Enumeration of planar constellations

MIREILLE BOUSQUET-MÉLOU and GILLES SCHAEFFER
LaBRI, Université Bordeaux 1
351 cours de la Libération
33405 Talence Cedex, FRANCE
bousquet,schaeffe@labri.u-bordeaux.fr

Abstract

The enumeration of transitive ordered factorizations of a given permutation is a combinatorial problem related to singularity theory. Let $n \geqslant 1$, $m \geqslant 2$, and let σ_0 be a permutation of $\mathfrak{S}_n$ having d_i cycles of length i, for $i \geqslant 1$. We prove that the number of m-tuples $(\sigma_1, \ldots, \sigma_m)$ of permutations of $\mathfrak{S}_n$ such that:

- $\sigma_1 \sigma_2 \cdots \sigma_m = \sigma_0$,
- the group generated by $\sigma_1, \ldots, \sigma_m$ acts transitively on $\{1, 2, \ldots, n\}$,
- $\sum_{i=0}^{m} c(\sigma_i) = n(m-1) + 2$, where $c(\sigma_i)$ denotes the number of cycles of σ_i,

is

$$m \frac{[(m-1)n-1]!}{[(m-1)n-c(\sigma_0)+2]!} \prod_{i \geqslant 1} \left[i \binom{mi-1}{i} \right]^{d_i}.$$

A one-to-one correspondence relates these m-tuples to some rooted planar maps, which we call constellations and enumerate via a bijection with some bicolored trees. For $m = 2$, we recover a formula of Tutte for the number of Eulerian maps. The proof extends the method applied in [16] to the latter case, and relies on the idea that maps are conjugacy classes of trees.

Our result might remind the reader of an old theorem of Hurwitz, giving the number of m-tuples of *transpositions* satisfying the above conditions. Indeed, we show that our result implies Hurwitz' theorem.

Résumé

L'énumération des factorisations ordonnées transitives d'une permutation est un problème combinatoire lié à la théorie des singularités. Soient $n \geqslant 1$, $m \geqslant 2$ et soit σ_0 une permutation de $\mathfrak{S}_n$ ayant d_i cycles de longueur i, pour tout $i \geqslant 1$. Nous montrons que le nombre de m-uplets $(\sigma_1, \ldots, \sigma_m)$ de permutations de $\mathfrak{S}_n$ telles que :

- $\sigma_1 \sigma_2 \cdots \sigma_m = \sigma_0$,
- le groupe engendré par $\sigma_1, \ldots, \sigma_m$ agit transitivement sur $\{1, 2, \ldots, n\}$,
- $\sum_{i=0}^{m} c(\sigma_i) = n(m-1) + 2$, où $c(\sigma_i)$ désigne le nombre de cycles de σ_i,

est

$$m \frac{[(m-1)n-1]!}{[(m-1)n-c(\sigma_0)+2]!} \prod_{i \geqslant 1} \left[i \binom{mi-1}{i} \right]^{d_i}.$$

Ces m-uplets sont en bijection avec des cartes planaires enracinées que nous appelons les constellations et que nous dénombrons à l'aide d'arbres bicoloriés. Pour $m = 2$, nous retrouvons une formule due à Tutte pour le nombre de cartes eulériennes. La preuve étend la méthode appliquée dans [16] à ce dernier cas, et s'appuie sur l'idée que les cartes sont des classes de conjugaison d'arbres.

Notre résultat ressemble à un théorème d'Hurwitz, qui donne le nombre de m-uplets de *transpositions* satisfaisant les conditions précédentes. Nous montrons de fait que notre résultat implique celui d'Hurwitz.

1 Introduction

Let σ_0 be a permutation in the symmetric group $\mathfrak{S}_n$. An *ordered factorization* of σ_0 is an m-tuple $(\sigma_1, \dots, \sigma_m)$ of permutations of $\mathfrak{S}_n$ such that $\sigma_1 \sigma_2 \cdots \sigma_m = \sigma_0$.

The enumeration of ordered factorizations of a fixed permutation is a widely studied problem. Its numerous different motivations make it very versatile, and give rise to different kinds of conditions that can be imposed on the factors. Here are some conditions often met in the literature.

- **The cyclic type of the factors.** One can decide that each factor σ_i must be taken inside a prescribed conjugacy class of $\mathfrak{S}_n$: in this case, one is merely trying to compute the *connection coefficients* of the symmetric group. A very general formula can be given in terms of characters [18, p.68]. The *rank* $n - c(\sigma)$ of a permutation σ gives the length of the shortest ordered factorization of σ into transpositions. The rank being clearly sub-additive, we observe that the connection coefficient is zero unless

$$\sum_{i=1}^{m} [n - c(\sigma_i)] \geqslant n - c(\sigma_0),$$

where $c(\sigma_i)$ denotes the number of cycles of σ_i (which only depends on its conjugacy class). Equivalently[1],

$$\sum_{i=1}^{m} c(\sigma_i) \leqslant n(m-1) + c(\sigma_0). \qquad (1)$$

- **The general minimality condition.** One can focus on the extremal case:

$$\sum_{i=1}^{m} c(\sigma_i) = n(m-1) + c(\sigma_0), \qquad (2)$$

which is minimal in terms of the lengths of the factors. This problems amounts to computing the *top connection coefficients* of the symmetric group [6]. The most celebrated result in this field corresponds to the case where all factors are transpositions and σ_0 is an n-cycle. The extremality condition (2) becomes $m = n - 1$, and the number of such factorizations is n^{n-2}, the number of Cayley trees [3, 7, 15].

- **The transitivity condition** requires that the group generated by $\sigma_1, \dots, \sigma_m$ acts transitively on $\{1, 2, \dots, n\}$. This condition finds its origin in the link between ordered factorizations and branched coverings of Riemann surfaces: roughly speaking, the transitivity condition is implied by the connectedness of the surfaces. This condition is widely considered, and will also be adopted in this paper: our factorizations will correspond to branched coverings of the two-dimensional sphere by itself.

- **The transitive minimality condition.** Most importantly, the upper bound on $\sum c(\sigma_i)$ given by (1) is no longer sharp under the transitivity condition. For instance, all transitive factorizations of a permutation σ_0 into m transpositions satisfy the following inequality [4, 19]:

$$m \geqslant n + c(\sigma_0) - 2 \qquad (3)$$

which is stronger than the inequality $m \geqslant n - c(\sigma_0)$ provided by (1). From (3), we easily derive the following inequality, valid for all transitive factorizations of σ_0:

$$\sum_{i=1}^{m} c(\sigma_i) \leqslant n(m-1) - c(\sigma_0) + 2,$$

which is stronger than (1). It can be understood in terms of the genus of the underlying Riemann surfaces.

We shall focus on extremal *transitive* factorizations. The case where all factors are transpositions was solved long time ago by Hurwitz [10] (see also [4, 19]).

[1] Condition (1) is necessary, but not sufficient, for the corresponding connection coefficient to be non zero.

Theorem 1.1 (Hurwitz) *Let $n \geqslant 1$. Let σ_0 be a permutation of $\mathfrak{S}_n$ having d_i cycles of length i, for $i \geqslant 1$. Then the number of m-tuples $(\tau_1, \dots, \tau_m)$ of transpositions of $\mathfrak{S}_n$ such that:*

- $\tau_1\tau_2\cdots\tau_m = \sigma_0$,
- *the group generated by $\tau_1, \dots, \tau_m$ acts transitively on $\{1, 2, \dots, n\}$,*
- $m = n + c(\sigma_0) - 2$, *where $c(\sigma_0)$ denotes the number of cycles of σ_0,*

is

$$H_{\sigma_0} = n^{c(\sigma_0)-3}(n + c(\sigma_0) - 2)! \prod_{i\geqslant 1} \left[\frac{i^i}{(i-1)!}\right]^{d_i}.$$

In this paper, we count extremal transitive factorizations *regardless of the cyclic type of the factors.* Our main theorem follows. We shall see that it implies Hurwitz' theorem.

Theorem 1.2 *Let $n \geqslant 1$. Let σ_0 be a permutation of $\mathfrak{S}_n$ having d_i cycles of length i, for $i \geqslant 1$. For $m \geqslant 0$, let $G_{\sigma_0}(m)$ denote the number of m-tuples $(\sigma_1, \dots, \sigma_m)$ of permutations of $\mathfrak{S}_n$ such that:*

- $\sigma_1\sigma_2\cdots\sigma_m = \sigma_0$,
- *the group generated by $\sigma_1, \dots, \sigma_m$ acts transitively on $\{1, 2, \dots, n\}$,*
- $\sum_{i=0}^{m} c(\sigma_i) = n(m-1) + 2$, *where $c(\sigma_i)$ denotes the number of cycles of σ_i.*

Then for $m \geqslant 2$,

$$G_{\sigma_0}(m) = m\, \frac{[(m-1)n-1]!}{[(m-1)n - c(\sigma_0) + 2]!} \prod_{i\geqslant 1} \left[i\binom{mi-1}{i}\right]^{d_i}.$$

Let us call an ordered factorization *proper* if none of its factors is the identity. The inclusion-exclusion principle implies that the number of proper minimal transitive m-factorizations of σ_0 is, for $n \geqslant 2$ and $m \geqslant 0$,

$$F_{\sigma_0}(m) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} G_{\sigma_0}(k). \tag{4}$$

Observe that a proper factorization $(\sigma_1, \dots, \sigma_m)$ satisfies $\sum_{i=0}^{m} c(\sigma_i) \leqslant c(\sigma_0) + m(n-1)$. If it is also transitive and minimal, then $\sum_{i=0}^{m} c(\sigma_i) = n(m-1)+2$ and thus, $m \leqslant n + c(\sigma_0) - 2$. Moreover, the choice $m = n + c(\sigma_0) - 2$ forces $c(\sigma_i)$ to be $n-1$, for $1 \leqslant i \leqslant m$, so that each factor is a transposition. This shows that the number of minimal transitive factorizations into transpositions, evaluated by Hurwitz, is

$$H_{\sigma_0} = F_{\sigma_0}(d)$$

where $d = n + c(\sigma_0) - 2$.

Theorem 1.2 provides, for each $\sigma_0 \in \mathfrak{S}_n$, an explicit polynomial $P(x) \in \mathbb{Q}[x]$, of degree $d = n + c(\sigma_0) - 2$ such that $G_{\sigma_0}(m) = P(m)$ for all $m \geqslant 0$. Defining the difference operator Δ by $\Delta P(x) = P(x+1) - P(x)$, we can rewrite (4) as follows:

$$F_{\sigma_0}(m) = \Delta^m P(0).$$

Observe that $\Delta^d(x^k) = 0$ if $k < d$ and $\Delta^d(x^d) = d!$. This implies that H_{σ_0} is, up to a factorial, the leading coefficient of $P(x)$:

$$\begin{aligned} H_{\sigma_0} &= F_{\sigma_0}(d) \\ &= \Delta^d P(0) \\ &= d![x^d]P(x) \\ &= (n + c(\sigma_0) - 2)!\, n^{c(\sigma_0)-3} \prod_{i\geqslant 1} \left[\frac{i^i}{(i-1)!}\right]^{d_i}. \end{aligned}$$

This is exactly Hurwitz' theorem.

Many of the enumeration problems mentioned above have an alternative description in terms of trees, maps, or hypermaps. Our theorem is not an exception to this rule: in Section 2, we describe a family of maps, called *constellations*, which are in one-to-one correspondence with minimal transitive factorizations. The rest of the paper focuses on constellations: we first define and enumerate a family of trees (Section 3), then we describe a correspondence between these trees and constellations (Section 4). This correspondence is one-to-one so that we obtain the number of constellations, and hence, of minimal transitive factorizations. The proof that it is indeed one-to-one is omitted due to space limitations (see [1]).

2 Constellations and their relatives

A *planar map* is a 2-cell decomposition of the oriented sphere into vertices (0-cells), edges (1-cells), and faces (2-cells). Loops and multiple edges are allowed. The degree of a vertex (or a face) is the number of edges incident to this vertex. Two maps are *isomorphic* if there exists an orientation preserving homeomorphism of the sphere that maps cells of one of the maps onto cells of the same type of the other map and preserves incidences. We shall consider maps up to isomorphism.

Definition 2.1 *Let $m \geqslant 2$. An m-constellation is a planar map whose faces are coloured black and white in such a way that*

- *all faces adjacent to a given white face are black, and vice-versa,*
- *the degree of any black face is m,*
- *the degree of any white face is a multiple of m.*

A constellation is rooted *if one of its edges, called the root edge, is distinguished.*

The black faces of a constellation will often be called its *polygons* or its *m-gons*. In what follows, we will mainly consider *rooted* constellations, and the word "rooted" will often be omitted. Observe that it is possible to label the vertices of an m-constellation with $1, 2, \ldots, m$ in such a way the vertices of any m-gon are labelled $1, 2, \ldots, m$ in counterclockwise order. We adopt the convention that the ends of the root edge are labelled 1 and 2: this determines the *canonical labelling* of the constellation (Fig. 1).

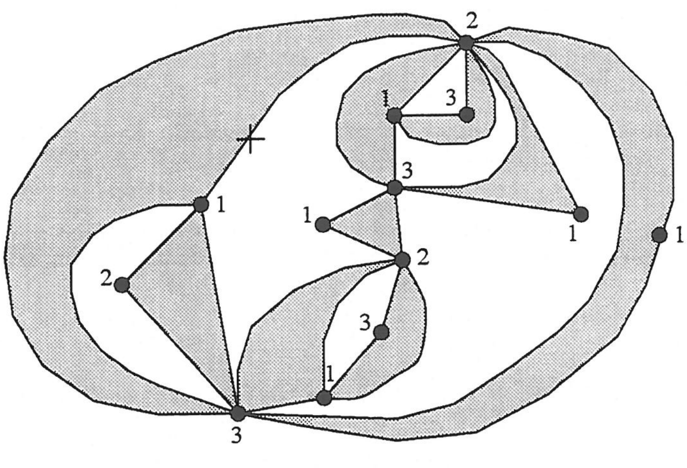

Figure 1: A rooted 3-constellation and its canonical labelling.

One can object that our maps do not look very much like real constellations. The terminology[2], which is due to Alexander Zvonkin, becomes more transparent if we replace each m-gon by an m-star (Fig. 2): we thus obtain a connected set of stars, which is undoubtly a constellation [8].

Our interest in constellations originates in the theory of "dessins d'enfants" (see for instance [12] and references therein). For more details, and an application of our result in this context, see [1].

Proposition 2.2 *Let $n \geqslant 1$ and $m \geqslant 2$. There exists a one-to-one correspondence between m-tuples $(\sigma_1, \ldots, \sigma_m)$ of permutations of $\mathfrak{S}_n$ such that:*

[2]Note that the word "constellation" was formerly used by Jacques with the meaning of "map" [11].

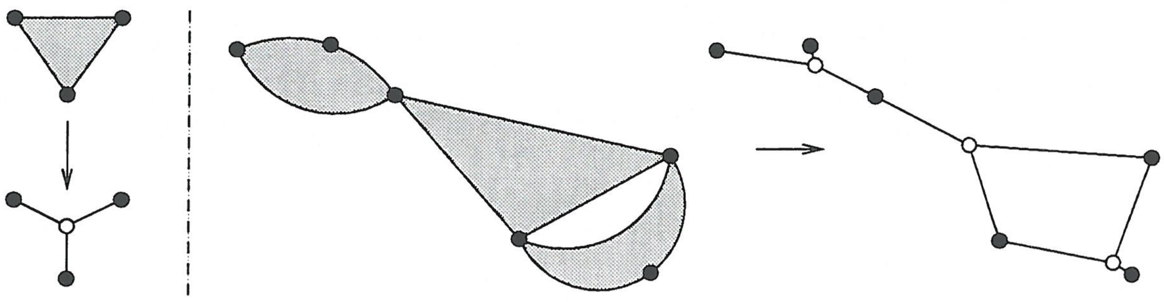

Figure 2: How constellations appear.

- *the group generated by* $\sigma_1, \ldots, \sigma_m$ *acts transitively on* $\{1, 2, \ldots, n\}$,
- $\sum_{i=0}^{m} c(\sigma_i) = n(m-1) + 2$, *where* $\sigma_0 = \sigma_1 \sigma_2 \cdots \sigma_m$,

and rooted m*-constellations formed of* n *polygons, labelled from* 1 *to* n *in such a way the polygon containing the root edge has label* 1. *Moreover, if the constellation has* d_i *white faces of degree* mi*, then* σ_0 *has* d_i *cycles of length* i.

Proof. Let C be a rooted m-constellation formed of n polygons labelled from 1 to n. Recall there is a canonical labelling (by $1, 2, \ldots, m$) of the vertices of C. For $1 \leqslant i \leqslant m$, each m-gon is adjacent to exactly one vertex of label i: hence, turning clockwise around vertices of label i defines a permutation of the n polygons, denoted σ_i, which we identify with a permutation of $\mathfrak{S}_n$.

As the constellation is connected, the group generated by $\sigma_1, \ldots, \sigma_m$ acts transitively on $\{1, 2, \ldots, n\}$.

Moreover, let W be a white face of degree mi: it has exactly i vertices of label m. Let $B_1, B_2, \ldots, B_i$ denote the i black faces adjacent to W by an edge labelled $(1, m)$, arranged in counterclockwise order around W. Then the permutation[3] $\sigma_0 = \sigma_1 \sigma_2 \cdots \sigma_m$ maps B_j onto B_{j+1} for $1 \leqslant j \leqslant i$ (with $B_{i+1} = B_1$). Hence, each cycle of σ_0 corresponds to a white face of C, and the cycle type of σ_0 is given by the degrees of the white faces.

Finally, the number of vertices of C is $v = \sum_{i=1}^{m} c(\sigma_i)$, the number of its faces is $f = n + c(\sigma_0)$ and the number of its edges is $e = nm$. The constellation C is drawn on the sphere, so that Euler's characteristic formula $v + f = e + 2$ reads $\sum_{i=0}^{m} c(\sigma_i) = n(m-1) + 2$.

Conversely, let $(\sigma_1, \ldots, \sigma_m)$ be an m-tuple of permutations as described in the proposition. We consider elementary black m-gons with vertices labelled from 1 to m in counterclockwise order, and white polygons of degree mi for $i \geqslant 1$, the vertices of which are labelled $1, 2, \ldots, m, 1, 2, \ldots, m$, *etc.* in clockwise order. We take n black m-gons, labelled from 1 to n, and $c(\sigma_0)$ white polygons, d_i of which are of degree mi. The m-tuple $(\sigma_1, \ldots, \sigma_m)$ describes an incidence relation on these $n + c(\sigma_0)$ polygons. Following this relation, we glue polygons together by identifying edges. According to general topology theory [14, chap.1], this yields a unique 2-cell decomposition of a compact connected surface without boundary. The condition $\sum_{i=0}^{m} c(\sigma_i) = (m-1)n + 2$ ensures, via Euler's characteristic formula, that this surface is the sphere, and hence that the map we have obtained is a planar constellation.

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Example. For the labelled rooted 3-constellation C of Fig. 3, we find $\sigma_1 = (1)(2,3)$, $\sigma_2 = (1,2,3)$ and $\sigma_3 = (1,3)(2)$. We compute $\sigma_0 = \sigma_1 \sigma_2 \sigma_3 = (1)(2)(3)$ which fits with the fact that C has three white faces, each of degree 3.

As the m-gons of a rooted constellation formed of n polygons can be labelled in $(n-1)!$ different ways and $n!/\prod_{i \geqslant 1} [i^{d_i} d_i!]$ permutations have exactly d_i cycles of length i, Proposition 2.2 implies the equivalence between Theorem 1.2 and Theorem 2.3 below, on which we shall focus from now on.

Theorem 2.3 *Let* $m \geqslant 2$. *The number of rooted* m*-constellations* C *having* d_i *white faces of degree* mi*, for* $i \geqslant 1$*, is*

$$m(m-1)^{f-1} \frac{[(m-1)n]!}{[(m-1)n - f + 2]!} \prod_{i \geqslant 1} \frac{1}{d_i!} \binom{mi-1}{i-1}^{d_i},$$

[3] We multiply permutations from right to left, as we compose functions.

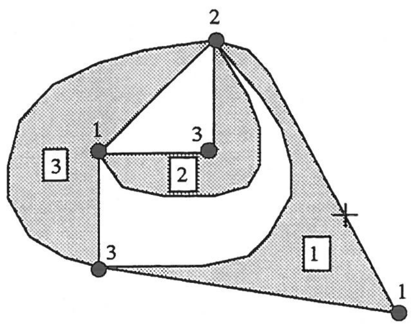

Figure 3: A 3-constellation with labelled 3-gons.

where $n = \sum i d_i$ is the number of m-gons, and $f = \sum d_i$ the number of white faces of C.

We can derive right now two interesting corollaries.

Corollary 2.4 *Let $n \geqslant 1$ and $m \geqslant 2$. The number of rooted m-constellations formed of n polygons is*

$$C_m(n) = \frac{(m+1)m^{n-1}}{[(m-1)n+2][(m-1)n+1]} \binom{mn}{n}.$$

Proof. There is a simple one-to-one correspondence, which preserves the number of polygons, between m-constellations and $(m+1)$-constellations whose white faces have degree $m+1$. Our result will thus follow from Theorem 2.3, by replacing m by $m+1$ and setting $d_1 = n$, $d_i = 0$ for $i \geqslant 2$.

To describe this correspondence, we use once again the canonical labelling of the vertices. We add at the center of each white face a new vertex labelled $m+1$, and pull the center of each edge $(m, 1)$ of the face so that it coincides with the new vertex (Fig. 4). We obtain an $(m+1)$-constellation whose white faces have degree $m+1$ (as each of them contains exactly one vertex labelled $m+1$), and the construction is clearly reversible.

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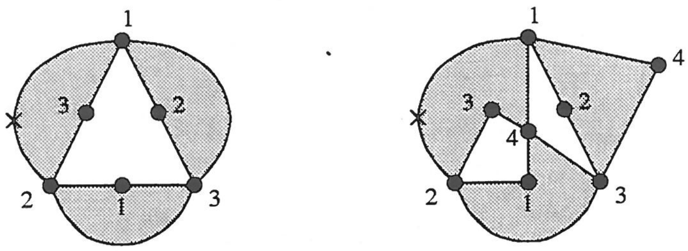

Figure 4: From a 3-constellation to a 4-constellation with all faces of degree 4.

Dual maps[4] of constellations will be called *m-Eulerian maps*. The definition of constellations provides the following characterization for m-Eulerian maps (Fig. 5).

Definition 2.5 *A planar map is m-*Eulerian *if it is bipartite (with black and white vertices), and*

- *the degree of any black vertex is m,*
- *the degree of any white vertex is a multiple of m.*

The case $m = 2$ justifies our terminology: if we remove all black vertices from a 2-Eulerian map, we obtain a map having only vertices of even degree; such maps are usually called *Eulerian*.

Of course, counting m-Eulerian maps is equivalent to counting m-constellations. In particular, Theorem 2.3 gives the number of rooted m-Eulerian maps having d_i white vertices of degree mi, for $i \geqslant 1$. When $m = 2$, we recover an old result of Tutte [2, 16, 20].

[4] Recall that the dual map C^* of a map C describes the incidence relation between the faces of C: in particular, the vertices (resp. faces) of C^* are the faces (resp. vertices) of C.

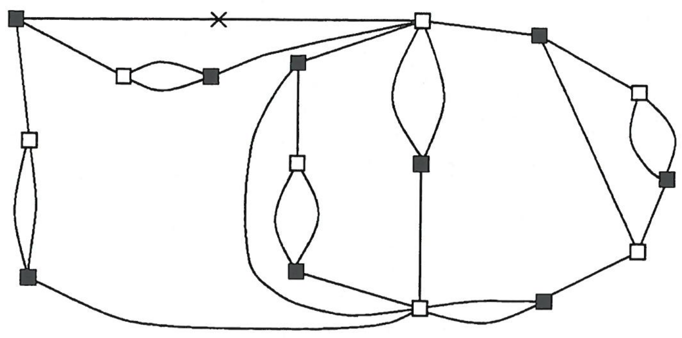

Figure 5: A rooted 3-Eulerian map, dual of the 3-constellation of Fig. 1.

3 Eulerian trees

A *planted tree* is a plane tree with a marked leaf (also called the *root*). In our figures, planted trees hang from their marked leaves. The (total) *degree* of a vertex is the degree in the context of graph theory, *i.e.*, one more than the arity in the functional representation of trees. Vertices of degree 1 are referred to as *leaves*, the others as *inner vertices*. The *inner degree* of a vertex is the number of inner vertices adjacent to it. The *depth* of a vertex is its distance to the root. The *left-to-right prefix order* (lr-prefix for short) on the vertices of a planted tree T is obtained recursively by visiting first the root of T, and then its subtrees $T_1, \dots, T_k$, taken from left to right, in lr-prefix order. The right-to-left prefix (rl-prefix) order is defined symmetrically. The number of planted trees having $n+1$ edges is the famous Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. More generally, the Lagrange inversion formula (see [5] for instance) or encodings by Lukaciewicz words [13, p.221] give the following classical result, first proved by Harary, Prins and Tutte [9].

Theorem 3.1 *The number of planted plane trees having d_i inner vertices of degree $i+1$ for $i \geqslant 1$, is*

$$\frac{(e-1)!}{(\ell-1)!} \prod_{i\geqslant 1} \frac{1}{d_i!},$$

where $e = 1 + \sum i d_i$ is the number of edges and $\ell = 2 + \sum (i-1) d_i$ the number of leaves of such trees.

Definition 3.2 *A bicolored (black and white) tree, planted at a black leaf, is said to be m-*Eulerian *if*

- *all neighbors of a white vertex are black, and vice-versa,*
- *all inner black vertices have total degree m and inner degree 1 or 2,*
- *all inner white vertices have total degree mi, for some $i \geqslant 1$, and have exactly $i-1$ inner neighbors of inner degree 1.*

Figure 6 shows a 3-Eulerian tree (plain lines).

Proposition 3.3 *Let $m \geqslant 2$. The number of m-Eulerian trees having d_i white vertices of degree mi, for $i \geqslant 1$, is*

$$(m-1)^{f-1} \frac{[(m-1)n]!}{[(m-1)n - f + 1]!} \prod_{i\geqslant 1} \frac{1}{d_i!} \binom{mi-1}{i-1}^{d_i},$$

where $n = \sum i d_i$ and $f = \sum d_i$. Such trees have exactly:

- *f inner white vertices,*
- *$n-1$ inner black vertices,*
- *$(m-1)n - f - m + 2$ white leaves,*
- *$(m-1)n - f + 2$ black leaves.*

Proof. We can construct all m-Eulerian trees having d_i white vertices of degree mi as follows.
1. We start with a planted tree T_1 having white inner vertices and black leaves, such that all vertices have degree 1 modulo $m-1$. More precisely, let d_i be the number of (inner) vertices having degree $(m-1)i+1$, for $i \geqslant 1$. According to Theorem 3.1, the number of such trees is

$$T(d_1, d_2, \ldots) = \frac{[(m-1)n]!}{[(m-1)n-f+1]!} \prod_{i \geqslant 1} \frac{1}{d_i!}.$$

2. In the middle of each inner edge of T_1, add a black vertex of total degree m. This vertex has $m-2$ white leaves, which can be displayed in $m-1$ different ways. As T_1 has $f-1$ inner edges, the number of trees T_2 thus obtained is $(m-1)^{f-1}T(d_1, d_2, \ldots)$.
3. To each of the d_i white vertices of T_2 of degree $(m-1)i+1$, add $i-1$ black children of total degree m. The position of these children can be chosen in $\binom{mi-1}{i-1}$ different ways, and this observation concludes the proof.

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Let T be an m-Eulerian tree. Let us arrange its leaves cyclically by reading them in lr-prefix order. For the tree of Fig. 6, starting from the root we obtain the (cyclic) word *bwbbwbbwbbwwwbbbwbbwwbw*, where b (resp. w) denotes a black (resp. white) leaf. We now match the letters w and b of this word as if they were respectively opening and closing brackets:

$$b\ w\ b\ b\ w\ b\ b\ w\ b\ b\ w\ w\ w\ b\ b\ b\ w\ b\ b\ w\ w\ b\ w$$

More precisely, at step 1, each letter w that is followed by a b is matched with this occurrence of b. We then forget all matched letters and repeat the procedure until no more matches are possible. We match accordingly the leaves of T (Fig. 6). As there are more black leaves than white leaves, some black leaves — exactly m of them — remain unmatched: we call them *single*.

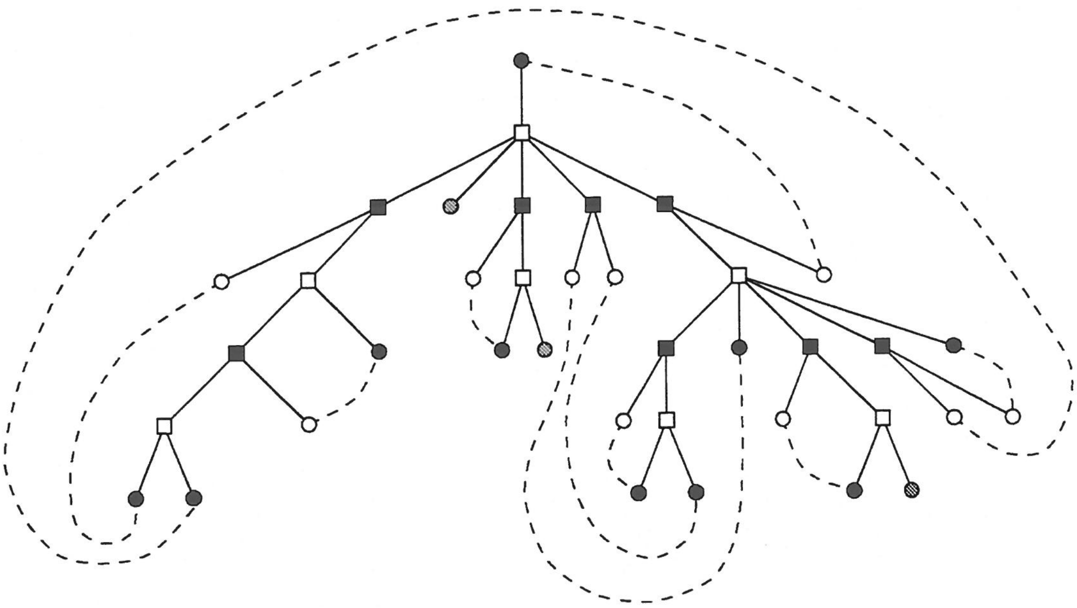

Figure 6: Matching the leaves of a 3-Eulerian tree (circles represent leaves, squares represent inner vertices).

Definition 3.4 *An m-Eulerian tree is said to be balanced if its root remains single after the matching procedure.*

Proposition 3.5 *Let $m \geqslant 2$. The number of balanced m-Eulerian trees having d_i white vertices of degree mi for $i \geqslant 1$ is*

$$m(m-1)^{f-1}\frac{[(m-1)n]!}{[(m-1)n-f+2]!}\prod_{i\geqslant 1}\frac{1}{d_i!}\binom{mi-1}{i-1}^{d_i},$$

where $n = \sum i d_i$ and $f = \sum d_i$.

Proof. Let A denote the number given by Proposition 3.3. Then mA can be understood either as the number of m-Eulerian trees having a single leaf distinguished, or, by planting the tree at this leaf, as the number of balanced m-Eulerian trees having a black leaf distinguished. As an m-Eulerian tree having d_i white vertices of degree mi has $(m-1)n - f + 2$ black leaves, the proposition follows.

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Observe that the expressions given in Theorem 2.3 and Proposition 3.5 are identical: hence Theorem 2.3 will follow from Proposition 3.5 via a one-to-one correspondence between balanced Eulerian trees and constellations.

4 The bijection between balanced Eulerian trees and constellations

4.1 From trees to constellations: the transformation Φ

The transformation of a balanced Eulerian tree T into a constellation $C = \Phi(T)$ is easy to describe. Actually, most of the work has been done already. The construction is exemplified on Fig. 7.

We form a first planar map E_1 by adding edges between the matched leaves of T. We thus obtain the dashed edges of Fig. 7*a*. Exactly m black leaves remain single. By construction, all of them lie in the same face of E_1; in what follows, we shall often consider E_1 as map on the plane (rather than on the sphere) by taking the convention that the single leaves lie in the infinite face.

We add in the infinite face of E_1 an extra star, having a black center and m rays. Each ray ends with a white leaf. We match these m white leaves with the m single vertices of the tree (dotted lines in Fig. 7*a*) in cyclic order to obtain a planar map. We mark the dotted edge that ends at the root of the tree. We finally erase all leaves of the underlying tree T and replace dashed and dotted lines by plain lines. By construction, the map we have obtained is a rooted m-Eulerian map E. Taking the dual of E gives a constellation C which we define to be $\Phi(T)$.

Observe that the Eulerian map associated with the tree of Fig. 7 is the map of Fig. 5 and that its dual is the constellation of Fig. 1.

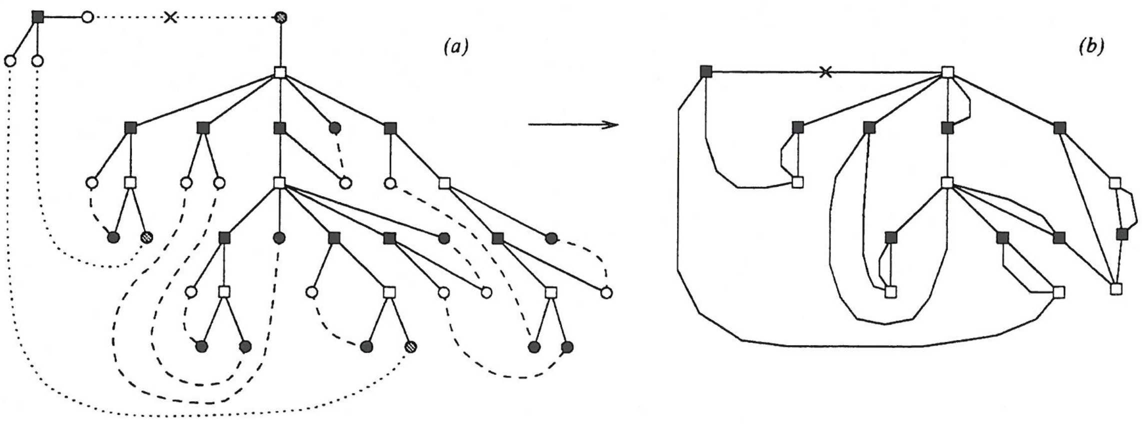

Figure 7: From a balanced 3-Eulerian tree to a 3-Eulerian map.

We wish to prove that the transformation Φ is a bijection between balanced m-Eulerian trees and m-constellations. What can the reverse bijection be? Imagine we start with a rooted m-constellation

C (or its dual map E, which is m-Eulerian) and try to construct the corresponding m-Eulerian tree T. What we need to do is select – in a clever way – a set S of edges of E, add two vertices on each of them in such a way the resulting map remains bicolored, and then delete the part of the edge that links these two vertices; this must yield two connected components: an m-star and a balanced m-Eulerian tree. Thus, the central difficulty of the reverse bijection consists in describing the set S of edges of E we need to *open*.

Let us consider again the Eulerian map of Fig. 7*b*. Looking at Fig. 7*a* tells us what the set S has to be. Let us draw the set of dual edges, denoted S' (Fig. 8, thick lines). We observe that S' is formed of the root m-gon of the constellation C, on which m trees, denoted $T_1, \dots, T_m$ are planted. These m trees cover all vertices of C. We shall see that this is a general phenomenon: describing the reverse bijection of Φ boils down to defining a certain covering forest of a constellation, which will be called its *rank forest.*

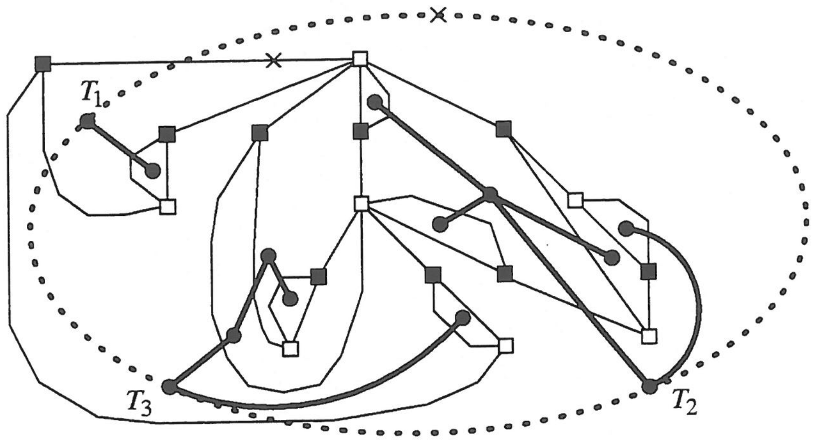

Figure 8: The dual edges of the dashed and dotted edges of Fig. 7*a*.

4.2 From constellations to trees: the transformation Ψ

Let C be a rooted constellation; let us draw it on the plane in such a way the infinite face is the root m-gon. Let $\bar{C}$ be obtained by orienting the edges of C in clockwise direction around white faces. We define the *rank* $r(v)$ of a vertex v as the length of the shortest (oriented) path of $\bar{C}$ going from a vertex of the root m-gon to v (Fig. 9). The rank of v should not be mixed up with its label $\ell(v) \in \{1, 2, \dots, m\}$, given by the canonical labelling defined in Section 2. The following lemma tells us how to construct the *rank forest* of a constellation. The principle is simple: we start from the root m-gon and proceed by breadth first search, from right to left. The reader is advised to practice on the example of Fig. 9.

Lemma 4.1 *Let C be a rooted constellation. There exists a unique covering forest F of C, consisting of m trees T_a, $1 \leqslant a \leqslant m$, respectively planted at the vertex labelled a of the root polygon, that satisfies the following four properties.*

1. *The orientation of edges of F induced by the trees T_a (from the roots to the leaves) coincides with their orientation in the oriented map $\bar{C}$.*

2. *The rank increases by one along each edge of F. In other words, the depth of a vertex of T_a is given by its rank.*

Let u be a vertex of C. Properties 1 and 2 imply that u belongs to T_a, where $a = \ell(u) - r(u)$ mod m.

3. *Assume $r(u) > 0$. All the vertices of label $\ell(u) - 1$ and rank $r(u) - 1$ occur in the same tree T_a, where $a = \ell(u) - r(u)$ mod m. If we visit them in rl-prefix order, the first one that is adjacent to u is the father of u in T_a.*

4. *Let v be the father of u in T_a. Let e be the edge of T_a that links v to its father. If we visit the edges of C adjacent to v in clockwise order, starting from e, the first one that ends at u belongs to T_a.*

This covering forest will be called the rank forest *of C.*

Proof. We construct F inductively, adding at step k all vertices of rank k (Fig. 9). At step 0, for $1 \leqslant a \leqslant m$, the tree T_a is reduced to the vertex labelled a that belongs to the root polygon. We plant T_a by attaching to this vertex a short extra edge that lies in the infinite face of C.

Assume that, after step k, the forest we have obtained is not yet covering C. Let u be a vertex of rank $k+1$. All vertices of rank k and label $\ell(u)-1$ belong to the same tree T_a. We choose the father v of u according to Property (3) of our lemma, and the edge of T_a joining v to u according to Property (4). ■

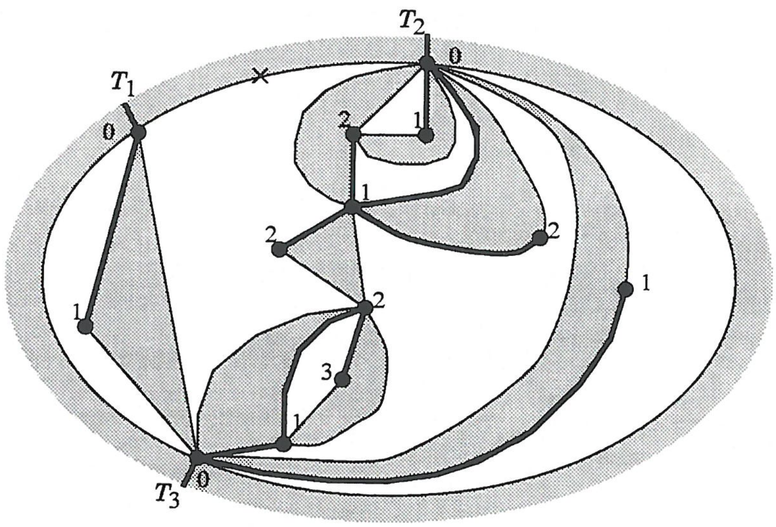

Figure 9: A rooted 3-constellation: the ranks of the vertices and the rank forest.

Once the rank forest of C is constructed, the Eulerian tree $\Psi(C)$ is easy to obtain. Let S' be the set of edges of C that belong either to the rank forest, or to the root m-gon. Let E be the dual map of C, and S be the dual set of S'. On each edge e of S, we add two vertices in such a way the resulting map remains bicolored; we then delete the part of e that links these two vertices. We claim that this provides an m-star and a balanced m-Eulerian tree, which we plant by the root edge of E.

Example. Starting from the constellation of Fig. 9, we obtain the tree of Fig. 7a.

The two constructions Φ and Ψ we have described achieve our main objective: giving a one-to-one correspondence between balanced m-Eulerian trees and m-Eulerian maps.

Theorem 4.2 *The transformation Φ is a bijection from balanced m-Eulerian trees to m-constellations. The reverse bijection is Ψ. Moreover, if $\Phi(T) = C$ and T has d_i white vertices of degree mi, then C has d_i white faces of degree mi.*

This result is far from immediate. Its proof is given in [1].

Our bijection illustrates a general idea that is developed in [17]: *natural families of rooted planar maps are canonical representatives of conjugacy classes of planted plane trees.* Here, we say that two trees are *conjugated* if one is obtained from the other by changing the root. This implies that conjugacy classes are simply plane trees, but the terminology originates in the analogy with words. Indeed, conjugating a tree results in conjugating the word obtained by a prefix ordering of its leaves, so that Proposition 3.5 can be seen as an application of Raney's theorem. The motto of [17] could then be stated: *if applying Raney's theorem to words yields trees, applying it to planted trees yields maps.* Besides constellations, planar maps, Eulerian planar maps, nonseparable planar maps and cubic nonseparable maps can indeed be obtained from suitable balanced trees by some matching procedure very similar to Φ.

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