COUNTING 1–VERTEX TRIANGULATIONS OF ORIENTED SURFACES

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Abstract

A $1-$*vertex triangulation* of an oriented compact surface S of genus g is an embedded graph $T \subset S$ having only one vertex such that all connected components of $S \setminus T$ are adjacent to exactly 3 edges of T (i.e. are triangles).

The aim of this paper is to give formulas for the number of such triangulations (up to equivalence) on an oriented surface of given genus).

Une triangulation à un sommet d'une surface orientable compacte de genre g est un graphe $T \subset S$ qui a un seul sommet et dont toutes les faces sont incidentes à trois arrêtes de T (ce sont des triangles).

Le but de cette article est de donné des formules explicite pour le nombre de cettes triangulations.

0. Introduction

Definition 0.1. A $1-$*vertex triangulation* of an oriented compact surface S of genus g is an embedded graph $T \subset S$ having only one vertex such that all connected components of $S \setminus T$ are adjacent to exactly 3 edges of T (i.e. are triangles).

Two such triangulations $T \subset S$ and $T' \subset S'$ are *isomorphic* (or *equivalent*) if there exists an orientation-preserving homeomorphism $\varphi : S \longrightarrow S'$ such that $\varphi(T) = T'$.

The aim of this paper is to give formulas for the number of such triangulations (up to equivalence) on an oriented surface of given genus (such triangulations exist in every genus $g \geq 1$).

In section 2 we introduce relevant tools (so-called oriented Wicks forms which are cellular decompositions with only one face of oriented surfaces) for our method.

Section 3 is devoted to the proof of our main results using the combinatorial tools introduced in the previous section.

We note, that J.Brenner and R.Lyndon considered such triangulations from a combinatorial point of view for studying non-parabolic subgroups of the modular group [**BL**].

1. Main results

Let $T \subset S$ be a $1-$vertex triangulation (1-VT for short) of an oriented surface of genus g. Denote by e the number of edges and by v the number of triangles in T. The Euler characteristic formula

$$\chi(S) = 2 - 2g = 1 - e + v$$

and the obvious identity $2e = 3v$ show that we have

$$\begin{array}{l} e = 3(2g-1) \quad , \\ v = 2(2g-1) \end{array}$$

and all 1-vertex triangulations of genus g consist hence of exactly $2(2g-1)$ triangles.

Triangles of a 1-VT T are of two types. Indeed, let Δ be a triangle of T and let a, b, c be the three edges of Δ. The triangle Δ is a *positive triangle* if the cyclic word defined by the labels of edges around the unique vertex of T is of the form

$$ab \ldots bc \ldots ca \ldots \qquad \text{or} \qquad ac \ldots cb \ldots ba \ldots$$

and Δ is *negative* if this word is of the form

$$ab\dots ca\dots bc\dots \quad \text{or} \quad ac\dots ba\dots cb\dots \quad .$$

It is shown in [M], that a 1-VT of genus g has $2g-2$ positive and $2g$ negative triangles. (But we will formulate and prove this fact in combinatorial terminology in Propositin 2.6.) Also, part (i) of the Theorem 1.1 was already proved in a more general form in [BC], but we will prove it by another method.

The automorphism group $\mathrm{Aut}(T)$ of a 1-VT T is the group of all permutations of oriented edges in T which are restrictions to the edge set of orientation-preserving homeomorphisms which leave T invariant. $\mathrm{Aut}(T)$ is always isomorphic to a subgroup of the cyclic group of order $2e = 6(2g-1)$ what can be seen by considering an appropriate small neighbourhood $U \subset S$ of the unique vertex in T (the graph $T \cap U \subset U$ has the form of a star with $2e$ edges).

Let us introduce the sets

$\mathcal{T}_1^g$: all 1-VT's of genus g (up to equivalence),

$\mathcal{T}_2^g(r) \subset \mathcal{T}_1^g$: all 1-VT's having an automorphism of order 2 leaving r edges of T invariant by reversing their orientation. (This automorphism is the half-turn with respect to the "midpoints" of these edges and exchanges the two adjacent triangles of an invariant edge.)

$\mathcal{T}_3^g(s,t) \subset \mathcal{T}_1^g$: all 1-VT's having an automorphism of order 3 leaving exactly s positive and t negative triangles invariant (this automorphism permutes cyclically the edges incident to the invariant triangles).

$\mathcal{T}_6^g(3r;2s,2t) = \mathcal{T}_2^g(3r) \cap \mathcal{T}_3^g(2s,2t)$: all 1-VT's having an automorphism γ of order 6 with γ^3 leaving $3r$ edges invariant and γ^2 leaving $2s$ positive and $2t$ negative triangles invariant (it is useless to consider the set $\mathcal{T}_6^g(r';s',t')$ defined analogously since 3 divides r' and 2 divides s', t' if $\mathcal{T}_6^g(r';s',t') \neq \emptyset$).

We define now the *masses* of these sets as

$$\begin{aligned}
m_1^g &= \sum_{T\in\mathcal{T}_1^g} \frac{1}{|\mathrm{Aut}(T)|} \ , \\
m_2^g(r) &= \sum_{T\in\mathcal{T}_2^g(r)} \frac{1}{|\mathrm{Aut}(T)|} \ , \\
m_3^g(s,t) &= \sum_{T\in\mathcal{T}_3^g(s,t)} \frac{1}{|\mathrm{Aut}(T)|} \ , \\
m_6^g(3r;2s,2t) &= \sum_{T\in\mathcal{T}_6^g(3r;2s,2t)} \frac{1}{|\mathrm{Aut}(T)|} \ .
\end{aligned}$$

Theorem 1.1. *(i)* $\mathrm{Aut}(T)$ *is cyclic of order* 1, 2, 3 *or* 6 *for every 1-vertex triangulation* T.

(ii) $m_1^g = \frac{2}{1}\left(\frac{1}{12}\right)^g \frac{(6g-5)!}{g!(3g-3)!}$.

(iii) $m_2^g(r) > 0$ *(with* $r \in \mathbf{N}$*) if and only if* $f = \frac{2g+1-r}{4} \in \{0,1,2,\dots\}$ *and we have then*

$$m_2^g(r) = \frac{2}{2}\left(\frac{2^2}{12}\right)^f \frac{1}{r!}\frac{(6f+2r-5)!}{f!(3f+r-3)!} \ .$$

(iv) $m_3^g(s,t) > 0$ *if and only if* $f = \frac{g+1-s-t}{3} \in \{0,1,2,\dots\}$, $s \equiv 2g+1 \pmod 3$ *and* $t \equiv 2g \pmod 3$ *(which follows from the two previous conditions). We have then*

$$m_3^g(s,t) = \frac{2}{3}\left(\frac{3^2}{12}\right)^f \frac{1}{s!t!}\frac{(6f+2s+2t-5)!}{f!(3f+s+t-1)!} \ .$$

(v) $m_6^g(3r;2s,2t) > 0$ *if and only if* $f = \frac{2g+5-3r-4s-4t}{12} \in \{0,1,2,\dots\}$, $2s \equiv 2g+1 \pmod 3$ *and* $2t \equiv 2g \pmod 3$ *(follows in fact from the previous conditions). We have then*

$$m_6^g(3r;2s,2t) = \frac{2}{6}\left(\frac{6^2}{12}\right)^f \frac{1}{r!s!t!}\frac{(6f+2r+2s+2t-5)!}{f!(3f+r+s+t-3)!} \ .$$

Set

$$\begin{aligned}
m_2^g &= \textstyle\sum_{r\in\mathbf{N},\ (2g+1-r)/4\in\mathbf{N}} m_2^g(r) \ , \\
m_3^g &= \textstyle\sum_{s,t\in\mathbf{N},\ (g+1-s-t)/3\in\mathbf{N},\ s\equiv 2g+1 \pmod 3} m_3^g(s,t) \ , \\
m_6^g &= \textstyle\sum_{r,s,t\in\mathbf{N},\ (2g+5-3r-4s-4t)/12\in\mathbf{N},\ 2s\equiv 2g+1 \pmod 3} m_6^g(3r;2s,2t)
\end{aligned}$$

(all sums are finite) and denote by M_d^g the number of equivalence classes of 1-VT's on an oriented genus g surface having an automorphism of order d (i.e. an automorphism group with order divisible by d).

Theorem 1.2. *We have*

$$\begin{aligned} M_1^g &= m_1^g + m_2^g + 2m_3^g + 2m_6^g \quad , \\ M_2^g &= 2m_2^g + 4m_6^g \quad , \\ M_3^g &= 3m_3^g + 3m_6^g \quad , \\ M_6^g &= 6m_6^g \end{aligned}$$

and $M_d^g = 0$ if d is not a divisor of 6.

The number M_1^g of this Theorem is of course the number of inequivalent 1-vertex triangulations on an oriented compact connected surface of genus g. See Table at the end of this paper for the first 15 values of M_1^g.

The following is an immediate consequence of Theorem 1.2.

Corollary 1.3. *There are exactly*
M_6^g inequivalent 1-VT's with 6 automorphisms,
$M_3^g - M_6^g$ inequivalent 1-VT's with 3 automorphisms,
$M_2^g - M_6^g$ inequivalent 1-VT's with 2 automorphisms and
$M_1^g - M_2^g - M_3^g + M_6^g$ inequivalent 1-VT's without non-trivial automorphisms.

Let us note that the formula (ii) can be obtained from the paper [**WL**], by formula (9) on page 207 and formula on the top of page 211, or from the Theorem 2.1 of [**G Sch**] with $\lambda = 2^{6g-3}$ and $\mu = 3^{4g-2}$. We will give our own proof of this fact. Let us note also, that $\lambda_g(6g-3) = (12g-6)m_1^g$, where $\lambda_g(n)$ is the number of ways to obtain an orientable genus g surface from $2n$-gon, which was defined in [**HZ**].

2. Oriented Wicks forms

The objects considered in this section are dual to 1-vertex triangulations. They are slightly easier to handle since they carry some combinatorial structures more immediately.

Definition 2.1. An *oriented Wicks form* is a cyclic word $w = w_1w_2\dots w_{2l}$ (where cyclic means that we consider equivalence classes of words under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots$ of letters $a_1, a_2, \dots$ and their inverses $a_1^{-1}, a_2^{-1}, \dots$ such that

(i) if a_i^ϵ appears in w (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in w also,

(ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_ia_i^{-1}$ or $a_i^{-1}a_i$ (no cancellation),

(iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of w then $a_j^{-\delta}a_i^{-\epsilon}$ is not a cyclic factor of w (no substitution of the form $a_i^\epsilon a_j^\delta \longmapsto x, \quad a_j^{-\delta}a_i^{-\epsilon} \longmapsto x^{-1}$ is possible).

An oriented Wicks form $w = w_1w_2\dots$ in an alphabet A is *isomorphic* to $w' = w_1'w_2'$ in an alphabet A' if there exists a bijection $\varphi : A \longrightarrow A'$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots$ define the same cyclic word.

An oriented Wicks form w is an element of the commutator subgroup when considered as an element in the free group G with free generators $a_1, a_2, \dots$. We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a conjugates of commutators in G.

The *topological genus* $g_t(w)$ of an oriented Wicks form $w = w_1\dots w_{2e}$ is defined as the topological genus of the oriented compact connected surface obtained by labelling and orienting the edges of a $2e$–gone (which we consider as a subset of the oriented plane) according to w and by identifying the edges in the obvious way.

Proposition 2.2. *The algebraic and the topological genus of an oriented Wicks form coincide.*[**C**],[**CE**]

We define thus the *genus* $g(w)$ of an oriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface S associated to an oriented Wicks form $w = w_1\dots w_{2e}$. This surface carries an immerged graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, condition (ii) and (iii) on Wicks form imply that Γ contains no vertices of degre 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1–gones or

2−gones). Also,this construction works in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on an oriented compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ. The associated oriented Wicks form is defined as the word which appears in this way on the contour of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms and the associated immerged graphs $\Gamma \subset S$, speaking of vertices and edges of oriented Wicks form.

The Euler characteristic formula

$$\chi(S) = 2 - 2g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges of $\Gamma \subset S$) shows that an oriented Wicks form of genus g has at least length $4g$ (the associated graph has a unique vertex of degree $4g$ and $2g$ edges) and at most length $6(2g-1)$ (the associated graph has $2(2g-1)$ vertices of degree three and $3(2g-1)$ edges).

We call an oriented Wicks form of genus g *maximal* if it has length $6(2g-1)$. Oriented maximal Wicks forms are dual to 1-vertex triangulations. This can be seen by cutting the oriented surface S along Γ, hence obtaining a polygon P with $2e$ sides. We draw a star T on P which joins an interior point of P with the midpoints of all its sides. Regluing P we get back S which carries now a 1-vertex triangulation given by T and each 1-vertex triangulation is of this form for some oriented maximal Wicks form (the immerged graphs $T \subset S$ and $\Gamma \subset S$ are dual to each other: ie. faces of T correspond to vertices of Γ and vice-versa. Two faces of T share a common edge if and only if the corresponding vertices of Γ are adjacent). This construction shows that we can work indifferently with 1-vertex triangulations or oriented maximal Wicks forms.

Similarly, cellular decompositions of oriented surfaces with one vertex and one face are the same as oriented minimal Wicks forms. Their number was found in [CM]. In this case there are two constructions which yield a bijection between these two sets. Indeed, the dual of an oriented minimal Wicks form is again an oriented minimal Wicks form (generally not equivalent to the former).

The following definitions are merely restatements in terms of oriented maximal Wicks forms of the corresponding definitions for 1-vertex triangulations given in section 1.

A vertex V (with oriented edges a, b, c pointing toward V) is *positive* if

$$w = ab^{-1} \dots bc^{-1} \dots ca^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots cb^{-1} \dots ba^{-1} \dots$$

and V is *negative* if

$$w = ab^{-1} \dots ca^{-1} \dots bc^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots ba^{-1} \dots ab^{-1} \dots \quad .$$

The duality between oriented maximal Wicks forms and 1-vertex triangulations sets up a bijection between vertices of given sign in forms and triangles of the same sign in triangulations.

For any oriented Wicks form $w = w_1 w_2 \dots w_{2e}$ of genus g and length $2e$ its *automorphism group* $\mathrm{Aut}(w)$ is the group of cyclic permutations μ of the linear word $w_1 w_2 \dots w_{2e}$ such that w and $\mu(w)$ are isomorphic as linear words (ie. $\mu(w)$ is obtained from w by permuting the letters of the alphabet). The group $\mathrm{Aut}(w)$ is clearly a subgroup of the group $\mathbf{Z}/2e\mathbf{Z}$ of all cyclic permutations of words of length $2e$.

The automorphism group $\mathrm{Aut}(w)$ of an oriented Wicks form can of course also be described in terms of permutations on the oriented edge set induced by orientation-preserving homeomorphisms of S leaving Γ invariant. In particular an oriented maximal Wicks form and the associated dual 1-vertex triangulation have isomorphic automorphism groups.

Given a finite set W of isomorphism classes of oriented Wicks forms with genus g we define its *mass* $m(W)$ as

$$m(W) = \sum_{w \in W} \frac{1}{|\mathrm{Aut}(w)|} \quad .$$

We define the sets W_1^g, $W_2^g(r)$, $W_3^g(s,t)$ and $W_6(r;s,t)$ in the obvious way: $W_d^g(*)$ is the set of equivalence classes with an automorphism of order d having perhaps parameters $*$ defined as in section 1 (replacing the word "triangle" by "vertex".) The masses of the sets $W_d^g(*)$ are given by the numbers $m_d^g(*)$ introduced in section 1.

Let V be a negative vertex of an oriented maximal Wicks form of genus $g > 1$. There are three possibilities for the local configuration around V. We call these configurations type α, β and γ (see Figure 1).

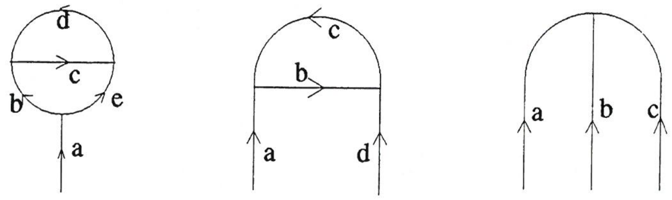

Figure 1.

Type α. The vertex V has only two neighbours which are adjacent to each other. This implies that w is of the form

$$w = x_1abcdb^{-1}ec^{-1}d^{-1}e^{-1}a^{-1}x_2u_1x_2^{-1}x_1^{-1}u_2$$

(where u_1, u_2 are subfactors of w) and is w obtained from the maximal oriented Wicks form

$$w' = xu_1x^{-1}u_2$$

of genus $g-1$ by the substitution $x \longmapsto x_1abcdb^{-1}ec^{-1}d^{-1}e^{-1}a^{-1}x_2$ and x^{-1} with $x_2^{-1}x_1^{-1}$ (this construction is called the α construction in [V]).

Type β. The vertex V has two non-adjacent neighbours. The word w is then of the form

$$w = x_1abca^{-1}x_2u_1y_1db^{-1}c^{-1}d^{-1}y_2u_2$$

(where perhaps $x_2 = y_1$ or $x_1 = y_2$ see [V] for all the details). The word w is then obtained by a β−construction from the word $w' = xu_1yu_2$ which is an oriented maximal Wicks form of genus $g-1$.

Type γ. The vertex V has three distinct neighbours. We have then

$$w = x_1ab^{-1}y_2u_1z_1ca^{-1}x_2u_2y_1bc^{-1}z_2u_3$$

(some identifications among x_i, y_j and z_k may occur, see [V] for all the details) and the word w is obtained by a so-called γ−construction from the word $w' = x\tilde{u}_2y\tilde{u}_1z\tilde{u}_3$.

Definition 2.3. We call the application which associates to an oriented maximal Wicks form w of genus g with a choosen negative vertex V the oriented maximal Wicks form w' of genus $g-1$ defined as above the *reduction* of w with respect to the negative vertex V.

An inspection of figure 1 shows that reductions with respect to vertices of α or β are always paired since two doubly adjacent vertices are negative, of the same type (which is α or β) and yield the same reduction.

The above constructions of type α, β and γ can be used to construct the list of all maximal orientable Wicks forms of genus $g > 1$ recursively.

Definition 2.4. Consider an oriented maximal Wicks forms $w = w_1 \dots w_{12g-6}$ of genus g. To any edge x of w we associate a transformation of w called the *IH transformation on the edge x*.

We start by considering the two subfactors axb and $cx^{-1}d$ of the (cyclic) word w. Geometric considerations and Definition 2.1 show that $b \neq a^{-1}, c \neq b^{-1}, d \neq a^{-1}, d \neq c^{-1}$ and $(c,d) \neq (a^{-1}, b^{-1})$.

According to the remaining possibilities we consider now the following transformation:

Type 1. $c \neq a^{-1}$ and $d \neq b^{-1}$. This implies that $d^{-1}a^{-1}$ and $b^{-1}c^{-1}$ appear as subfactors in the cyclic word w. The IH transformation on the edge x is then defined by the substitutions

$$\begin{array}{ccc} axb & \longmapsto & ab \\ cx^{-1}d & \longmapsto & cd \\ d^{-1}a^{-1} & \longmapsto & d^{-1}ya^{-1} \\ b^{-1}c^{-1} & \longmapsto & b^{-1}y^{-1}c^{-1} \end{array}$$

in the cyclic word w.

Type 2a. Suppose $c^{-1} = a$. This implies that $b^{-1}axb$ and $d^{-1}a^{-1}x^{-1}d$ are subfactors of the cyclic word w. Define the IH transformation on the edge x by

$$\begin{array}{lcl} b^{-1}axb & \longmapsto & b^{-1}yab \\ d^{-1}a^{-1}xd & \longmapsto & d^{-1}y^{-1}a^{-1} \end{array} .$$

Type 2b. Suppose $d^{-1} = b$. Then $axba^{-1}$ and $dx^{-1}b^{-1}d^{-1}$ are subfactors of the cyclic word w and we define the IH transformation on the edge x by

$$\begin{array}{lcl} axba^{-1} & \longmapsto & abya^{-1} \\ dx^{-1}b^{-1}d^{-1} & \longmapsto & db^{-1}y^{-1}d^{-1} \end{array} .$$

Lemma 2.5. *(i) IH transformations preserve oriented maximal Wicks forms of genus g.*
(ii) Two oriented maximal Wicks forms related by a IH transformation of type 2 are equivalent.

Proof. This results easily by considering the associated graph $\Gamma \subset S$.

Indeed, the IH transformation on the edge x amounts to the following: Contract first the edge x. This produces a vertex of degree 4 which can be "opened" in a unique different way preserving the cyclic order of the 4 incident edges.

Graphically this amounts to the replacement of a (deformed) letter I by a (not less deformed) letter H. Assertions (i) and (ii) follow easily. QED

Proposition 2.6. *An oriented maximal Wicks form of genus g has exactly $2(g-1)$ positive and $2g$ negative vertices.*

Lemma 2.7. *An α or a β construction increases the number of positive and negative vertices by 2.*

The proof is easy.

Lemma 2.8. *The number of positive or negative vertices is constant under IH transformations.*

Proof of Lemma 2.8. The Lemma holds for IH transformations of type 2 by Lemma 2.5 (ii). Let hence w, w' be two oriented maximal Wicks forms related by an IH transformation of type 1 with respect to the edge x of w respectively y of w'. This implies that w contains the four subfactors

$$axb \quad , \quad cx^{-1}d \quad , \quad d^{-1}a^{-1} \quad , \quad b^{-1}c^{-1}$$

and w' contains the subfactors

$$ab \quad , \quad cd \quad , \quad d^{-1}ya^{-1} \quad , \quad b^{-1}y^{-1}c^{-1}$$

in the same cyclic order and they agree everywhere else. It is hence enough to check the lemma for the six possible cyclic orders of the above subfactors.

One case is

$$\begin{array}{lclccccccc} w & = & axbu & \dots & cx^{-1}d & \dots & d^{-1}a^{-1} & \dots & b^{-1}c^{-1} & \dots \ , \\ w' & = & abu & \dots & cd & \dots & d^{-1}ya^{-1} & \dots & b^{-1}y^{-1}c^{-1} & \dots \ , \end{array} .$$

In this case the two vertices of w incident in x and the two vertices of w' incident in y have opposite signs. All other vertices are not involved in the IH transformation and keep their sign and the Lemma holds hence in this case.

The five remaining cases are similar and left to the reader. QED

Proof of Proposition 2.6. The result is true in genus 1 by inspection (the cyclic word $a_1a_2a_3a_1^{-1}a_2^{-1}a_3^{-1}$ is the unique oriented maximal Wicks form of genus 1 and has two negative vertices.)

Consider now an oriented maximal Wicks form of genus $g+1$. Choose an oriented embedded loop λ of minimal (combinatorial) length in Γ.

First case. If λ is of length 2 there are two vertices related by a double edge in Γ. This implies that they are negative and of type α or β. The assertion of Proposition 2.6 holds hence for w by Lemma 2.7 and by induction on g.

Second case. We suppose now that λ is of length ≥ 3. The oriented loop λ turns either left or right at each encountered vertex. If it turns on the same side at two consecutive vertices V_i and V_{i+1} the IH transformation with respect to the edge joining V_i and V_{i+1} relates w to a form w' of the same genus but containing a shorter loop. If λ does not contain two consecutive vertices V_i and V_{i+1} with the above property (ie. if λ turns first left, then right, then left etc.) choose any edge $\{V_i, V_{i+1}\}$ in λ and make an IH transformation with respect to this edge. This produces a form w' which contains a loop λ' of the same length as λ but turning on the same side at the two consecutive vertices V_{i-1}, V_i or V_{i+1}, V_{i+2}. By induction on the length of λ we can hence relate w by a sequence of IH transformation to an oriented maximal Wicks form of genus $g+1$ containing a loop of length 2. Hence, we are reduced to the first case. QED

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We proof the corresponding assertions for oriented maximal Wicks forms. The translation in terms of 1-vertex triangulations is immediate.

Let w be an oriented maximal Wicks form with an automorphism μ of order d. Let p be a prime dividing d. The automorphism $\mu' = \mu^{d/p}$ is hence of order p. If $p \neq 3$ then μ' acts without fixed vertices on w and proposition 2.6 shows that p divides the integers $2(g-1)$ and $2g$ which implies $p = 2$. The order d of μ is hence of the form $d = 2^a 3^b$. Repeating the above argument with the prime power $p = 4$ shows that $a \leq 1$.

All orbits of μ^{2^a} on the set of positive (respectively negative) vertices have either 3^b or 3^{b-1} elements and this leads to a contradiction if $b \geq 2$. This shows that d divides 6 and proves (i).

Proof of (ii). An element of W_1^{g+1} (which designes the set of equivalence classes of oriented minimal Wicks forms with genus $g+1$) can be obtained by applying an α, β or γ construction to an element in W_1^g.

There are respectively $2\binom{6g-3}{1}$, $4\binom{6g-3}{2} + 4\binom{6g-3}{1}$ and $8\binom{6g-3}{3} + 8(6g-3)(6g-4) + 8\binom{6g-3}{1}$ different possibilities for these constructions starting with a given element in W_1^g. On the other hand, Proposition 2.6 shows that we can construct $2(g+1)$ oriented maximal Wicks forms in W_1^g by applying reduction with respect to a negative vertex to a given element in W_1^{g+1}. The numbers of such "augmentations" and "reductions" coincide after weighting with the correct coefficients. These weights have to take care of automorphisms and the fact that type α and β constructions give rise to 2 negative vertices with the same "inverse". A carefull analysis shows that

$$\left(4\binom{6g-3}{1} + 8\binom{6g-3}{2} + 8\binom{6g-3}{1} + 8\binom{6g-3}{3} + 16\binom{6g-3}{2} + 8\binom{6g-3}{1}\right)m_1^g = 2(g+1)m_1^{g+1}$$

which proofs (ii) by induction since the function

$$g \longmapsto 2\frac{(6g-5)!}{12^g\, g!(3g-3)!}$$

satisfies the same equation and we have equality for $g = 1$ (since $m_1^g = \frac{1}{6} = 2\frac{1!}{12\ 1!0!}$).

Proof of (iii). First case: $r < 2g+1$ and hence $f = \frac{2g+1-r}{4} \geq 1$. Let w be an oriented maximal Wicks form of genus g with an automorphism μ of order 2 reversing the orientation of exactly r edges. There are $\frac{6g-3-r}{2}$ orbits of (unoriented) edges not invariant under μ. Consider the graph obtained by removing all $\mu-$invariant edges from the quotient graph Γ/μ. After removing leaves and vertices of degree 2 we get an oriented maximal Wicks form $\tilde{w}$ with $\frac{6g-3-r}{2} - r = \frac{3(2g-r-1)}{2}$ edges and hence of genus $f = \frac{2g+1-r}{4} \geq 1$ (recall that an oriented maximal Wicks form of genus f has $3(2f-1)$ edges).

More precisely, let w be represented by the word $w_1 w_2 \ldots w_{12g-6}$. The subword $w_1 w_2 \ldots w_{6g-3}$ contains exactly one representant of each orbit for the action of μ on oriented edges. Remove from the word $w_1 \ldots w_{6g-3}$ all letters w_k with $w_{k+6g-3} = w_k^{-1}$ (they correspond to edges reversed by μ). The resulting word w' has length $6g-3-r$ and has the property that if w_k appears in w' then either w_k^{-1} or w_{k+6g-3}^{-1} appears exactly once in w' also. Replacing w_{k+6g-3}^{-1} by w_k^{-1} we get a word which satisfies (i) of Definition

2.1. Removing from this word (and of the resulting ones) all cyclic subfactors of the form $w_k w_k^{-1}$ we get a word w'' satisfying also condition (ii). Cancel w_i and its inverse (or w_j and its inverse) if $w_i w_j$ and $w_j^{-1} w_i^{-1}$ both occur as cyclic subfactors. This produces ultimately an oriented maximal Wicks form $\tilde{w}$. A counting argument shows that it has genus f. (A good way to understand what happens is to write the word w along two concentric circles related by radial segments indexed by invariant edges).

An oriented maximal Wicks form $\tilde{w}$ obtained as above has an extra structure defined as follows. Given an oriented edge $\tilde{a}$ of $\tilde{w}$, choose a preimage a in w (recall that we constructed $\tilde{w}$ starting from w by deleting letters and replacing other letters by elements in the same orbit under α). We have then $w = a u_1 a^{-1} u_2$. Since $\mu(a) \neq a^{-1}$, the two subfactors u_1 and u_2 cannot have the same length. Denote by u the shorter one. Set $\varphi(a) \equiv \varphi(a^{-1}) \equiv l \pmod 2$ if u contains l letters representing edges reversed under μ. We get thus a function φ with values in $\mathbf{Z}/2\mathbf{Z}$. This function satisfies

$$\varphi(a) + \varphi(b) + \varphi(c) \equiv 0 \pmod 2$$

whenever a, b, c are 3 edges incident in a common vertex of $\tilde{w}$. Such function is called a $\mathbf{Z}/2\mathbf{Z}$*-flow* on the graph $\tilde{\Gamma}$.

Conversely, for given an oriented maximal Wicks form $\tilde{w}$ of genus $f = \frac{2g+1-r}{4}$ and a $\mathbf{Z}/2\mathbf{Z}$-flow φ on its graph $\tilde{\Gamma}$, we can construct

$$\frac{(12f-6)(12f-2)\cdots(12f-10+4r)}{r!}$$

oriented maximal Wicks forms of genus g having an automorphism μ of order 2 reversing r edges associated to the pair $(\tilde{w}, \varphi)$. Indeed, we have $(12f-6)$ possibilities to attach the first edge reversed by μ, $(12f-2)$ choices for the second edge and so on. Since there are $r!$ possible orderings of the μ-invariant edges we have to divide by $r!$. Finally, the $\mathbf{Z}/2\mathbf{Z}$ flow shows how to glue together preimages of orbits under μ.

The set of $\mathbf{Z}/2\mathbf{Z}$-flows is a vector space over $\mathbf{Z}/2\mathbf{Z}$ of dimension $2f$. This implies that we have

$$2^{2f}\frac{(12f-6)(12f-2)\cdots(12f-10+4r)}{r!} m_1^f = 2\, m_2^g(r)$$

(the factor 2 on the right hand side comes from the fact that the Wicks forms contributing to m_1^f are essentially weighted with weight 1 while they have weight $\frac{1}{2}$ in m_2^g). This equation is also satisfied by replacing m_1^f with $2\frac{(6f-5)!}{12^f f!(3f-3)!}$ and $m_2^g(r)$ with $\frac{(6f+2r-5)!}{3^f r! f!(3f+r-3)!}$ (recall that $g = \frac{4f+r-1}{2}$) and this proofs (iii) in the first case.

Second case: $f = 0$ (the construction of $\tilde{w}$ as above shows that we cannot have $f < 0$). The idea is the same as in the first case. Here we have to glue a first invariant edge on an empty word (1 possibility) for the second and the third invariant edge we have 2 possibilities, for the forth there are 6 possibilities etc. Since there are no flows on an empty graph we get

$$2m_2^g(2g+1) = 2\frac{2\cdot 6\cdots(4r-10)}{r!}$$

which is readily checked.

Proof of (iv). First case: $t > 0$. Let w be an oriented maximal Wicks form having an automorphism of order 3 fixing s positive and $t > 0$ negative vertices. The t fixed negative vertices give rise to t possible reductions producing oriented Wicks forms w' of genus $g-1$ invariant under an automorphism of order 3. The parameters of w' are then $(t-1, s)$. On the other hand, for given an oriented Wicks form w' of genus $g-1$ with an automorphism of order 3 and parameters $(t-1, s)$ there are $2(2g-3)$ γ-constructions yielding a Wicks form of genus g with an automorphism of order 3 and parameters (s, t) (choose the midpoint of any of the $6(2g-3)$ oriented edges in w' and make the γ-construction with respect to its orbit). We have hence

$$2(2g-3)m_3^{g-1}(t-1, s) = t m_3^g(s, t)$$

which is also satisfied by the righthand side of formula (iii) in Theorem 1.1.

Let us now consider the case $t = 0$ (no invariant vertices of negative type). The proof of this case is very similar to the proof of (iii) .

We can suppose $g > 1$ since there are only two vertices of negative type in genus 1 . We consider hence an oriented maximal Wicks form w of genus g with an automorphism μ of order 3 fixing s positive and no negative vertices. Since μ leaves no edge invariant, there are $\frac{6g-3}{3} = 2g-1$ orbits of invariant edges. In genus $g > 1$, invariant vertices under an automorphisme μ of order 3 are nether adjacent. There are hence s orbits of edges of w incident in a vertex fixed under μ. Removing their orbits from the orbits of edges leaves us with a graph on the orbit space which has s vertices of degree 2. Removing these vertices of degree 2 yields an oriented maximal Wicks form $\tilde{w}$ of genus $f = \frac{g+1-s}{3}$ (all vertices are of degree 3, there is one face and there are $2g - 1 - 2s = 6f - 3$ edges). The construction of this form is completely analogous to the construction in the proof of (ii). As in the proof of (ii) this form has an extra structure. This extra structure is here a $\mathbf{Z}/3\mathbf{Z}$–flow, ie. an application φ of the set of oriented edges of $\tilde{w}$ into $\mathbf{Z}/3\mathbf{Z}$ such that $\varphi(e) \equiv -\varphi(-e) \pmod 3$ and $\varphi(a) + \varphi(b) + \varphi(c) \equiv 0 \pmod 3$ for three oriented edges a, b, c pointing toward a common vertex of $\tilde{w}$.

Conversely, given an oriented maximal Wicks form $\tilde{w}$ of genus f together with the above extra structure (a $\mathbf{Z}/3\mathbf{Z}$–flow on its graph $\tilde{\Gamma}$) there are

$$\frac{(12f-6)(12f-2)\cdots(12f-10+4s)}{s!}$$

possibilities to "extend" it into an oriented maximal Wicks form w of genus g which has an automorphism μ of order 3 fixing exactly s positive and no negative vertices.

Since the set of $\mathbf{Z}/3\mathbf{Z}$–flows on $\tilde{\Gamma}$ is a $\mathbf{Z}/3\mathbf{Z}$–vector space of dimension $2f$ we get

$$3^{2f}\frac{(12f-6)(12f-2)\cdots(12f-10+4s)}{s!}m_1^f = 3\ m_3^g(s,0) = 3\ m_3^{3f+s-1}(s,0) \quad .$$

A routine calculation shows that this equation is also satisfied with m_1^f replaced by $2\frac{(6f-5)!}{12^f f!(3f-3)!}$ and $m_3^{3f+s-1}(s,0)$ replaced by $\frac{2}{3}\left(\frac{3}{4}\right)^f \frac{(6f+2s-5)!}{s!(3f+s-1)!f!}$ and this proofs (iv) in the case $f \geq 1$. The proof for $f = 0$ is similar to the analogous proof of (iii).

Proof of (v). We apply again the idea used in the proof of (iii). Let w be an oriented maximal Wicks form with an automorphism μ of order 6. Considering the automorphism μ^3 of order 2 and applying the reduction used in the proof of (iii) we get an oriented maximal Wicks form $\tilde{w}$ of genus $h = \frac{2g+1-3r}{4}$ together with a $\mathbf{Z}/2\mathbf{Z}$–flow φ on $\tilde{\Gamma}$. This form $\tilde{w}$ is however an element of $W_3^h(s,t)$ and has hence an automorphism $\tilde{\mu}$ of order 3 which leaves φ invariant. Analogously to the proof of (iii) we use this data to produce elements in $W_6^g(3r; 2s, 2t)$ by making all constructions $\tilde{\mu}$–invariant. We must understand the vector space of $\tilde{\mu}$–invariant $\mathbf{Z}/2\mathbf{Z}$–flows:

Lemma 3.1. *Let $\tilde{w} \in W_3^h(s,t)$ be an oriented maximal Wicks form with an automorphism $\tilde{\mu}$ of order 3 (having parameters s,t). The space of $\tilde{\mu}$–invariant $\mathbf{Z}/2\mathbf{Z}$–flows on $\tilde{\Gamma}$ is then of dimension $\frac{h+1-s-t}{3}$.*

The lemma and a counting argument show then that

$$3^{(h+1-s-t)/3}\frac{(4h-6)(4h-2)\cdots(4h-10+4r)}{r!}m_3^h(s,t) = 2m_6^g(3r; 2s, 2t)$$

and a routine calculation implies assertion (v).

Proof of Lemma 3.1. Let $\tilde{\varphi}$ be a $\tilde{\mu}$–invariant $\mathbf{Z}/2\mathbf{Z}$–flow on $\tilde{\Gamma}$. We remark that $\tilde{\varphi}(a) \equiv \tilde{\varphi}(b) \equiv \tilde{\varphi}(c) \equiv 0 \pmod 2$ if a, b, c are three edges incident in a $\tilde{\mu}$–fixed vertex. This shows that all reductions used in the proof of (iv) can also be applied to the flow $\tilde{\varphi}$ and these constructions are injective on $\tilde{\mu}$–invariant $\mathbf{Z}/2\mathbf{Z}$–flows.

Theorem 1.1 is proved. QED

Table. Number of 1-vertex triangulations or of oriented maximal Wicks forms in genus 1 — 15:

1	1
2	9
3	1726
4	1349005
5	2169056374
6	5849686966988
7	23808202021448662
8	136415042681045401661
9	1047212810636411989605202
10	10378926166167927379808819918
11	129040245485216017874985276329588
12	1966895941808403901421322270340417352
13	36072568973390464496963227953956789552404
14	783676560946907841153290887110277871996495020
15	19903817294929565349602352185144632327980494486370

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