Enumeration of m-ary cacti according to their color and degree distributions*

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Résumé

Le but de cet article est de dénombrer diverses classes de cactus plans m-gonaux cycliquement colorés, appelés cactus m-aires. Ce problème combinatoire est motivé par la classification topologique des polynômes complexes ayant au plus m valeurs critiques, étudiée par Zvonkin et autres. Nous obtenons des formules explicites pour les cactus m-aires enracinés et non enracinés, selon i) le nombre de polygones, ii) la distribution des sommets de chaque couleur, iii) la distribution des degrés des sommets de chaque couleur. Nous dénombrons également les cactus m-aires selon l'ordre du groupe d'automorphismes. Par une généralisation de la formule d'Otter, nous exprimons l'espèce des cactus m-aires non enracinés en termes de cactus enracinés et de cactus pointés en un sommet. Une variante de l'inversion de Lagrange m-dimensionnelle est alors utilisée pour le dénombrement de ces structures.

Abstract

The purpose of this paper is to enumerate various classes of cyclically colored m-gonal plane cacti, called m-ary cacti. This combinatorial problem is motivated by the topological classification of complex polynomials having at most m critical values, studied by Zvonkin and others. We obtain explicit formulae for both rooted and unrooted m-ary cacti, according to i) the number of polygons, ii) the vertex-color distribution, iii) the vertex degree distribution of each color, and also for m-ary cacti according to the order of their automorphism group. Using a generalization of Otter's formula, we express the species of (unrooted) m-ary cacti in terms of rooted and of pointed cacti. A variant of the m-dimensional Lagrange inversion is used to enumerate these structures.

1 Introduction

A *cactus* is a connected simple graph in which each edge lies in exactly one elementary cycle. It is equivalent to say that all blocks (2-connected components) of a cactus are elementary cycles, i.e., polygons. An *m-cactus* is a cactus all of whose polygons are m-gons, for some fixed $m \geq 2$.

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By convention, a 2-cactus is simply a tree. These graphs are also called "Husimi trees", and their definition was given by Harary and Uhlenbeck [12] following a paper by Husimi [13] on the cluster integrals in the theory of condensation in statistical mechanics. See also Uhlenbeck and Ford [20], and Riddell [17].

A *plane* m-cactus is an embedding of an m-cactus into the plane so that every edge is incident with the unbounded region. An *m-ary cactus* is a plane m-cactus whose vertices are cyclically m-colored $1, 2, \ldots, m$ counterclockwise within each m-gon. For technical reasons, we also consider a single vertex colored in any one of the m colors to be an m-ary cactus. A quaternary cactus is shown on Figure 1.

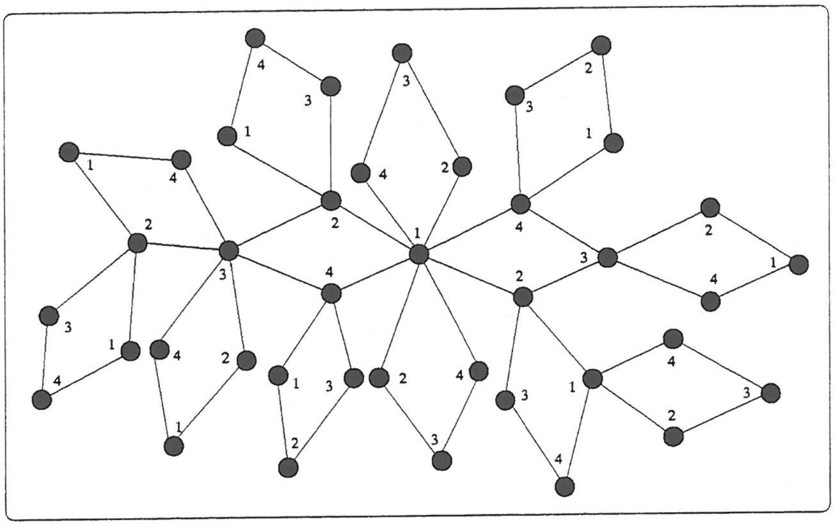

Figure 1: A quaternary cactus.

Our goal is to enumerate m-ary cacti according to the number of polygons, the vertex distribution of each color, and the degree distributions of vertices of each color, and also according to the order of their automorphism group. See [2] for a more complete version of this paper. This extends previous results [14] of two of the authors on the enumeration of bicolored plane trees according to their degree distributions, to general $m \geq 2$.

This study is motivated by the topological classification of complex polynomials having at most m critical values. Indeed, the preimage, under such a polynomial, of an m-gon joining the m "critical values" $z_1, z_2, \ldots, z_m$ yields an m-ary cactus whose degree distributions correspond to the multiplicities of the critical points. See [7] for more details.

Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of nonnegative integers and set $n = \sum_i n_i$. We say that a m-ary cactus C has *vertex-color distribution* $\vec{n}$ if C has n_i vertices of color i, for $i = 1, \ldots, m$. The integer n is then the total number of vertices in C. We define the *degree* of a vertex in a cactus to be the number of m-gons adjacent to that vertex. Note that it is half the number of edges adjacent to the given vertex. Let k_{ij} be the number of vertices of color i and degree j, of C. This vertex degree distribution is then represented by a $m \times \infty$ matrix K, whose ij^{th} entry is k_{ij}. The vector $\vec{k}_i = (k_{i0}, k_{i1}, k_{i2}, \ldots)$, which is the i^{th} row of the matrix K and represents the vertex degree distribution the vertices of color i. Hence, the number of vertices of degree i is given by $n_i = \sum_j k_{ij}$ Finally, let p be the number of polygons in C.

For example, in Figure 1, the vertex-degree distributions are

$$\vec{k}_1 = (0,7,1,0,1,0,\ldots) = 1^7 2^1 4^1, \quad \vec{k}_2 = (0,7,3,0,0,0,\ldots) = 1^7 2^3,$$
$$\vec{k}_3 = (0,8,1,1,0,0,\ldots) = 1^8 2^1 3^1, \quad \vec{k}_4 = (0,9,2,0,0,0,\ldots) = 1^9 2^2,$$
$$p = 13, \ n_1 = 9, \ n_2 = 10, \ n_3 = 10, \ n_4 = 11, \text{ and } n = 40.$$

2 Coherence conditions

We now state necessary and sufficient conditions for the existence of an m-ary cactus having a given vertex-color distribution and vertex degree distribution of each color.

Lemma 1 *There exists an m-ary cactus having vertex-color distribution $\vec{n} = (n_1, n_2, \ldots, n_m)$ if and only if*

1. $p = (n-1)/(m-1)$ is an integer, where $n = \sum_{i=1}^{m} n_i$,
2. $p \geq 1 \Rightarrow n_i \leq p$, for $i = 1, \ldots, m$.

□

Lemma 2 *There exists an m-ary cactus having n vertices, p polygons and whose vertex degree distribution is given by the matrix $K = (k_{ij})$, with $n = \sum_{ij} k_{ij}$, if and only if*

1. $\sum_j j k_{ij} = p$, *for all* i,
2. $n = (m-1)p + 1$,
3. $p \geq 1 \Rightarrow k_{i0} = 0$ *for all* i.

□

3 Main results

The main results in this paper are the three following theorems on the enumeration of unlabelled (and unrooted) m-ary cacti:

Theorem 3 *The number $|\tilde{\mathcal{K}}_{m,p}|$ of m-ary cacti having p m-gons is given by*

$$|\tilde{\mathcal{K}}_{m,p}| = \frac{1}{p}\left(\frac{1}{(m-1)p+1}\binom{mp}{p} + \sum_{\substack{d|p \\ d<p}} \phi(p/d)\binom{md}{d}\right), \tag{1}$$

where ϕ is the Euler function. □

Theorem 4 *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector satisfying the coherence conditions 1 and 2 of Lemma 1. The number $|\tilde{\mathcal{K}}_{m,\vec{n}}|$ of m-ary cacti having vertex-color distribution $\vec{n}$ is given by*

$$|\tilde{\mathcal{K}}_{m,\vec{n}}| = \frac{1}{p^2}\left(\prod_{i=1}^{m}\binom{p}{n_i} + \sum_{i=1}^{m} \sum_{\substack{d|(p,\vec{n}-\vec{e}_i) \\ d>1}} \phi(d)(p-n_i+1)\binom{p/d}{(n_i-1)/d}\prod_{j\neq i}\binom{p/d}{n_j/d}\right), \tag{2}$$

where $p = (n-1)/(m-1)$ is the number of polygons, $\vec{e}_i$ is the vector having 1 as its i^{th} component and 0 elsewhere, and the notation $d|(p, \vec{n}-\vec{e}_i)$ means that d divides p and all components of $\vec{n}-\vec{e}_i$. □

Theorem 5 *Let $K = (k_{ij})$ be a $m \times \infty$ matrix of nonnegative integers and set $n = \sum_{ij} k_{ij}$ and $p = (n-1)/(m-1)$. Suppose that K, n and p satisfy the coherence conditions 1, 2, and 3 of Lemma 2. Then the number $|\widetilde{\mathcal{K}}_{m,K}|$ of m-ary cacti having k_{ij} vertices of color i and of degree j, is given by*

$$|\widetilde{\mathcal{K}}_{m,K}| = p^{m-2}\left(\prod_{i=1}^{m}\frac{1}{n_i}\binom{n_i}{\vec{k}_i} + \sum_{(i,j)\in\mathrm{supp}(K)}\ \sum_{\substack{d>1\\ d\in\mathrm{Div}(j,K-E_{ij})}} \frac{\phi(d)}{\prod_{\ell\neq i} n_\ell}\binom{(n_i-1)/d}{(\vec{k}_i-\vec{e}_j)/d}\prod_{\ell\neq i}\binom{n_\ell/d}{\vec{k}_\ell/d}\right), \quad (3)$$

where $\vec{k}_i$ denotes the i^{th} row in the matrix K, $n_i = \sum_j k_{ij}$, E_{ij} is the matrix whose ij^{th} entry is 1, all others being 0, supp(K) *denotes the* support *of K, that is,* $\mathrm{supp}(K) = \{(i,j) \mid k_{ij} \neq 0\}$, *$d \in \mathrm{Div}(j, K - E_{ij})$ means that d divides j and all entries in the matrix $K - E_{ij}$.* □

Moreover, the numbers $|\overline{\mathcal{K}}_{m,p}|$, $|\overline{\mathcal{K}}_{m,\vec{n}}|$, and $|\overline{\mathcal{K}}_{m,K}|$ of asymmetric cacti in the corresponding classes are obtained by replacing the Euler ϕ function in formulas (1), (2), and (3) by the Möbius function μ.

4 Functional equations for m-ary cacti

We consider the class $\mathcal{K}$ of m-ary cacti as a species on m sorts of vertices, one for each color. Note that the plane embedding of an m-ary cactus C is completely characterized by the specification, for each vertex v of C, of a circular permutation on the polygons adjacent to v. We now introduce the following subsidiary m-sort species:

- $\mathcal{K}^{\diamond}$: species of *rooted* (that is, pointed at a polygon) m-ary cacti (see Figure 2),
- $\mathcal{K}^{\bullet_i}$: species of m-ary cacti, *pointed* at a vertex of color i (see Figure 3),
- $\mathcal{A}_i$: species of m-ary cacti, *planted* at a vertex of color i (see Figure 4).

A *planted* cactus is similar to a pointed cactus except that a pair of half-edges is attached to the pointed vertex, thus contributing 1 more to the degree of this vertex, and preventing the adjacent polygons to fully rotate around it. We have the following isomorphisms of species, for $i = 1, \ldots, m$:

Proposition 6

$$\mathcal{A}_i = X_i L(\widehat{\mathcal{A}}_i), \quad (4)$$

$$\mathcal{K}^{\bullet_i} = X_i(1 + C(\widehat{\mathcal{A}}_i)), \quad (5)$$

$$\mathcal{K}^{\diamond} = \mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_m, \quad (6)$$

where X_i denotes the species of singletons of sort (or color) i, $\widehat{\mathcal{A}}_i := \prod_{j\neq i}\mathcal{A}_j$ denotes the product of all $\mathcal{A}_j$ except $\mathcal{A}_i$, C denotes the species of (non-empty) circular permutations and L denotes that of (possibly empty) permutations, or, more precisely, of linear orders (or lists). □

Remark that equations (4) and (6) are essentially due to Goulden and Jackson [9]. The following result is used to express unrooted m-ary cacti in terms of pointed and rooted ones. It is closely related to Otter's dissimilarity characteristic formula for trees [16].

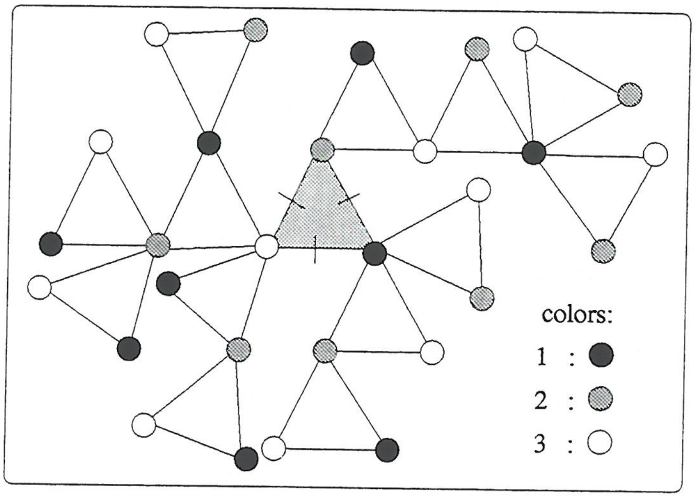

Figure 2: A rooted ternary cactus.

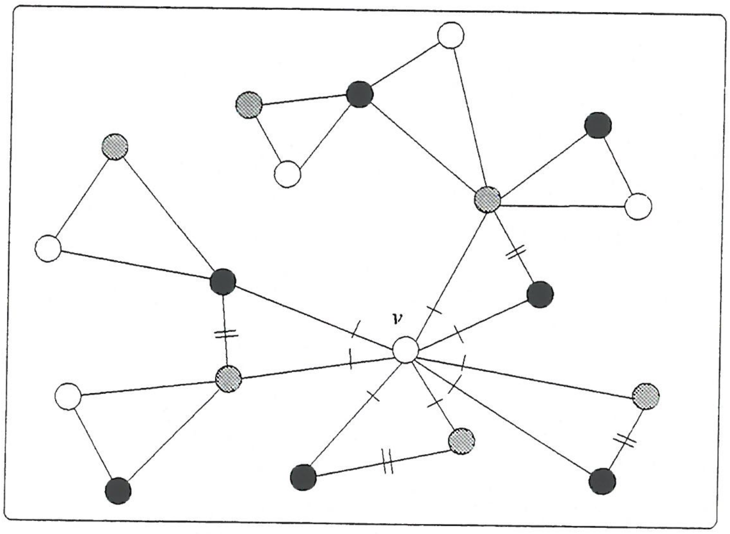

Figure 3: A ternary cactus pointed at vertex v.

Theorem 7 DISSYMMETRY THEOREM FOR m-ARY CACTI. *There is an isomorphism of species*

$$\mathcal{K}^{\bullet_1} + \mathcal{K}^{\bullet_2} + \cdots + \mathcal{K}^{\bullet_m} = \mathcal{K} + (m-1)\mathcal{K}^{\diamond}. \tag{7}$$

□

Corollary 8 *The species $\mathcal{K}$ of m-ary cacti can be written as*

$$\mathcal{K} = \sum_{i=1}^{m} \mathcal{K}^{\bullet_i} - (m-1)\mathcal{K}^{\diamond} = \sum_{i=1}^{m} X_i(1+C)(\widehat{\mathcal{A}}_i) - (m-1)\prod_{i=1}^{m} \mathcal{A}_i. \tag{8}$$

□

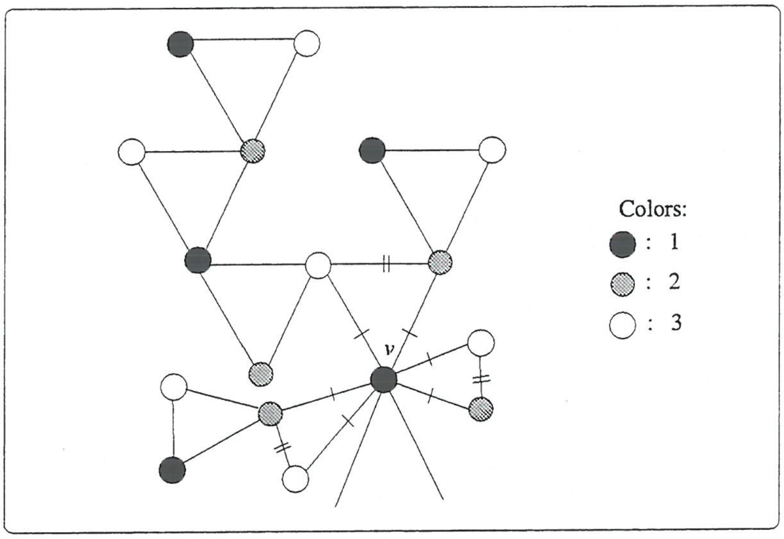

Figure 4: A planted ternary cactus.

The combinatorial equations presented in Proposition 6, Theorem 7 and in Corollary 8 lead to functional equations involving multivariable generating series. In order to solve them, we use a special form of multidimensional Lagrange inversion, which can be directly applied to m-ary cacti. It is based on the standard form of Lagrange inversion, due to Good; see Theorem 1.2.9, 1, of [8] or the equivalent formula (28b) of [1]. There is a particularly simple two-dimensional case of this formula, the alternating case, which we call the *Chottin formula.* In the papers [5] [6], Chottin worked extensively on the two-dimensional Lagrange inversion and its combinatorial proof. We extend this result into m dimensions.

Theorem 9 GENERALIZED CHOTTIN FORMULA. *Let $A_1, A_2, \ldots, A_m$ be formal power series in the variables $x_1, x_2, \ldots, x_m$ such that for $i = 1, \ldots, m$, the relations $A_i = x_i\Phi_i(\widehat{A}_i)$ are satisfied, where the Φ_i are given formal power series of one variable, and $\widehat{A}_i = \prod_{j\neq i} A_j$. Also let $\mathbf{n} = (n_1, \ldots, n_m)$ with $n_i \geq 1$ and let $\alpha_1, \ldots, \alpha_m$ be nonnegative integers. Set $n = \sum_{i=1}^m n_i$ and $\alpha = \sum_{i=1}^m \alpha_i$. Suppose that the following coherence conditions are satisfied:*

$$n_i \geq \alpha_i, \qquad \frac{n-\alpha}{m-1} = \beta \text{ is an integer.}$$

Then

$$[x_1^{n_1} \cdots x_m^{n_m}]A_1^{\alpha_1} \cdots A_m^{\alpha_m} = D \cdot [s_1^{\beta_1} \cdots s_m^{\beta_m}]\Phi_1^{n_1}(s_1) \cdots \Phi_m^{n_m}(s_m), \tag{9}$$

where

$$D = \prod_{i=1}^m (1 + \frac{\beta_i}{n_i}) - \sum_{j=1}^m \frac{\beta_j}{n_j} \prod_{i\neq j} (1 + \frac{\beta_i}{n_i}), \tag{10}$$

and $\beta_i = \beta - n_i + \alpha_i$.

Remark. The case where $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$ was derived by Goulden and Jackson [9].

5 Enumeration of m-ary cacti

5.1 Rooted m-ary cacti

The first result in this section is proved in [3]. It makes use of a bijection between m-ary cacti and unlabelled bicolored plane trees whose black vertices all have degree m.

Proposition 10 *The number $|\widetilde{\mathcal{K}}^{\diamond}_{m,p}|$ of rooted m-ary cacti having p m-gons is given by*

$$|\widetilde{\mathcal{K}}^{\diamond}_{m,p}| = \frac{1}{(m-1)p+1}\binom{mp}{p}. \tag{11}$$

□

For the enumeration of rooted m-ary cacti with a prescribed vertex-color distribution, we consider the combinatorial equations $\mathcal{A}_i = X_i L(\widehat{\mathcal{A}}_i)$ and $\mathcal{K}^{\diamond} = \mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_m$, given in Proposition 6. These combinatorial equations lead to the following functional equations:

$$\mathcal{A}_i(\mathbf{x}) = x_i \frac{1}{1-\widehat{\mathcal{A}}_i(\mathbf{x})} \tag{12}$$

and

$$\mathcal{K}^{\diamond}(\mathbf{x}) = \mathcal{A}_1(\mathbf{x})\cdots\mathcal{A}_m(\mathbf{x}), \tag{13}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. It can easily be shown that the species $\mathcal{A}_i$ is asymmetric, which implies that $\widetilde{\mathcal{A}}_i(\mathbf{x}) = \mathcal{A}_i(\mathbf{x})$, that is, the generating series of labelled and unlabelled $\mathcal{A}_i$-structures are equal. The same observation applies for the species $\mathcal{K}^{\diamond}$, so that $\mathcal{K}^{\diamond}(\mathbf{x}) = \widetilde{\mathcal{K}^{\diamond}}(\mathbf{x})$. The number $|\widetilde{\mathcal{K}}^{\diamond}_{m,\vec{n}}|$ of (unlabelled) rooted m-ary cacti having $\vec{n}$ as vertex-color distribution is given by

$$|\widetilde{\mathcal{K}}^{\diamond}_{m,\vec{n}}| = [x_1^{n_1}\cdots x_m^{n_m}]\mathcal{K}^{\diamond}(\mathbf{x}). \tag{14}$$

The next result follows from Theorem 9.

Proposition 11 *Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector satisfying conditions 1 and 2 of Lemma 1. The number $|\widetilde{K}^{\diamond}_{m,\vec{n}}|$ of rooted m-ary cacti having $\vec{n}$ as vertex-color distribution is given by*

$$|\widetilde{K}^{\diamond}_{m,\vec{n}}| = \frac{1}{p}\prod_{i=1}^{m}\binom{p}{n_i}, \tag{15}$$

where $p = (\sum_i n_i - 1)/(m-1)$ denotes the number of polygons in such a cactus.

The problem of enumerating rooted m-ary cacti according to the vertex degree distribution has been solved by Goulden and Jackson:

Proposition 12 [9] *Let $K = (k_{ij})$ be a $m \times \infty$ matrix of nonnegative integers, $n = \sum_{ij} k_{ij}$ and $p = (n-1)/(m-1)$. Suppose that K, n and p satisfy the coherence conditions 1, 2 and 3 of Lemma 2. Then the number $|\widetilde{\mathcal{K}}^{\diamond}_{m,K}|$ of rooted m-ary cacti having k_{ij} vertices of color i and degree j is given by*

$$|\widetilde{\mathcal{K}}^{\diamond}_{m,K}| = \frac{p^{m-1}}{\prod_{i=1}^{m} n_i}\cdot\prod_{i=1}^{m}\binom{n_i}{\vec{k}_i}, \tag{16}$$

where $\vec{k}_i$ denotes the i^{th} row of K and $n_i = \sum_j k_{ij}$. □

5.2 Unrooted m-ary cacti

In order to enumerate unrooted m-ary cacti, two methods can be applied. One is adapted from Liskovets [15] for the enumeration of unrooted planar maps. It uses the concept of quotient maps. Proofs of Theorems 3–5 using Liskovet's method can be found in [3] and [4].

The other approach uses the Dissymmetry Theorem (Theorem 7). By Corollary 8, we have

$$\widetilde{\mathcal{K}}(\mathbf{x}) = \sum_{i=1}^{m} \widetilde{\mathcal{K}^{\bullet_i}}(\mathbf{x}) - (m-1)\mathcal{K}^{\diamond}(\mathbf{x}), \tag{17}$$

where

$$\widetilde{\mathcal{K}^{\bullet_i}}(\mathbf{x}) = x_i(1 + \sum_{d\geq 1} \frac{\phi(d)}{d} \log \frac{1}{1 - \widehat{\mathcal{A}}_i(\mathbf{x}^d)}), \tag{18}$$

and $\mathbf{x}^d := (x_1^d, x_2^d, \ldots, x_m^d)$. Theorem 4, which enumerates unrooted m-ary cacti having a prescribed vertex-color distribution $\vec{n}$ follows by applying Theorem 9.

In order to enumerate m-ary cacti according to their degree distributions, we use weights in the form of monomials $w(\mathcal{C}) = \prod_{i,j} r_{ij}^{k_{ij}}$ with $i = 1, \ldots, m$ and $j \geq 0$, for a cactus $\mathcal{C}$ having degree distribution $K = (k_{ij})$. In other words, the variable r_{ij} acts as a counter for (or marks) vertices of color i and degree j. We also use the notation $\mathbf{r}_i$ to denote the sequence $(r_{i0}, r_{i1}, \ldots)$. We denote by $\mathcal{K}_w$, $\mathcal{K}_w^{\diamond}$, and $\mathcal{K}_w^{\bullet_i}$ the corresponding species of m-ary cacti, weighted in this manner. We denote by $\mathcal{A}_{i,\mathbf{r}}$ the species of planted (at a vertex of color i) m-ary cacti similarly weighted by degree. The functional equations (4)–(8) can then be extended as follows:

$$\mathcal{A}_{i,\mathbf{r}} = X_i(r_{i,1} + r_{i,2}\widehat{\mathcal{A}}_{i,\mathbf{r}}^2 + r_{i,3}\widehat{\mathcal{A}}_{i,\mathbf{r}}^3 + \cdots), \tag{19}$$

where $\widehat{\mathcal{A}}_{i,\mathbf{r}} = \prod_{j\neq i} \mathcal{A}_{i,\mathbf{r}}$,

$$\mathcal{K}_w^{\bullet_i} = X_i(r_{i,0} + r_{i,1}C_1(\widehat{\mathcal{A}}_{i,\mathbf{r}}) + r_{i,2}C_2(\widehat{\mathcal{A}}_{i,\mathbf{r}}) + \cdots), \tag{20}$$

where C_k denotes the species of circular permutations of length k,

$$\mathcal{K}_w^{\diamond} = \prod_{i=1}^{m} \mathcal{A}_{i,\mathbf{r}}, \tag{21}$$

and

$$\mathcal{K}_w = \sum_{i=1}^{m} \mathcal{K}_w^{\bullet_i} - (m-1)\mathcal{K}_w^{\diamond}. \tag{22}$$

The important point here is that the weights behave multiplicatively, with respect to the operations of product and partitional composition. The consequences for the labelled and unlabelled generating functions are as follows:

$$\mathcal{A}_{i,\mathbf{r}}(\mathbf{x}) = x_i(r_{i,1} + r_{i,2}\widehat{\mathcal{A}}_{i,\mathbf{r}}(\mathbf{x}) + r_{i,3}\widehat{\mathcal{A}}_{i,\mathbf{r}}^2(\mathbf{x}) + \cdots), \tag{23}$$

$$\widetilde{\mathcal{A}_{i,\mathbf{r}}}(\mathbf{x}) = \mathcal{A}_{i,\mathbf{r}}(\mathbf{x}), \tag{24}$$

$$\mathcal{K}_w^{\bullet_i}(\mathbf{x}) = x_i(r_{i,0} + \sum_{h\geq 1} \frac{r_{i,h}}{h} \widehat{\mathcal{A}}_{i,\mathbf{r}}^h(\mathbf{x})), \tag{25}$$

$$\widetilde{\mathcal{K}_w^{\bullet_i}}(\mathbf{x}) = x_i(r_{i,0} + \sum_{h\geq 1} \frac{r_{i,h}}{h} \sum_{d|h} \phi(d) \widehat{A}_{i,\mathbf{r}^d}^{h/d}(\mathbf{x}^d)), \tag{26}$$

where $\mathbf{r}^d$ denotes the set of variables $\{r_{i,j}^d\}$, for $i = 1, \ldots, m$, $j \geq 0$. We also have

$$\mathcal{K}_w^{\circ}(\mathbf{x}) = \widetilde{\mathcal{K}_w^{\circ}}(\mathbf{x}) = \prod_{i=1}^{m} \mathcal{A}_{i,\mathbf{r}}(\mathbf{x}), \tag{27}$$

and finally,

$$\widetilde{\mathcal{K}_w}(\mathbf{x}) = \sum_{i=1}^{m} \widetilde{\mathcal{K}_w^{\bullet_i}}(\mathbf{x}) - (m-1)\mathcal{K}_w^{\circ}(\mathbf{x}). \tag{28}$$

Theorem 5, which enumerates m-ary cacti according to the vertex degree distribution of each color, then follows from applying Theorem 9 to the previous equation.

5.3 Unrooted m-ary cacti according to their automorphisms

Since automorphisms of m-ary cacti are required to preserve colors, the only possible symmetries of an m-ary cacti are rotations around a central vertex. See Figure 5. Let $s \geq 2$ be an integer. Let $\mathcal{K}_{w,\geq s}$ and $\mathcal{K}_{w,=s}$ denote the species of m-ary cacti whose automorphism group (necessarily cyclic) is of order a multiple of s, and exactly s, respectively. Then, following the notation of [14], section 3, we have

$$\mathcal{K}_{w,\geq s} = \sum_{i=1}^{m} X_i C_{\mathbf{r}_i,\geq s}(\widehat{\mathcal{A}}_{i,\mathbf{r}}), \tag{29}$$

$$\mathcal{K}_{w,=s} = \sum_{i=1}^{m} X_i C_{\mathbf{r}_i,=s}(\widehat{\mathcal{A}}_{i,\mathbf{r}}). \tag{30}$$

We can determine the unlabeled generating series $\widetilde{\mathcal{K}}_{w,\geq s}(\mathbf{x})$ and $\widetilde{\mathcal{K}}_{w,=s}(\mathbf{x})$ by formulas (3.2) and (3.3) of [14], essentially due to Stockmeyer. See [1], Exercise 4.4.16, and [19]. We find the following.

Proposition 13 *Let $s \geq 2$ and $p \geq 1$ be integers, and $K = (k_{ij})$, a matrix of nonnegative integers satisfying conditions 1, 2 and 3 of Lemma 2 with $n_i = \sum_{j\geq 0} k_{ij}$ and $n = \sum_i n_i$. Then the numbers $\widetilde{\mathcal{K}}_{\geq s}(K)$ and $\widetilde{\mathcal{K}}_{=s}(K)$ of unrooted m-ary cacti with vertex degree distribution K, and automorphism group of order a multiple of s, and exactly s, respectively, are given by*

$$\widetilde{\mathcal{K}}_{\geq s}(K) = \widetilde{\mathcal{F}}_{\geq s}(K) + \widetilde{\mathcal{F}}_{\geq s}(\sigma K) + \cdots + \widetilde{\mathcal{F}}_{\geq s}(\sigma^{m-1}K), \tag{31}$$

where σ acts on K by putting the first row of K in the last position so that the new first row is now the previous second row, and

$$\widetilde{\mathcal{K}}_{=s}(K) = \widetilde{\mathcal{F}}_{=s}(K) + \widetilde{\mathcal{F}}_{=s}(\sigma K) + \cdots + \widetilde{\mathcal{F}}_{=s}(\sigma^{m-1}K), \tag{32}$$

where

$$\widetilde{\mathcal{F}}_{\geq s}(K) = \frac{p^{m-2}s}{\prod_{j\neq 1} n_j} \sum_{h,d} \phi(d/s) \binom{(n_1-1)/d}{(\vec{k}_1 - \vec{e}_h)/d} \prod_{j\neq 1} \binom{n_j/d}{\vec{k}_j/d} \tag{33}$$

and

$$\widetilde{\mathcal{F}}_{=s}(K) = \frac{p^{m-2}s}{\prod_{j\neq 1} n_j} \sum_{h,d} \mu(d/s) \binom{(n_1-1)/d}{(\vec{k}_1 - \vec{e}_h)/d} \prod_{j\neq 1} \binom{n_j/d}{\vec{k}_j/d}, \tag{34}$$

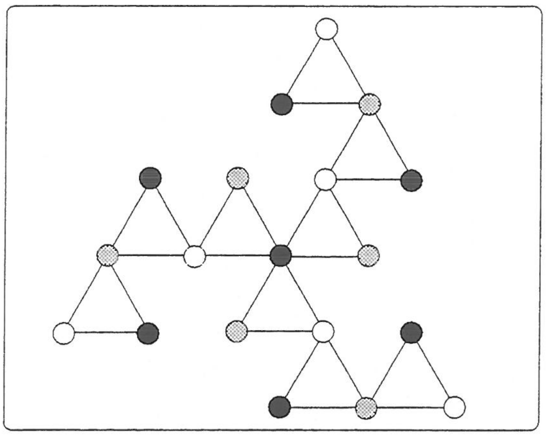

Figure 5: A ternary cactus with a symmetry of order 3.

where $\vec{k}_i$ is the i^{th} row in K, the sum being taken over all pairs of integers $h, d \geq 1$ such that $h \in \mathrm{Supp}(\vec{k}_1)$, $s|d$, and $d \in \mathrm{Div}(h, K - E_{1h})$. $\square$.

Similar but simpler computations yield the following for the vertex color distribution.

Proposition 14 *Let $s \geq 2$ be an integer, and let $\vec{n} = (n_1, \ldots, n_m)$ be a vector of positive integers satisfying conditions 1 and 2 of Lemma 1. Then the numbers $\widetilde{\mathcal{K}}_{\geq s}(\vec{n})$ and $\widetilde{\mathcal{K}}_{=s}(\vec{n})$ of unrooted m-ary cacti with vertex-color distribution $\vec{n}$ and automorphism group of order a multiple of s, and exactly s, respectively, are given by*

$$\widetilde{\mathcal{K}}_{\geq s}(\vec{n}) = \sum_{i=1}^{m} \frac{s(p - n_i + 1)}{p^2} \sum_d \phi(d/s) \binom{p/d}{(n_i - 1)/d} \prod_{j \neq i} \binom{p/d}{n_j/d}, \tag{35}$$

$$\widetilde{\mathcal{K}}_{=s}(\vec{n}) = \sum_{i=1}^{m} \frac{s(p - n_i + 1)}{p^2} \sum_d \mu(d/s) \binom{p/d}{(n_i - 1)/d} \prod_{j \neq i} \binom{p/d}{n_j/d}, \tag{36}$$

the second summation being taken over all integers d such that $s|d$ and $d \in \mathrm{Div}(p, \vec{n} - \vec{\mathbf{e}}_i)$. $\square$

$\vec{n}$	$\widetilde{\mathcal{K}}^{\diamond}_{\vec{n}}$	$\widetilde{\mathcal{K}}_{\vec{n}}$	$\overline{\mathcal{K}}_{\vec{n}}$
$(7,7)$	226512	17424	17424
$(5,6)$	5292	536	523
$(6,6,7)$	28224	3138	3135
$(4,4,5)$	225	39	36
$(5,6,8)$	10584	1176	1176
$(5,5,5)$	1323	189	189
$(4,6,7)$	1960	248	242
$(5,6,6)$	5488	692	680
$(3,4,4,5)$	50	10	10
$(6,6,6,7)$	21952	2752	2736

$\vec{n}$	$\widetilde{\mathcal{K}}^{\diamond}_{\vec{n}}$	$\widetilde{\mathcal{K}}_{\vec{n}}$	$\overline{\mathcal{K}}_{\vec{n}}$
$(1,3,3)$	1	1	0
$(2,2,3)$	3	1	1
$(1,4,4)$	1	1	0
$(2,3,4)$	6	2	1
$(3,3,3)$	16	4	4
$(3,3,5)$	20	4	4
$(1,3,3,3)$	1	1	0
$(2,2,3,3)$	3	1	1
$(2,3,4,4)$	6	2	1
$(4,4,4,4)$	125	25	25

Table 1: The number of unlabelled m-ary cacti
(rooted, plain, asymmetric) according to their vertex-color distribution.

m	N	$\widetilde{\mathcal{K}}_N^{\bullet_i}$, $i=1,\cdots,m$	$\widetilde{\mathcal{K}}_N^{\diamond}$	$\widetilde{\mathcal{K}}_N$	$\overline{\mathcal{K}}_N$
2	$(1^5 3^2, 2^7)$	(8, 7)	14	1	1
2	$(1^2 2^2 4^1, 1^2 2^4)$	(76, 90)	150	16	14
3	$(1^3 2^3, 1^3 2^3, 1^6 3^1)$	(600, 600, 702)	900	102	99
3	$(1^2 2^1, 1^2 2^1, 1^2 2^1)$	(12, 12, 12)	16	4	4
3	$(4, 1^4, 1^4)$	(1,1,1)	1	1	0
3	$(2^2, 1^2 2, 1^4)$	(1,2,2)	2	1	0
3	$(1^1 3^1, 1^2 2, 1^4)$	(2,3,4)	4	1	1
3	$(1^2 2^2, 1^2 2^2, 1^4 2^1)$	(54, 54, 69)	81	15	12
3	$(1^3 2^1 4^1, 1^3 2^3, 1^7 2^1)$	(600, 720, 960)	1080	120	120
3	$(1^3 2^2, 1^3 2^2, 1^3 2^2)$	(280, 280, 280)	392	56	56
3	$(1^2 3^2, 1^4 2^2, 1^6 2^1)$	(120, 180, 212)	240	32	28
3	$(2^4, 1^4 2^2, 1^6 2^1)$	(20, 30, 36)	40	6	4
3	$(1^4 4^1, 1^4 2^2, 1^4 2^2)$	(252, 300, 300)	400	52	48
3	$(1^2 2^3, 1^4 2^2, 1^4 2^2)$	(504, 600, 600)	800	104	96
4	$(1^4 2^2, 1^4 2^2, 1^4 2^2, 1^6 2^1)$	(6000, 6000, 6000, 7008)	8000	1008	992

Table 2: The number of unlablelled m-ary cacti
(rooted, plain, asymmetric) according to their vertex-degree distributions.

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