The Solution of a Conjecture of Stanley and Wilf for all layered patterns

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Abstract

Proving a conjecture of Wilf and Stanley in hitherto the most general case, we show that for any layered pattern q there is a constant c so that q is avoided by less than c^n permutations of length n. This implies the solution of this conjecture for at least 2^k patterns of length k, for any k.

Résumé

En prouvant la spécialisation la plus générale d'une conjecture de Wilf et Stanley à ce jour, nous démontrons que le nombre des permutations de $[n]$ évitant un motif de type étagère est plus petit de c^n, pour une constante c, qui dépend uniquement du motif. Cela prouve la conjecture pour 2^{k-1} motifs de longueur k.

1 Background and Definitions

Let $q = (q_1, q_2, \ldots, q_k) \in S_k$ be a permutation, and let $k \leq n$. We say that the permutation $p = (p_1, p_2, \cdots, p_n) \in S_n$ contains a subsequence (or pattern) of type q if there is a set of indices $1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_k} \leq n$ such that $p(i_1) < p(i_2) < \cdots < p(i_k)$. Otherwise we say that p is q-avoiding.

For example, a permutation is 132-avoiding if it doesn't contain three (not necessarily consecutive) elements among which the leftmost is the smallest and the middle one is the largest.

It is a long-studied and very hard problem to determine the number $S_n(q)$ of permutations in S_n (or in what follows, n-permutations) which avoid a certain pattern q. The general conjecture [11]

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claims that only very few of them, namely less than c^n, where c is some constant depending on q. It is also conjectured that the limit $(S_n(q))^{1/n}$ always exists. However, efforts to prove this have been unsuccesful for most patterns. The conjecture has been proven for patterns of length 3 [8] and 4 [2], [3], [4], and for monotonic patterns of any length [7]. Given that this conference is taking place in Barcelona, we point out that if q is of length 3, then $S_n(q)$ is equal to the nth *Catalan* number.

Apart from this, there are some other scattered results giving the answer for some particular permutations by bijectively proving that the number of n-permutations avoiding them equals the number of those which avoid the monotonic pattern. For this kind of results, see [10], [9]. It is possible [5] to explain by the means of complexity theory why this problem is so difficult; in short, there are is no efficient algorithm to decide whether a permutation contains another one as a pattern.

In this paper we prove the $S_n(q) < c^n$ conjecture for the most general class so far, the *layered patterns*. A pattern is called layered if it consists of the disjoint union of substrings (the *layers*) so that the entries decrease within each layer, and increase between the layers. For example, 321654879 is a layered pattern with layers 321, 54, 876, and 9. Layered patterns are thoroughly examined from a different aspect in [6]. The diagram of a generic layered pattern is shown on Figure 1.

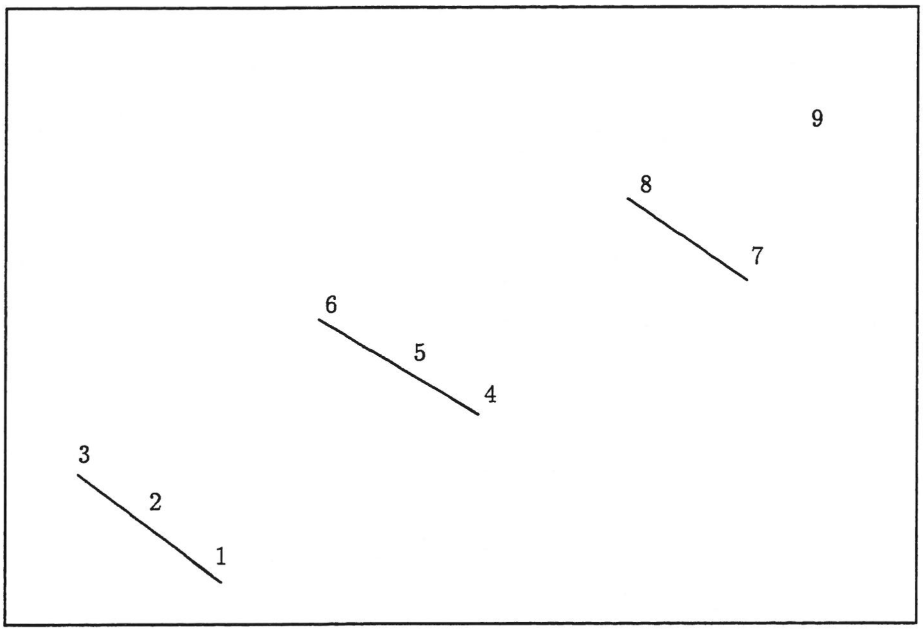

Fig 1: a layered permutation

Clearly, there is a natural bijection between layered patterns of length k and vectors with positive integer coordinates whose sum is k, by taking the length of the ith layer to be the ith coordinate of the corresponding vector. Therefore, the number of layered patterns of length k is just the number of compositions of k, that is, 2^{k-1}. So our result will yield the proof of the Wilf-Stanley conjecture for at least 2^k patterns as it is obvious that $S_n(q) = S_n(q')$, where q' is the reverse of q. In most cases, the *complement* q'' of q, obtained by subtracting each entry of q from $k+1$ is yet a different pattern, and clearly $S_n(q) = S_n(q'')$. Previous results proved the conjecture for only a constant number patterns of length k.

After the completion of present paper, [1], the conjecture has been proved for another family of patterns of size 2^{k-1}, as we will discuss it in Theorem 2. That theorem, our result, and Lemma 3 together will make it possible for us to prove the conjecture for some patterns which start as a layered pattern and as a unimodal pattern.

Denote $P(k_1, k_2, \cdots, k_t)$ the unique layered pattern of length $k = \sum_{i=1}^{t} k_i$ whose ith layer is of length k_i. Thus $P(3, 2, 3, 1)$ denotes our previous example, the pattern 3 2 1 5 4 8 7 6 9.

If the $S_n(q) < c^n$ conjecture is true for some pattern q, then we will say that q is a *good* pattern. Our proof will proceed as follows.

First, we prove the conjecture for all patterns of the form $Q_k = P(1, k-2, 1)$, for any $k \geq 3$. This is the heart of our proof. We are going to achieve this by showing that the *growth rate* $g_n = S_n(Q_k)/S_{n-1}(Q_k)$ is bounded. Then we use a lemma first proved in [2] to "replace" the last element of a pattern $P(1, k_1, 1)$ by the pattern $P(1, k_1, k_2, 1)$ and still get a good pattern. Then we will iterate this procedure to reach the good pattern $P(1, k_1, k_2, \cdots, k_t, 1)$. As subsequences of good patterns are certainly good patterns, too, this will imply that $P(k_1, k_2, \cdots, k_t)$, completing the proof.

2 The pattern $1\, k-1\, k-2 \cdots 3\, 2\, k$

As we mentioned in the Introduction, the case of monotonic patterns has been solved by Regev. He has proved the following strong result.

Lemma 1 *[7] For all n, $S_n(1234\cdots k)$ asymptotically equals*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = \gamma_k^2 \int_{x_1 \geq} \int_{x_2 \geq} \cdots \int_{\geq x_k} [D(x_1, x_2, \cdots, x_k) \cdot e^{-(k/2)x^2}]^2 dx_1 dx_2 \cdots dx_k,$$

where $D(x_1, x_2, \cdots, x_k) = \Pi_{i<j}(x_i - x_j)$, and $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

The following theorem is an important tool in our efforts to prove that all layered patterns are good.

Theorem 1 *For all $k \geq 3$, the pattern $Q_k = P(1, k-2, 1)$ is good.*

Proof: Note that $Q_3 = P(1,1,1) = 123$, so $S_n(Q_3) = C_n = \binom{2n}{n}/(n+1) < 4^n$ [8]. Also note that $Q_4 = P(1,2,1) = 1324 < 32^n$, as proved in [2]. The proof of the general case is somewhat tedious, though conceptionally not very difficult. If the reader does not want to break the course of the proof of our main result, he may want to take this theorem for granted, and continue with the next section.

We are going to prove our theorem by induction on k. If $k \leq 4$, then the statement is true. Now suppose we know the statement for all positive integers $t < k$, and prove it for k.

We first need to show that $Q'_k = (k-2)(k-3)\cdots 21(k-1)$, that is, the pattern obtained from Q_k by deleting its first element, is a good pattern. Fortunately, this is a direct consequence of the following general theorem. For readers who do not want to rely on this yet unpublished theorem, we will indicate in the Remark after the proof of Theorem 1 how to prove the much weaker statement that Q'_k is a good pattern. We also note that it is widely conjectured that $S_n(Q'_k) = S_n(12\cdots k-1)$, which would certainly imply that Q'_k is a good pattern.

A pattern $q = q_1 q_2 \cdots q_k$ is called *unimodal* if its entries first increase steadily, then decrease steadily. In other words, there exists an i so that $q_1 < q_2 < \cdots < q_i > q_{i+1} > \cdots > q_k$. Then the following Theorem is the most recent addition to the theory of pattern avoidance.

Theorem 2 *[1] Every unimodal pattern is a good pattern.*

In other words, if the pattern q starts with the increasing string of its smallest r elements, than this string can be reversed without changing the value of $S_n(q)$. In particular, if $r = k-1$, we get the pattern which will be useful for us.

Corollary 1 *For any $k \geq 3$, we have $S_n(12\cdots k) = S_n((k-1)(k-2)\cdots 21k)$. So $S_n((k-1)(k-2)\cdots 21k) < (k-1)^{2n}$.*

Recall that elements in a permutation which are smaller than any elements they are preceded by are called *left-to-right minima.* Similarly, we will say that an element is a *right-to-left maximum* if it is larger than any element it precedes. Note that the right-to-left maxima form a decreasing subsequence and so do the left-to-right minima, too. Entries of permutations which are neither left-to-right minima nor right-to-left maxima will be called *remaining entries.*

Example 1 In the permutation 351264, the entries 3 and 1 are the left-to-right minima, the entries 6 and 4 are the right-to-left maxima, and the entries 5 and 2 are the remaining entries.

We will use the left-to-right minima to classify all n-permutations in a very useful way.

Definition 1 *Two n-permutations x and y are said to be in the same* weak class *if the left-to-right minima of x are the same as those of y, and they are in the same positions.*

For example, 3 4 1 2 5 and 3 5 1 2 4 are in the same weak class. The number of weak classes is easy to determine:

Lemma 2 *The number of nonempty weak classes is $c_n = \binom{2n}{n}/(n+1) < 4^n$.*

Proof: Each weak class contains exactly one 123-avoiding permutation which is obtained by writing all the entries which are not left-to-right minima in decreasing order. The number of 123-avoiding permutations is known to be $C_n = \binom{2n}{n}/(n+1)$ and the proof is complete. $\diamond$

Therefore, it suffices to show that no weak class W can contain more than W_k^n Q_k-avoiding permutations; this would imply $S_n(Q_k) < C_n \cdot W_k^n < (4W_k)^n$. Our induction hypothesis thus says that $S_n(Q_{k-1}) < (4W_{k-1})^n$.

If W has only one left-to-right minimum, that is, permutations in W start with their entry 1, then the rest of any such permutation must be Q'_k-avoiding. There are less than $(k-2)^{2n} < C_k^n$ such permutations by corollary 1, so our claim is true.

Suppose for now (for clarity) that W has only two left-to-right minima, a, which is the leftmost element, and $b = 1$. Let $p \in W$ be a Q_k-avoiding n-permutation. Entries of p larger than a will be called *large* entries and entries larger than 1 but smaller than a will be called small entries. Clearly, all small entries are located on the right of $b = 1$, otherwise a third left-to-right minimum would exist.

The string of small entries p_S must be a Q'_k-avoiding permutation as any Q'_k-pattern on them would form a Q_k-pattern completed with the entry 1. The same goes for the string of large entries, p_L, the entry a replacing the entry 1. So if we have n_1 small entries and n_2 large entries, then we have less than $(k-2)^{2n_1}$ choices for the string of small entries and $(k-2)^{2n_2}$ choices for that of large entries. As $n_1 + n_2 = n - 2$, this yields altogether less than $(k-2)^{2n-4}$ choices for these two substrings.

What is left to do is to find a strong enough upper bound for the number of ways these two substrings (together with the left-to-right minima) can be *merged* together, without creating any Q_k-pattern. The number of all mergings could be as high as $\binom{n_1+n_2}{n_1}$, but we will see that most mergings actually do create Q_k-patterns. (It would be a pitfall to say that $\binom{n_1+n_2}{n_1} < 2^n$, so we are done, because this argument does not work when W has more than a constant number of left-to-right minima). What we need is a method which works for any number of left-to-right minima.

We will see that as k grows, it is not getting much easier to find good mergings. In fact, we are going to see that when passing from Q_{k-1}-avoidance to Q_k-avoidance, the ratio of good mergings vs. all mergings grows only by an exponential factor. Because of our induction hypothesis, this ratio was so small it resulted in an exponential number of good mergings only, this implies that for Q_k-avoiding permutations, there is only an exponential number of good mergings as well. Indeed, the number of all mergings of several strings is independent of the conditions imposed on each string, it suffices to examine how the ratio of good mergings can grow if k grows.

Intuitively, the fact that this ratio does not grow fast (in reality, it seems to grow even much slower than we could establish it), is not surprising. For if k grows by one, both the substring of small and large entries is allowed to contain Q_{i+1} instead of Q_i, but the pattern we have to avoid after their merging grows only by one, not by two. In the reality, it seems that $\sqrt[n]{S_n(Q_k)}$ is close to $(k-1)^2$. It is known that in general $\sqrt[n]{S_n(Q_k)}$ cannot converge to a smaller number than $(k-1)^2$ because $S_n(Q_4) > S_n(1234)$ for all $n \geq 7$ [3].

Let $L_k = \{$ all possible strings of large entries in W which completed with a form a Q_k-avoiding permutation $\}$, and let $S_k = \{$all possible strings of small entries in W which, completed with b form a Q_k-avoiding permutation $\}$. Define $l_k = |L_k|$ and $s_k = |S_k|$. Suppose that there are t positions between a and b, so only $n_2 - t$ large entries have a chance to be preceded by a small entry. Then the number of *all* permutations in W in which the string of large entries is from L_k and that of small entries is from S_k is clearly

$$c_k = l_k \cdot s_k \cdot \binom{n_1 + n_2 - t}{n_1} = l_k \cdot s_k \cdot \binom{n - 2 - t}{n_1}$$

On the other hand, the number of Q_k-avoiding permutations in W is clearly less than c_k, as some (in fact, most) of these mergings create a Q_k-pattern. Let g_k be the number of permutations in W which avoid Q_k. This implies that their small entries form a string in S_k, and their large entries form a string in L_k.

We will show that

$$\frac{g_k}{c_k} \leq \frac{g_{k-1}}{c_{k-1}} \cdot 4^{n_2} \cdot (k-2)^{2n_2},$$

for all $k \geq 5$. This clearly implies our claim by proving the number of good mergings for Q_k is only an exponential factor larger than that for Q_{k-1}, and we know that this latter is small by the induction hypothesis on k.

Let $p \in W$, and we want p to be Q_k-avoiding. As we said above, this implies that the string p_S of small entries as well as the string p_L of large entries avoids Q'_k. However, both of them may contain copies of Q'_{k-1}, and copies of any shorter Q'_r.

Take p_L. Choose any right-to-left maximum M on it, and consider all large entries which are smaller than M and which are on the left of M. Among these entries, take all left-to-right maxima, and color them blue. Do this for all right-to-left maxima. The significance of these points is that, as the reader can easily check, they are the starting points of the maximal (ie. not extendable) Q'_r-patterns, for any $r < k$.

Dually, take p_S, and choose any right-to-left maximum M, and consider all small entries which are smaller than M and which are on the left of M. Among these entries, take all right-to-left minima, and color them red. Do this for all right-to-left maxima. The significance of these points in turn is, that they play the role of 1 in any maximal (ie. not extendable) Q'_r-patterns.

Example 2 Let $k = 4$, then $Q'_k = 213$, and $Q_k = 1324$. Let $p = 341258967$. Then p has two left-to-right minima, the entries 3 and 1, one small entry, 2, and six large entries, 4,5,6,7,8,9. So p_L is the substring 458967. If M is chosen to be the right-to-left maximum 7, then the entries on the left of M and smaller than M are 4,5, and 6. All of these three are right-to-left minima, so they all get colored red.

Now we are in a position to find an upper bound for the number of good mergings of p_S and p_L. We decompose our merging procedure into two parts.

1. First try to merge p_L and the string of *uncolored* small entries, p_S^u, to get the permutation p'. Recall that the merging of two substrings contains the left-to-right minima of p in addition to these two substrings. Note that p_L^u is Q'_{k-1}-avoiding. Clearly the permutation we obtain will be Q_k-avoiding if and only if we merged p_S^u, and the string of *uncolored* large entries p_L^u together without creating a Q_{k-1}-pattern on them. (For we could find, by the definition of our coloring, a blue large entry to complete any such Q_{k-1}-pattern to a Q_k-pattern). But this is nothing else but the merging of two Q'_{k-1}-avoiding strings, ie. p_S^u and p_L^u. Recall that n_2 denotes the number of large entries. Now observe that we have less than $(4 \cdot (k-2)^2)^{n_2}$ choices for the set, position and permutation of the colored large entries. Indeed, we have less than 2^{n_2} choices for their set, we have less than 2^{n_2} choices for the positions in which they are, finally, they must form a Q'_k-avoiding permutation, (which means less than $(k-2)^{2n_2}$ choices, by proposition 1) otherwise together with a, they would form a Q_k-pattern.

   To summarize, if, instead of merging only p_L^u and p_S^u so we get a Q_{k-1}-avoiding permutation, we want to merge p_L and p_S^u so we avoid Q_k, the ratio of "good mergings vs. all mergings" can go up by no more than an exponential factor.

2. When we want to insert the red small entries as well, we face additional constraints because we risk creating Q_k-patterns, but we cannot loosen our rules as we did before, ie. when we passed from Q_{k-1}-avoidance to Q_k-avoidance. Therefore, as inserting new elements can only create new Q_k-patterns and cannot eliminate existing ones, the ratio of good mergings decreases.

By these two operations we used to extend the mergings of two Q'_k-avoiding permutations into that of two Q_k-avoiding permutations, the ratio g_k/c_k became at most $(4(k-2)^2)^{n_2}$ bigger, which is an exponential factor. Indeed, the first operation increased the ratio by at most this much, and the second one decreased it.

If W has $m > 2$ left-to-right minima, then the same argument holds except that the large entries must be defined as those larger than the left-to-right minima, and the small entries as all the rest. This way the large entries will still form a Q'_k-avoiding permutation as needed, and the small entries, together with their $m-1$ left-to-right minima, form a Q_k-avoiding permutation, which can be decomposed by this same procedure. During this procedure, the exponents of $(k-2)^2$, that is, those coming from the current set of large entries will add up to an integer less than n. (In an alternative way of speaking, we could say we do an inductive proof on m, the number of left-to-right minima). The above argument shows that the number of good mergings remains exponential. This completes the proof of our claim that any weak class contains less than $W_k^n = (k-2)^{2n} \cdot 4^n \cdot W_{k-1}^n = (4(k-2)^2 \cdot W_{k-1})^n$ Q_k-avoiding permutations, yielding $S_n(Q_k) < (4W_k)^n$. $\diamond$

Remark: To see that Q'_k is a good pattern without using Theorem 2, we can again use this same inductive procedure, the initial condition $S_n(Q'_4) = S_n(213) = C_n < 4^n$ being true. Taking reverse and complement, we see that $S_n(Q'_k) = S_n(1\,(k-1)\,(k-2)\cdots 32)$, and we can copy the above proof. In fact, this case is even simpler, as the substrings p_L^u and p_S^u need to avoid the decreasing pattern of length $k-2$, and that is easier to deal with than Q'_k.

3 The General Case

In this section we use the result of the previous section as well as recursive methods from [2] to prove that all layered patterns are good.

Proposition 1 *If q is a good pattern and q' is a pattern obtained from q by deletion of some elements, then q' is a good pattern as well.*

Proof: Every q'-avoiding permutation is q-avoiding, too. $\diamond$

Definition 2 *Let q be a pattern, and let y be an entry of q. Then* to replace y *by the pattern w is to add $y-1$ to all entries of w, and to all entries of q larger than y, then to delete y and to succesively insert the entries of w at its position.*

Example 3 Replacing the entry 1 in 1 4 2 3 by 1 3 2 4 results in the pattern 1 3 2 4 7 5 6.

The following lemma is our main tool in proving that all layered patterns are good.

Lemma 3 *("replacing an element by a pattern") Let q be a pattern and let y be an entry of q so that for any entry x preceding y we have $x < y$ and for any entry z preceded by y we have $y < z$. Suppose that $S_n(q) < K^n$ for some constant K and for all n.*

Let w be a pattern of length k starting with 1 and ending with k so that $S_n(w) < C^n$ holds for all n, for some constant C. Let q' be the pattern obtained by replacing the entry y by the pattern w in q. Then $S_n(q') < (2CK)^n$, thus q' is a good pattern.

Figure 2 is intended to help the reader visualize the definition of replacing an element by a pattern in the special case of lemma 3.

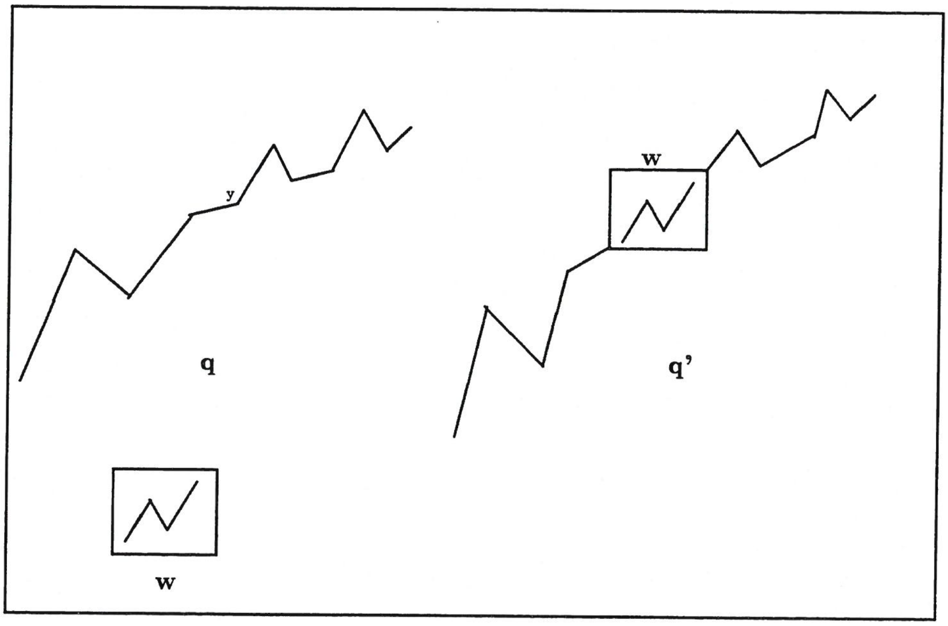

Fig. 2: Replacing an element by a pattern

Proof: Take an n-permutation p which avoids q'. Suppose it contains q. Then consider all copies of q in our permutation and consider their entries y. Clearly, these entries must form a permutation which does not contain w. For suppose they do, and denote y_1 and y_k the first and last elements of that purported copy of w. Then the initial segment of the copy of q which contains y_1 followed by the y_2 through y_{k-1} and the ending segment of the copy of q which contains y_k would form a copy of q'.

Therefore, if p avoids q', then it either avoids q, or the substring of its entries which can play the role of y in a copy of q avoids w. This shows that less than $(2C)^{n-1} \cdot K^n + K^n < (2CK)^n$ permutations of length n can avoid q'. $\diamond$

Now we are in a position to prove our main theorem.

Theorem 3 *Every layered pattern is a good pattern.*

Proof: Let $P = P(k_1, k_2, \ldots, k_t)$ be any layered pattern. Then $P_1 = P(1, k_1, 1)$ is a good pattern by Theorem 1. Now apply Lemma 3 to replace the last element of P_1 by the pattern $P(1, k_2, 1)$, which is in turn a good pattern by Theorem 1, to get the good pattern $P(1, k_1, 1, k_2, 1)$, then delete the middle layer of length 1 to get the good pattern $P_2 = P(1, k_1, k_2, 1)$. Then continue this way, that is, at the ith step, replace the last element of $P_i = P(1, k_1, k_2, \cdots k_i, 1)$ by the good pattern $P(1, k_{i+1}, 1)$, to get the good pattern $P_{i+1} = P(1, k_1, k_2, \cdots, k_{i+1}, 1)$. After t steps, we get that $P_n = P(1, k_1, k_2, \cdots, k_t, 1)$ is a good pattern. As P itself is contained in P_n, this implies that P is a good pattern. $\diamond$

Lemma 3 now implies the following Corollary as layered and unimodal patterns are good by Theorems 2 and 3.

Corollary 2 *Let q_1 be a layered pattern ending with its largest entry and let q_2 be a unimodal pattern starting with the entry 1. Let q be a pattern obtained by replacing the last entry of q_1 by q_2. Then q is a good pattern.*

Finally, note that half of all layered patterns of length k_1, that is, 2^{k_1-2} of them, end with their largest entry, whereas 2^{k_2-2} unimodal patterns of length k_2 start with their smallest entry. If k_1 runs through $1, 2, \cdots k$, where $k_1 + k_2 - 1 = k$, and k is fixed, then this way we obtain roughly $k \cdot 2^{k-3}$ patterns of length k which are now proved to be good patterns.

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