,q)$ of permutations in S^j according to their length, the number of their left minima and the number of their non inversions is:*

$$S^j(x,y,q) = x^j y f_j(x,y,q) \prod_{i=1}^{j-1} (y+q[i]_q) + \sum_{k=2}^{j} x^{k-1} y \prod_{i=1}^{k-2} (y+q[i]_q)\,; \quad j \geq 1;$$

with:

$$f_j(x,y,q) = \frac{\sum_{n\geq 0} \frac{(-1)^n x^n q^{jn+\binom{n+1}{2}}}{(q)_n (x(y+q[j-1]_q))_{n+1}}}{\sum_{n\geq 0} \frac{(-1)^n x^n q^{jn+\binom{n}{2}}}{(q)_n (x(y+q[j-1]_q))_n}}.$$

We denote the nominator and the denominator of $f_j(x,y,q)$ by $A_j(x,y,q)$ and $B_j(x,y,q)$ respectively. After some computations we obtain:

Lemma 4.5 *The functions $A_j(x,y,q)$ and $B_j(x,y,q)$ verify the following equalities:*

$$A_j(xq,y,q) = \frac{(1-x(y+q[j-1]_q))}{x^2 q^{j+1} (y+q[j-1]_q)} \left[\left(1 - x(y+q[j-1]_q + q^j)\right) A_j(x,y,q) - B_j(x,y,q) \right],$$

$$B_j(xq,y,q) = (1-x(y+q[j-1]_q))\, A_j(x,y,q).$$

Lemma 4.5 allows us to find the q–equation satisfied by $f_j(x,y,q)$:

$$x^2 q^{j+1} (y+q[j-1]_q) f_j(xq,y,q) f_j(x,y,q) - \left(1 - x(y+q[j-1]_q + q^j)\right) f_j(x,y,q) + 1 = 0. \quad (4)$$

If $y = q = 1$ then equation (4) gives:

$$f_j(x,1,1) = \frac{1-(j+1)x - \sqrt{1-2(j+1)x+(j-1)^2x^2}}{2jx^2},$$

so from Theorem 4.4 we obtain:

$$S^j(x,1,1) = x^{j-2}(j-1)! \frac{1-(j+1)x - \sqrt{1-2(j+1)x+(j-1)^2x^2}}{2} + \sum_{k=1}^{j-1} k! x^k.$$

This means that the generating function of S^j permutations according to their length is algebraic and quadratic, except for $j \to \infty$. In this case we obtain the expected result for the generating function of permutations according to their length, their left minima number and the number of their non inversions, that is:

$$S^\infty(x,y,q) = \sum_{n\geq 1} x^n y \prod_{m=1}^{n-1} (y+q[m]_q)\,.$$

References

[1] E. Babson, J. West, The permutations $123p_4...p_l$ and $321p_l...p_1$ are Wilf equivalent, (preprint).

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, From Motzkin to Catalan permutations: a "discrete continuity", submitted to *Disc. Math.*.

[3] J. Berstel, Axel Thue's papers on repetitions in words: translation, *Publication du LaCIM*, 19, (1994).

[4] M. Bóna, Permutations avoiding certain patterns. The case of length 4 and some generalizations, *Disc. Math.*, 175, (1997), 55–67.

[5] M. Bóna, Exact enumeration of 1342–avoiding permutations; a close link with labelled trees and planar maps, *J. Combinatorial Theory*, Ser. A, 80, (1997), 257–272.

[6] P. Bose, J.F. Buss, A. Lubiw, Pattern matching for permutations, *Lec. Notes Comp. Sc.*, 709, (1993), 200-209.

[7] M. Bousquet–Mélou, A method for the enumeration of various classes of column–convex polygons, *Disc. Math.*, 154, (1996), 1–25.

[8] M. Bousquet-Mélou, Multi-statistic enumeration of two-stack sortable permutations, to appear in *Electronic J. Combinatorics.*

[9] T. Chow, J. West, Forbidden subsequences and Chebyshev polynomials (preprint).

[10] F.R.K. Chung, R.L. Graham, V.E. Hoggatt, M. Kleiman, The number of Baxter permutations, *J. Combinatorial Theory*, Ser. A, 24, (1978), 382–394.

[11] R. Cori, S. Dulucq, G. Viennot, Shuffle of parenthesis systems and Baxter permutations, *J. Combinatorial Theory* , Ser. A, 43 (1986), 1-22.

[12] S. Dulucq, S. Gire, J. West, Permutations à motifs exclus et cartes planaires non séparables, *Proc. of the* 5^{th} *FPSAC* (A. Barlotti, M. Delest, R. Pinzani eds.), Florence (1993), 165-178.

[13] S. Dulucq, O. Guibert, Mots de piles, tableaux standards et permutations de Baxter, *Proc. of the* 6^{th} *FPSAC*, Dimacs, (1994), 119-128.

[14] L. Euler, Institutiones calculi differentialis, in *Opera Omnia, Series Prima*, vol. X, Verlag von B.G. Teubner, Leipzig (1913).

[15] I.M. Gessel, Symmetric functions and P-recursiveness, *J. Combinatorial Theory (A)*, 53, (1990), 257-285.

[16] S. Gire, Arbres, permutations a motifs exclus et cartes planaires: quelques problemes algorithmiques et combinatoires, Thèse de l'Université de Bordeaux I (1993).

[17] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison–Wesley Pub. Co., Reading, Massachusetts, (1989).

[18] O. Guibert, Combinatoires des permutations a motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, Thèse de l'Université de Bordeaux I (1996).

[19] I. P. Goulden, J. West, Raney paths and a combinatorial relationship between rooted non-separable planar maps and two-stack-sortable permutations, *J. Combinatorial Theory (A)* (to appear).

[20] D.E. Knuth, *The art of computer programming*, vol. 3, Addison–Wesley, Reading, MA, (1973).

[21] M. Lothaire, Combinatorics and words, G.C. Rota editor, *Encyclopedia of Mathematics and its Applications* 17, (Addison-Wesley, Reading, MA, 1983).

[22] I.G. Macdonald, Notes on Schubert Polynomials, Publication du LaCIM, 6 (Montréal, 1991).

[23] P.A. Macmahon, *Combinatory Analysis*, vols. 1 and 2, Cambridge Univ. Press, Cambridge, (1915) (reprinted by Chelsea, New York, 1955).

[24] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Advances in Mathematics*, 41, (1981), 115-136.

[25] D. Rotem, Stack sortable permutations, *Disc. Math.*, 33, (1981), 185–196.

[26] R. Simion, F.W. Schmidt, Restricted permutations, *European Journal of Combinatorics*, 6, (1985), 383-406.

[27] Z. Stankova, Forbidden Subsequences, *Disc. Math.*, 132, (1994), 291–316.

[28] J. West, Permutations with forbidden subsequences and stack-sortable permutations, Ph.D. Thesis, M.I.T., Cambridge, MA (1990).

[29] J. West, Sorting twice through a stack, *Theo. Comp. Science*, 117, (1993), 303-313.

[30] J. West, Generating trees and the Catalan and Schröder numbers, *Disc. Math.*, 146, (1995), 247–262.

[31] J. West, Generating trees and forbidden subsequences, *Disc. Math.*, 157, (1996), 363–374.