The k-consecutive Arrangements

by

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Summary

We introduce a family of subspace arrangements in $\mathbb{R}^n$ called the *k-consecutive arrangements*. They are associated with certain Coxeter groups and related to the k-equal arrangements. For example, in the type A case the subspaces are those of the form

$$x_i = x_{i+1} = \ldots = x_{i+k-1}, \quad 1 \leq i \leq n-k+1.$$

We study the intersection lattices of these arrangements and show how the methods of NBB bases and of finite fields can be used to combinatorially explain their Möbius functions and characteristic polynomials, respectively.

Résumé

Nous definissons une famile d'arrangements des souséspaces en $\mathbb{R}^n$ que nous appelons les *arrangements k-consecutifs*. Ils sont associés aux groupes de Coxeter et reliés aux arrangements k-egals. Par exemple, dans le cas de type A, nous avons les souséspaces de la forme

$$x_i = x_{i+1} = \ldots = x_{i+k-1}, \quad 1 \leq i \leq n-k+1.$$

Nous étudions les treillis d'intersection de ces arrangements et nous démontrons comment les méthodes des bases NBB et des champs finis peuvent éxpliquer dans une manière combinatoire leurs fonctions Möbius et polynômes caractéristiques, respectivement.

1 Introduction

The k-equal subspace arrangements of type A were introduced by Björner, Lovász and Yao [4, 5] motivated by a problem in computational complexity. Computing the Möbius function of the intersection lattice was crucial for obtaining a lower bound on the complexity. Since then there has been a flurry of activity studying these arrangements and their analogs for the other infinite families of Coxeter groups [6, 7, 9, 14, 22, 23].

The purpose of this paper is to introduce a new family of subspace arrangements, the k-consecutive arrangements, which are closely related to the k-equal ones. Their intersection lattices

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are also interesting from a combinatorial standpoint. We show how the theory of NBB bases developed by Blass and Sagan [10] can be used to combinatorially explain the associated Möbius functions. The finite field method of Athanasiadis [1] is used to compute the characteristic polynomials.

The rest of this work is organized as follows. In Sections 2, 3, and 4 we discuss the coordinate, type A, and type B cases, respectively (type D is similar to the last). We consider the related k-circular arrangements in Section 5. Finally we end with some comments and open problems.

But first let us review some basic definitions that will be needed in the sequel. By an *arrangement* we will mean a finite set

$$\mathcal{A} = \{K_1, K_2, \ldots, K_m\}$$

of linear subspaces of $\mathbb{R}^n$. The corresponding *intersection lattice* is the set

$$L(\mathcal{A}) = \{X \mid X \text{ is an intersection of some of the } K_i\}$$

partially ordered by *reverse* containment, i.e., $X \leq Y$ if and only if $X \supseteq Y$. So $L(\mathcal{A})$ has as unique minimal element $\mathbb{R}^n$ and as unique maximal element $\cap_{i=1}^m K_i$. In an arbitrary finite lattice, L, these two elements are denoted $\hat{0}$ and $\hat{1}$, respectively. Furthermore let $\vee$ stand for the join (least upper bound) operation and $\wedge$ stand for the meet (greatest lower bound) in L. We will also be interested in the atom set of L, $A = A(L)$, which consists of all elements covering $\hat{0}$. Note that the Latin letter A is used for the atom set while a script $\mathcal{A}$ is used for an arrangement. In the case $L = L(\mathcal{A})$, the atoms are just the subspaces if there are no containments among them.

Associated with any partially ordered set P having a $\hat{0}$ is its *Möbius function*, $\mu : P \to \mathbb{Z}$, defined recursively by

$$\sum_{y \leq x} \mu(y) = \delta_{x,\hat{0}} \tag{1}$$

where $\delta_{x,\hat{0}}$ is the Kronecker delta. If P also has a $\hat{1}$ we will often write $\mu(P)$ for $\mu(\hat{1})$. For the intersection lattice of an arrangement we also have the *characteristic polynomial* which is a polynomial in t given by

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

More about arrangements can be found in the book of Orlik and Terao [15] or the article of Björner [3]. Good sources for general information about Möbius functions and characteristic polynomials are Stanley's text [21] or Sagan's paper [19].

2 The coordinate case

Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. For $i, j, n \in \mathbb{Z}_{\geq 0}$ we will use the notation

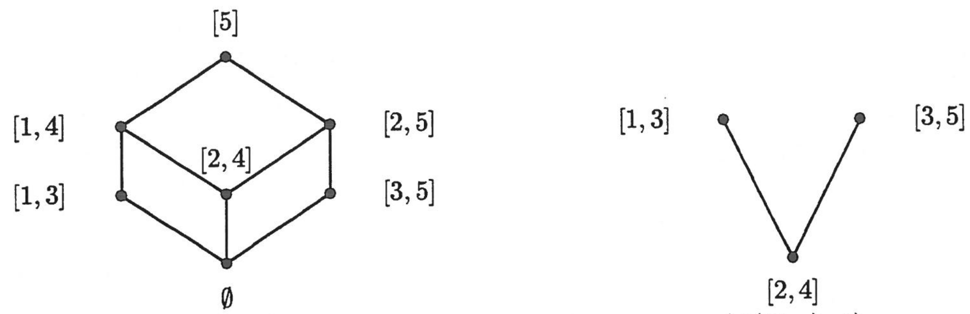
$$[n] = \{1, 2, \ldots, n\} \quad \text{and} \quad [i, j] = \{i, i+1, \ldots, j\}.$$

Define the *k-consecutive coordinate arrangement*, $\mathcal{K}_{n:k}$, to be the set of all subspaces in $\mathbb{R}^n$ of the form

$$x_i = x_{i+1} = \ldots = x_{i+k-1} = 0, \qquad 1 \leq i \leq n-k+1.$$

It is easy to see that the intersection lattice $L(\mathcal{K}_{n:k})$ is isomorphic to the poset $B_{n:k}$ generated by taking joins of intervals

$$[i, i+k-1], \qquad 1 \leq i \leq n-k+1$$



$(B_{5:3}, \leq)$ $(A(B_{5:3}), \trianglelefteq)$

Figure 1: $B_{5:3}$ and a partial order on its atoms

in the Boolean algebra of subsets of $[n]$, where a subset corresponds to the set of indices of variables set equal to zero. As an example, we have drawn the Hasse diagram of $B_{5:3}$ on the left in Figure 1.

Greene [13] first calculated the Möbius function of the $B_{n:k}$ using algebraic techniques. We will show how μ can be derived combinatorially by using the theory of NBB bases [10], which we now review. This is an extension to an arbitrary lattice of the concept of an NBC base in a geometric one [18]. In fact part of the motivation for this generalization was the current interest in subspace arrangements which, unlike arrangements of hyperplanes, can fail to have $L(\mathcal{A})$ geometric.

Given a lattice $(L, \leq)$, put an arbitrary partial order $\trianglelefteq$ on the atoms $A(L)$. Note that $\leq$ will be the partial order in L while $\trianglelefteq$ will be the one in $A(L)$. The latter can be anything from the total incomparability order induced by $\leq$ to a total order as would be used in the NBC case. We say that $D \subseteq A(L)$ is *bounded below (BB)* if, for every $d \in D$ there is an $a \in A(L)$ such that

1. $a \triangleleft d$, and
2. $a < \bigvee D$.

By way of illustration, let us go back to $B_{5:3}$ and put the partial order shown in Figure 1 on $A(B_{5:3})$. From the first condition of this definition, it is clear that if D contains a minimal element of $\trianglelefteq$ then it can not be BB. Also from the second requirement we see that if $|D| \leq 1$ then again D is not BB. So in our example, the only possible BB set is $D = \{[1,3], [3,5]\}$. It is easy to verify that it is indeed BB since

1. $[2,4] \triangleleft [1,3]$, $[2,4] \triangleleft [3,5]$, and
2. $[2,4] \triangleleft [1,3] \vee [2,4] = [5]$.

Now say that $B \subseteq A(L)$ is an *NBB base of* $x \in L$ if $\bigvee B = x$ and B does *not* contain any D which is BB. The main result about NBB bases is as follows.

Theorem 2.1 **([10])** *Let L be any finite lattice and let $\trianglelefteq$ be any partial order on $A(L)$. Then for all $x \in L$*

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x. ■

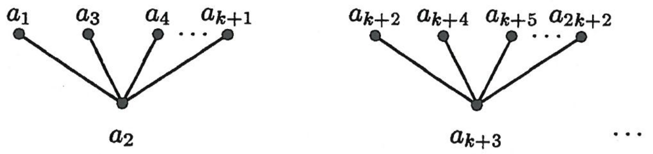


Figure 2: The order $\trianglelefteq$ for $B_{n:k}$

Returning again to our example, we see that $\mu([1,4]) = (-1)^2 = 1$ since $[1,4]$ has exactly one NBB base, namely $\{[1,3],[2,4]\}$. However $\mu([5]) = 0$ since $[5]$ has no NBB bases: If B were such a base then we would have to have $[1,3] \in B$ since this is the only atom containing the element 1. Dually we are forced to have $[3,5] \in B$ and so B must contain our forbidden BB set. It is easy to verify that these μ values are correct directly from the definition (1).

Proposition 2.2 ([13]) *In $B_{n:k}$ we have*

$$\mu([n]) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{k+1}, \\ -1 & \text{if } n \equiv -1 \pmod{k+1}, \\ 0 & \text{else.} \end{cases}$$

Proof. Let the atoms of $B_{n:k}$ be $a_1, \ldots, a_{n-k+1}$ where $a_i = [i, i+k-1]$. Define $\trianglelefteq$ as in Figure 2.

Let B be an NBB base of $[n]$ if it exists. Then $a_1 \in B$ since a_1 is the only atom containing 1. So none of $a_3, \ldots, a_{k+1}$ is in B since any of these atoms forms a BB set with a_1. The only available atom remaining which contains $k+1$ is a_2, forcing $a_2 \in B$. Iterating this argument we find that if B exists then it must be unique and

$$B = \{a_1, a_2, a_{k+2}, a_{k+3}, \ldots\}.$$

If $n \equiv 0$ or $-1 \pmod{k+1}$ then $\bigvee B = [n]$ so we have a base of even or odd cardinality, respectively. Otherwise $a_{n-k+1} \notin B$ and since this is the only atom containing n, B does not join to $[n]$. ■

We will now use the finite field method to calculate the characteristic polynomial $\chi(\mathcal{K}_{n:k}, t)$. This was not considered by Greene or others studying $B_{n:k} \cong L(\mathcal{K}_{n:k})$ because the poset is not ranked and so one needs to consider it as an intersection lattice to obtain the necessary powers of t.

The finite field method was developed by Athanasiadis [1] in an effort to generalize a theorem of Blass and Sagan [11]. In fact, it is based on an older result of Crapo and Rota [12] which was also rediscovered by Terao [24], but Athanasiadis was the first one to realize the theorem's power and apply it systematically. The basic idea is that one considers $\mathcal{A}$ as an arrangement in the vector space $\mathbb{F}_p^n$, where $\mathbb{F}_p$ is the Galois field with p elements and p is a large prime. Then to evaluate $\chi(\mathcal{A}, t)$ at $t = p$ one just counts the points of $\mathbb{F}_p^n$ that are not contained in any subspace of $\mathcal{A}$. For the precise statement of the result, let $|\cdot|$ denote cardinality.

Theorem 2.3 ([1, 12, 24]) *If $\mathcal{A}$ is a subspace arrangement in $\mathbb{R}^n$ defined over $\mathbb{Z}$ and hence over $\mathbb{F}_p$, then for large enough primes p*

$$\chi(\mathcal{A}, p) = |\mathbb{F}_p^n \setminus \bigcup \mathcal{A}|. \quad ■$$

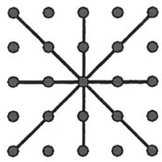


Figure 3: The arrangement $\mathcal{B}_2$ in $\mathbb{F}_5^2$

For example, consider the B_2 Coxeter arrangement which consists of the hyperplanes

$$\mathcal{B}_2 = \{x_1 = 0,\ x_2 = 0,\ x_1 = x_2,\ x_1 = -x_2\}.$$

Viewing $\mathcal{B}_2$ as an arrangement in $\mathbb{F}_5^2$, where we use $\mathbb{F}_5 = \{-2, -1, 0, 1, 2\}$, we get the picture in Figure 3. So, by inspection, the number of points on none of the hyperplanes is $|\mathbb{F}_5^2 \setminus \bigcup \mathcal{B}_2| = 8$. On the other hand, it is well known that $\chi(\mathcal{B}_2, t) = (t-1)(t-3)$. So $\chi(\mathcal{B}_2, 5) = 4 \cdot 2 = 8$, agreeing with the previous count.

To apply the method to our current situation we need to define some constants related to the binomial coefficients, namely

$$\binom{n}{i}_k := \#\text{ of } S \subseteq [n],\ |S| = i, \text{ no } k \text{ consecutive}.$$

For example, $\binom{6}{3}_2 = 4$ counting

$$\{1,3,5\};\ \{1,3,6\};\ \{1,4,6\};\ \{2,4,6\}.$$

Proposition 2.4 *The characteristic polynomial of $\mathcal{K}_{n:k}$ is*

$$\chi(\mathcal{K}_{n:k}, t) = \sum_i \binom{n}{i}_k (t-1)^{n-i}. \tag{2}$$

Furthermore, we have the divisibility relation

$$(t-1)^{\lfloor n/k \rfloor} \mid \chi(\mathcal{K}_{n:t}, t). \tag{3}$$

Proof. Let p be a sufficiently large prime and consider a point $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_p^n \setminus \bigcup \mathcal{K}_{n:k}$. If i of the coordinates are to be zero, there are $\binom{n}{i}_k$ ways to pick them. Then the remaining $n - i$ nonzero coordinates can be chosen in a total of $(p-1)^{n-i}$ ways. Summing on i, we see that equation (2) holds for an infinite number of values $t = p$ and so must be true for general t.

A largest subset of $[n]$ with no k consecutive is

$$[n] \setminus \{k, 2k, 3k, \ldots\}.$$

So $\binom{n}{i}_k = 0$ if $n - i < \lfloor n/k \rfloor$. Plugging this into the characteristic polynomial gives (3). ■

3 The type A case

Define the *k-consecutive arrangement of type A*, $\mathcal{A}_{n:k}$, to be all subspaces of $\mathbb{R}^n$ of the form

$$x_i = x_{i+1} = \ldots = x_{i+k-1}, \qquad 1 \le i \le n - k + 1.$$

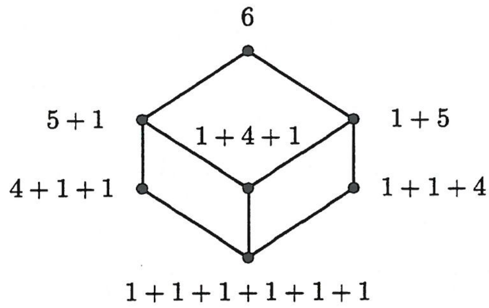


Figure 4: The lattice $C_{6:4}$

The intersection lattice $L(\mathcal{A}_{n:k})$ is isomorphic to the poset $C_{n:k}$ generated by taking joins of compositions (ordered partitions)

$$k+1+\cdots+1, 1+k+1+\cdots+1, \ldots, 1+\cdots+1+k.$$

in the poset of all compositions of n ordered by refinement. As an example, we have drawn $C_{6:4}$ in Figure 4. The reader will note a resemblance to $B_{5:3}$. This is not an accident as we will now see.

Define a lattice

$$B^*_{n:k} = \{[n] \setminus S \ : \ S \in B_{n:k}\}$$

where the elements are ordered by *reverse* inclusion. So $B^*_{n:k}$ is just a relabeling of $B_{n:k}$ with relabeling isomorphism $\alpha : B_{n:k} \to B^*_{n:k}$ where

$$\alpha(S) = [n] \setminus S.$$

If $S \subseteq [n]$ then we write $S = \{n_1, \ldots, n_l\}_<$ to mean that the elements of S are listed in increasing order. Now define $\beta : B^*_{n:k} \to C_{n+1:k+1}$ by

$$\beta(\{n_1, \ldots, n_l\}_<) = c_1 + c_2 + \cdots + c_{l+1}$$

where $c_i = n_i - n_{i-1}$ and by definition $n_0 = 0, n_{l+1} = n + 1$. It is easy to check that β is also a well-defined lattice isomorphism. So by composing these two maps

$$C_{n+1:k+1} \cong B_{n:k}.$$

Because of this isomorphism we can immediately write down the Möbius function and characteristic polynomial of $L(\mathcal{A}_{n:k})$.

Proposition 3.1 *The Möbius function of $C_{n:k}$ is*

$$\mu(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{k}, \\ -1 & \text{if } n \equiv 0 \pmod{k}, \\ 0 & \text{else.} \end{cases}$$

The characteristic polynomial of $\mathcal{A}_{n:k}$ is

$$\chi(\mathcal{A}_{n:k}, t) = t\chi(\mathcal{K}_{n-1:k-1}, t) = \sum_i \binom{n-1}{i}_{k-1} t(t-1)^{n-i-1}. \quad \blacksquare$$

Using the definition of $\mathcal{A}_{n:k}$ permits us to write down another expression for χ, this time in terms of the falling factorial basis for the polynomial algebra. We will use the notation

$$\langle t\rangle_i := t(t-1)(t-2)\cdots(t-i+1).$$

We also need to define a deformation of the Stirling numbers of the second kind parallel to the ones considered for the binomial coefficients. In particular, let

$$\begin{aligned} S_k(n:i) \quad := \quad & \#\text{ of partitions } B_1/\ldots/B_i \text{ of } [n] \text{ with} \\ & \text{no } B_j \text{ containing } k \text{ consecutive integers.} \end{aligned}$$

For example, $S_3(4:2) = 5$ counting

$$1,2/3,4;\ 1,3/2,4;\ 1,4/2,3;\ 1,2,4/3;\ 1,3,4/2.$$

Proposition 3.2 *The characteristic polynomial of $\mathcal{A}_{n:k}$ can be written*

$$\chi(\mathcal{A}_{n:k}, t) = \sum_i S_k(n:i)\langle t\rangle_i.$$

Proof. Let $p >> 0$. Any $\mathbf{x} \in \mathbb{F}_p^n \setminus \bigcup \mathcal{A}_{n:k}$ has i different coordinates for some i, $1 \le i \le n$. There are $\langle p\rangle_i$ ways to pick the values to be used and $S_k(n:i)$ ways to distribute these values among the coordinates. ■

Note that $S_k(n:i) > 0$ for $1 < i \le n$ so no nice divisibility relation can be derived.

4 The type B case

The *k-consecutive arrangements of type B*, $\mathcal{B}_{n:k}$, consist of all possible subspaces of the form

$$\begin{aligned} & x_j = 0, \quad 1 \le j \le n, \quad \text{and} \\ & \epsilon_i x_i = \epsilon_{i+1}x_{i+1} = \ldots = \epsilon_{i+k-1}x_{i+k-1}, \quad 1 \le i \le n-k+1. \end{aligned}$$

where $\epsilon_j = \pm 1$ for all j. Using the finite field method, we can obtain the generating function for the corresponding characteristic polynomials.

Proposition 4.1 *For fixed k we have the generating function*

$$\sum_{n\ge 1} \chi(\mathcal{B}_{n:k}, t)x^n = \frac{(t-1)x(1-2^{k-1}x^{k-1})}{1-2x-(t-3)x(1-2^{k-1}x^{k-1})}. \quad ■$$

This expression is too complicated to hope for a simple derivation of the Möbius function by NBB means. However, when $k=2$, the fraction simplifies and we can set $t=0$ to obtain the following result which also has an NBB proof.

Corollary 4.2 *For $n \ge 1$ we have*

$$\mu(\mathcal{B}_{n:2}) = (-1)^n 3^{n-1}. \quad ■$$

As Björner and Sagan [6] have done in the k-equal case, we can consider arrangements $\mathcal{B}_{n:k:h}$ with subspaces

$$\begin{aligned} & x_j = x_{j+1} = \ldots = x_{j+h-1} = 0, \qquad 1 \le j \le n-h+1, \qquad \text{and} \\ & \epsilon_i x_i = \epsilon_{i+1}x_{i+1} = \ldots = \epsilon_{i+k-1}x_{i+k-1}, \qquad 1 \le i \le n-k+1. \end{aligned}$$

Then $\mathcal{B}_{n:k:1} = \mathcal{B}_{n:k}$ and $\mathcal{B}_{n:k:k}$ is a k-consecutive analog of type D. One can write down generating functions like the one in Proposition 4.1 in this case as well.

5 The k-circular arrangements

Following a suggestion of Athanasiadis, we define the *k-circular coordinate arrangement*, $\mathcal{K}^{\circ}_{n:k}$, as all subspaces of $\mathbb{R}^n$ of the form

$$x_i = x_{i+1} = \ldots = x_{i+k-1} = 0, \qquad 1 \le i \le n$$

where the subscripts are taken modulo n. Circular analogs for other types are defined in the obvious way. Note that $L(\mathcal{K}^{\circ}_{n:k})$ is isomorphic to the poset $B^{\circ}_{n:k}$ generated by taking joins of intervals

$$[i, i+k-1], \qquad 1 \le i \le n$$

($i + k - 1$ taken modulo n) in the Boolean algebra of subsets of $[n]$.

We can use our usual methods to prove the next result, as well as ones for the other types.

Proposition 5.1 *In $B^{\circ}_{n:k}$, $n \ge k$, we have*

$$\mu([n]) = \begin{cases} k & \text{if } n \equiv 0 \pmod{k+1}, \\ -1 & \text{else.} \end{cases}$$

and

$$\chi(\mathcal{K}^{\circ}_{n:k}, t) = \sum_i \binom{n}{i}^{\circ}_{k} (t-1)^{n-i},$$

where $\binom{n}{i}^{\circ}_{k}$ is the number of $S \subseteq [n]$, $|S| = i$, with no k circularly consecutive. ■

6 Comments and open problems

I. Coefficients. The constants introduced as coefficients of the various characteristic polynomials have interesting properties. Consider, for example, the $\binom{n}{i}_k$. Clearly $\binom{n}{i}_k = \binom{n}{i}$ for $0 \le n < k$. And for small k we have

$$\begin{aligned} \binom{n}{i}_1 &= \delta_{i,0} \quad \text{(Kronecker)}, \\ \binom{n}{i}_2 &= \binom{n-i+1}{i}, \\ \binom{n}{i}_3 &= \sum_m \binom{m}{i-m-2}\binom{n-i+2}{m+2} + \binom{m}{i-m-3}\binom{n-i+1}{m+2} \end{aligned}$$

where the last expression is only true for $i \ge 3$ and does not have a closed form. We also have the recursion like the one for ordinary binomial coefficients except for an extra term.

Proposition 6.1 *For $n \ge k \ge 1$ we have*

$$\binom{n}{i}_k + \binom{n-k-1}{i-k}_k = \binom{n-1}{i}_k + \binom{n-1}{i-1}_k$$

where $\binom{-1}{i}_k = \delta_{i,0}$ and $\binom{n}{i}_k = 0$ for $i < 0$ or $i > n \ge 0$. ■

There are also many unanswered questions about these coefficients. For example, if two of the indices are held fixed and the third one varies, does one get a unimodal or even log concave sequence?

II. Topology and group actions. Let S^d be the sphere of dimension d and let $\Delta(L)$ be the order complex of lattice L. Using non-pure lexicographic shellings, Björner and Wachs proved the following strengthening of Proposition 2.2.

Theorem 6.2 **([8])** *We have*

$$\Delta(C_{n:k}) \simeq \begin{cases} S^{2(n-1)/k-2} & \text{if } n \equiv 1 \pmod{k}, \\ S^{2n/k-3} & \text{if } n \equiv 0 \pmod{k}, \\ \text{point} & \text{else.} \end{cases} \quad \blacksquare$$

A subset of an NBB base is NBB, so let NBB(L) be the simplicial complex of NBB bases of all of $x \in L, x \neq \hat{1}$. Segev has shown that there is a close relationship between the two complexes under consideration.

Theorem 6.3 **([20])** *There is a homotopy equivalence*

$$\Delta(L) \simeq \text{NBB}(L). \quad \blacksquare$$

It would be interesting to derive Theorem 6.2 from Theorem 6.3. This is non-trivial since the NBB bases of L do not form a matroid. For example, it is not always the case that an NBB base for some x can be extended to one for any $y \geq x$.

Another approach for using NBB bases to obtain topological information about $\Delta(L)$ would be to try and generalize Björner's construction of a homology basis in a geometric lattice [2]. Let L be geometric and let $B = \{a_1, \ldots, a_n\}$ be an NBC base of $\hat{1}$. Associate with B an element of $\Delta(L)$ defined by

$$\rho_B = \sum_{\pi \in \mathfrak{S}_n} (-1)^\pi (a_{\pi(1)}, a_{\pi(1)\vee\pi(2)}, \ldots, a_{\pi(1)\vee\cdots\vee\pi(n-1)})$$

where $\mathfrak{S}_n$ is the symmetric group on $[n]$ and $(-1)^\pi$ is the sign of $\pi \in \mathfrak{S}_n$. It is not hard to verify directly from their definition that the ρ_B are cycles. In fact these cycles can be used to compute the homology of $\Delta(L)$ (with integer coefficients).

Theorem 6.4 **([2])** *Let L be a geometric lattice of rank n. Then the only non-vanishing homology group of $\Delta(L)$ is in the top dimension, $n-2$, having as a homology basis*

$$\{\rho_B \ : \ B \text{ an NBC base for } \hat{1}\}. \quad \blacksquare$$

It would be very useful to have an analog of this theorem for NBB bases.

Finally, in the case of the k-circular arrangements there is an action of the cyclic group on the subspaces and hence on the intersection lattice and its homology. Sundaram and Welker [23] have studied the action of the symmetric group on the k-equal arrangements. No doubt the k-circular case will lead to interesting representations of the cyclic group.

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