ON DISCRETE MORSE FUNCTIONS AND COMBINATORIAL DECOMPOSITIONS

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ABSTRACT. This paper relates the recent theory of discrete Morse functions due to Forman [13] and combinatorial decompositions such as shellability, which are known to have many useful applications within combinatorics. First, we present the basic aspects of discrete Morse theory for regular cell complexes in terms of the combinatorial structure of their face posets. We introduce the notion of a *generalized shelling* of a regular cell complex and describe how to construct a discrete Morse function associated with such a decomposition. An application of Forman's theory gives us generalizations of known results about the homotopy properties of shellable complexes. We show that for simplicial complexes constructing generalized shellings is equivalent to constructing discrete Morse functions. We also discuss an application to a set of complexes related to matroids.

1. INTRODUCTION

This paper will focus on a recent development in topology – namely a discrete version of Morse theory developed by Forman [13] – and relate it to combinatorial decompositions such as shellings and interval partitions which have been studied extensively in combinatorics [2]. The primary purpose of this paper is to present the basic ideas of discrete Morse theory using combinatorial terminology and to show that it provides a unifying framework for some problems of topological combinatorics. We will discuss results and examples of discrete Morse theory that are readily accessible given some familiarity with the tools of topological combinatorics discussed in the survey [3]. We refer the reader interested in further topological details of this theory to Forman's paper [13].

One of the principal ideas of discrete Morse theory is to construct for a given finite cell complex (which we will assume to be *regular*), a "more efficient" cell-complex (which will not, in general, be regular), while retaining topological properties of the original space as much as possible. The construction of the more efficient complex depends on the existence of *discrete Morse functions* on the original regular cell complex. We will show that for a given *generalized* shelling of a regular cell complex there is a canonical discrete Morse function. A simple application of Forman's theory gives us a generalization of known results [6],[7] about the homotopy properties of shellable cell complexes. We show that for the special case of simplicial complexes constructing discrete Morse functions is, in a certain sense, equivalent to constructing generalized shellings. We conclude with an application to a class of complexes related to matroids. Further combinatorial applications of discrete Morse theory can found in the recent papers [1],[20] and [14].

2. Preliminaries

We will assume familiarity with the notion of cell complexes [3], which are traditionally called CW-complexes in standard algebraic topology texts such as Munkres [19] and Massey[16], [17]. Throughout the paper we will assume all such complexes to be finite. In a combinatorial context, it is most natural to consider *regular* cell complexes since with this additional property, the topology of the associated space is completely determined by the *face poset* of closed cells ordered with respect to containment. We refer the reader to [3] or [8] for further details and terminology. Hence forth, without change of notation we will also regard a regular cell-complex Σ as a poset, whose order and cover relation are denoted by $\leq$ and $\prec$, respectively, with $\geq$, $<$, $\succ$ etc. having the obvious interpretations. For $\sigma \in \Sigma$, let $\delta\sigma$ be the boundary subcomplex of σ and let $\bar{\sigma} = \{\sigma\} \cup \delta\sigma$. Recall that if Σ is a regular cell complex , $\bar{\sigma}$ is (homeomorphic to) the $dim\sigma$-ball while $\delta\sigma$ is a $(dim\sigma - 1)$-sphere. The dimension of Σ is the number $max\{dim\sigma : \sigma \in \Sigma\}$, and we will say that Σ is *pure* if all its maximal cells have the same dimension. When the regular cell complex is a simplicial complex, we will refer to its cells as its *faces* and its maximal cells as *facets.*

The property of shellability has been classically been studied only in the context of pure cell complexes and pure simplicial complexes. Recently, Björner and Wachs [6], [7] have undertaken a systematic study of shellability for general (non-pure) cell complexes and its applications. We now present this definition of shellable complexes.

Definition. An ordering $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the maximal cells of a d-dimensional regular cell complex Σ is a *shelling* if either $d = 0$ or if it satisfies the following conditions:

- (S1) There is an ordering of the maximal cells of $\delta\sigma_1$ which is a shelling.

- (S2) For $2 \leq j \leq m$, $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k)$ is pure and $(dim\sigma_j - 1)$-dimensional
- (S3) For $2 \leq j \leq m$, there is an ordering of the maximal cells of $\delta\sigma_j$ which is a shelling and further, the maximal cells of $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k)$ appear first in this ordering.

A regular cell complex is said to be *shellable* if it admits a shelling. In the general non-pure context, the following result is due to Björner and Wachs [6], [7] and it describes the primary topological consequence of shellability.

Theorem 2.1. *If a regular cell complex Σ is shellable then it is homotopy equivalent to a wedge of spheres.*

Next, we define an even more general decomposition property for regular cell complexes which has a natural relation to the discrete Morse theory of Forman.

Definition. An ordering $\sigma_1, \sigma_2, \ldots, \sigma_m$ of distinct (not necessarily maximal) cells of a regular cell complex Σ is a *generalized shelling* if satisfies the following two conditions *and* (S1), (S2) and (S3):

- (G1) $\Sigma = \cup_{i=1}^{m} \bar{\sigma}_m$.
- (G2) If $\sigma_i \in \delta\sigma_j$ then $i < j$.

Hence if $\sigma_1, \sigma_2, \ldots, \sigma_m$ are maximal cells of Σ, then we get the definition of Björner and Wachs [6]. We will show later that there are examples of complexes that are not shellable but admit non-trivial generalized shellings that are, in some sense, canonical. We should point out that this notion of generalized shelling is quite different from one defined for posets by Kozlov in [15].

The next proposition relates the existence of generalized shellings in simplicial complexes to interval-partitions and provides a non-recursive definition for generalized shellings in this context. We omit the proof, which is quite routine. Note that, as is traditional in combinatorial literature, the empty set is also considered to be a ((−1)-dimensional) face.

Proposition 2.2. *Let Σ be a simplicial complex. Then for an ordered subset $F_1, F_2, \ldots F_m$ of faces of Σ the following are equivalent:*

(i) $F_1, F_2, \ldots F_m$ is a generalized shelling for Σ.

(ii) There exist faces $G_1, G_2, \ldots G_m$ with $G_i \subseteq F_i$ such that the sequence $\{[G_i, F_i], i = 1, \ldots, m\}$ of intervals, partitions Σ and further $\cup_{i=1}^{k} [G_i, F_i]$ is a simplicial complex for $k = 1, 2, \ldots, m$.

Following [9], we will refer to the ordered sequence of intervals $\{[G_i, F_i], i = 1, \ldots, m\}$ as an *S-partition* of Σ.

3. Elements of discrete Morse theory

We will derive results for the homotopy type of complexes which admit non-trivial generalized shellings by applying the theory of discrete Morse functions developed by Forman [13]. We begin with the definition of these functions.

Definition. Given a (finite) regular cell complex Σ, a *(discrete) Morse function* on Σ is a function $f : \Sigma \to R$ satisfying the following two conditions for every cell σ of Σ :

- $|\{\tau \prec \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$. (M1)
- $|\{\omega \succ \sigma : f(\omega) \leq f(\sigma)\}| \leq 1$. (M2)

A convenient way to think about a Morse function is to regard it as being "almost increasing" with respect to dimension. Clearly, any function which is increasing with respect to dimension would be an (uninteresting!) example of a discrete Morse function. Figure 1 shows an example of a discrete Morse function on a 2-dimensional cell complex. Note that this complex is not pure and it is not shellable in the sense of Björner and Wachs.

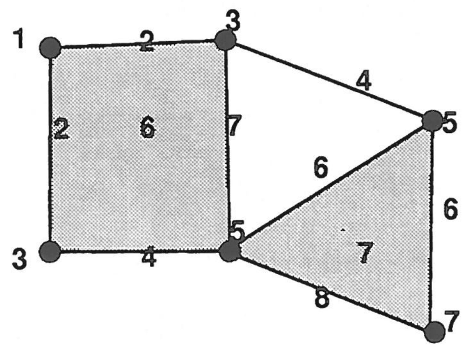

Figure 1. Example of a discrete Morse function

Definition. A p-dimensional cell σ of Σ is *critical* (with respect to a fixed Morse function f) if it satisfies each of the following conditions:

- $|\{\tau \prec \sigma : f(\tau) \geq f(\sigma)\}| = 0$. (C1)
- $|\{\omega \succ \sigma : f(\omega) \leq f(\sigma)\}| = 0$. (C2)

We will denote by $\mathcal{C}(f)$ – the set of critical cells of Σ with respect to f. The reader can verify that in the example of Figure 1, there are exactly two critical cells for the given Morse function – the vertex labelled with 1 and the edge labelled with 6 which has both vertices labelled with 5.

The following is one of the central theorems of discrete Morse theory.

Theorem 3.1. 1. *([13] Corollary 3.5) Suppose Σ is regular cell complex with a discrete Morse function. Then Σ is homotopy equivalent to a cell complex with exactly one cell of dimension p for each critical cell of Σ of dimension p.*

2. *([18], [13] Corollary 3.7) "Weak Morse Equalities": Let β_j be the j'th Betti number of Σ with coefficients in some fixed field and m_j be the number of j-dimensional critical cells, then $\beta_j \leq m_j$ for every j.*

We will indicate later an outline of Forman's proof of statement (1) of this theorem. If we take the dimension as a Morse function on any complex, every cell would be critical and hence the above theorem tells us nothing new. Hence, it is important to construct "efficient" Morse functions (that have few critical points), especially in view of (2) of the above theorem. For more details about both weak and *strong* Morse inequalities, we refer the reader to [18] and [13]. Observe that applying the above theorem to the example of Figure 1 shows that the complex is homotopy equivalent to the circle.

We will now restate the basic concepts of discrete Morse theory in graph-theoretic terms to emphasize the combinatorial nature of the theory for regular cell complexes. The discrete vector fields discussed in Forman's paper [13] and the work of Stanley [21] and Duval [12] on decompositions of simplicial complexes have underlying ideas that are similar in nature to what we will present. We begin with following simple lemma of Forman.

Lemma 3.2. *If f is a Morse function on a regular cell complex Σ and σ is any cell of Σ, then conditions (C1) and (C2) cannot both be false for σ.*

Proof. If possible, let $\omega \succ \sigma \succ \tau$ satisfy $f(\omega) \leq f(\sigma) \leq f(\tau)$. Now let α be a cell distinct from σ that also satisfies $\omega \succ \alpha \succ \tau$. The existence of such an α for a regular cell complex follows from the fact that $\delta\omega$ is a sphere. Applying condition (M1) to ω and (M2) to τ, we have $f(\omega) > f(\alpha) > f(\tau)$ which leads to a contradiction. $\square$

In particular, if a cell is not critical then it violates exactly one of (C1) and (C2). Now, we can regard the Hasse diagram of Σ as a directed graph, which we call $G(\Sigma)$, with the edges being cover relations directed from higher to lower dimensional cells. Clearly, $G(\Sigma)$ is acyclic in the directed sense. The above lemma implies that there exists a matching

$M(f)$ on the Hasse diagram of Σ associated with every discrete Morse function f of Σ such that the set of cells of Σ not incident to any edge of M is exactly $C(f)$. From the definition of a discrete Morse function, it is clear that $M(f)$ is precisely the set of the cover relations where f is non-increasing with respect to dimension. In general, for a set M (possibly empty) of edges of $G(\Sigma)$, we let $G_M(\Sigma)$ be the directed graph obtained from $G(\Sigma)$ by reversing the direction of edges in M. Then the following easy proposition gives an alternative description of Forman's framework for regular cell complexes.

Proposition 3.3. *A subset C of the cells of a regular cell complex Σ is the set of critical cells for some discrete Morse function f if and only if there exists a matching M on $G(\Sigma)$ such that $G_M(\Sigma)$ is acyclic and C is the set of nodes of $G(\Sigma)$ not incident to any edge in M.*

We will refer to a matching M satisfying the conditions of the above proposition as a *Morse matching* for $G(\Sigma)$. Since $G_M(\Sigma)$ corresponding to a given discrete Morse function f is acyclic, it must have a sink node. From the construction it is clear that this sink node must be a vertex, and this vertex must be critical. Thus any discrete Morse function on a nonempty cell complex has at least one critical vertex. Now we can also give an outline of Forman's proof of the first statement of Theorem 3.1 using the language of the above proposition. $G_M(\Sigma)$ must also have a source node, say a p-dimensional cell σ. Now assume that σ is *not* maximal in Σ. Then by the construction of $G_M(\Sigma)$, σ is contained in the boundary of exactly one $(p+1)$-dimensional cell, say τ and further σ and τ must be matched to each other in M. It is well known that the subcomplex of Σ, defined by $\Sigma \setminus \{\sigma,\tau\}$ is homotopy equivalent to Σ - more specifically, it is a deformation retract of Σ. Hence the proof for this case essentially follows by induction. Such a reduction $\Sigma \to \Sigma \setminus \{\sigma,\tau\}$ is referred to in the literature as an *elementary collapse* [13], [3]. Now suppose σ is a maximal cell. Then σ must be critical. We apply the result inductively to the subcomplex $\Sigma \setminus \{\sigma\}$, which completely contains the boundary of the cell σ. Then if we glue the open cell σ back on, along its boundary, the resulting complex has the desired properties. We remark that in this case, the resulting complex need not be regular, as the boundary of σ might be collapsed to a point.

We will say that Σ *collapses* to a subcomplex Γ, if Γ is obtained from Σ by a sequence of elementary collapses. This is true precisely when for some discrete Morse function f on Σ, the corresponding Morse matching matches every face of $\Gamma - \Sigma$ to some other face of $\Gamma - \Sigma$ (see [13]).

4. Generalized shellings and discrete Morse functions

In this section, we will construct discrete Morse functions for complexes with given generalized shellings. We will first prove these results for shellable *pseudomanifolds*. Recall that a d-pseudomanifold is a pure d-dimensional regular cell complex such that (i) every $(d-1)$-cell is contained in at most two d-cells, and (ii) for any two d cells β and τ there exists a sequence of d-cells $\beta = \sigma_1, \sigma_2, \ldots, \sigma_m = \tau$ such that σ_i and σ_{i+1} share a common $(d-1)$-cell for $1 \leq i \leq m-1$. The *boundary* of a d-pseudomanifold is the subcomplex generated by the set of $(d-1)$-cells which are contained in exactly one d-cell. A very fundamental and useful result of Bing ([11], [8] Chapter 4) asserts that a shellable pseudomanifold is a ball (a sphere) if it has a nonempty(empty) boundary. In the next proposition, we show that shellable pseudomanifolds are "nice" examples for discrete Morse theory in that we can construct the most efficient Morse functions for them. Following Björner [2], we will call the cell σ_j, for $j \geq 2$, a *homology cell* with respect to a fixed generalised shelling $\sigma_1, \sigma_2, \ldots, \sigma_m$ of a complex Σ if $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k) = \delta\sigma_j$. In the special case when Σ is a simplicial complex (as in Proposition 2.2), these homology faces derive from intervals in the associated S-partition for which $L_i = U_i$.

Proposition 4.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a shelling of a d-pseudomanifold Σ and let v be any vertex in $\bar{\sigma}_1$. Then, Σ admits a Morse function f such that*

(i) If Σ is the d-sphere then v and σ_m are only critical cells, while if Σ is a d-ball then v is the only critical cell.

(ii) When restricted to $\cup_{k=1}^{j}\bar{\sigma}_k$ for $1 \leq j < m$, the only critical cell of f is v.

Theorem 4.2. *Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a generalized shelling of a regular cell complex Σ and let v be any vertex in $\bar{\sigma}_1$. Then there exists a discrete Morse function f of Σ such that v is critical and further any other cell σ is critical if and only if it is a homology cell.*

Applying Theorem 3.1 to the Morse function of the above theorem we get the following result.

Corollary 4.3. *For a d-dimensional regular cell complex Σ, let m_j be the number of j-dimensional homology cells in some generalized shelling, $j = 0, 1, \ldots, d$ and suppose they are not all zero. Then we have the following:*

1. *Σ is homotopy equivalent to a cell complex with $m_0 + 1$ points and m_j j-dimensional cells for $j = 1, 2 \ldots, d$.*

2. *If the homology cells appear in non-increasing order of dimension in the generalized shelling then Σ is homotopy equivalent to a wedge of spheres consisting of m_j j-dimensional spheres, $j = 0, 1, 2, \ldots, d$.*

Remarks. The condition of non-increasing dimension for homology cell is inspired by the *rearrangement lemma (2.6)* of Björner and Wachs [6], [7]. They show that the maximal cells of a shellable regular cell complex can be rearranged to give a shelling order in which the maximal cells appear in non-increasing order of dimension. Hence for shellable complexes, Statement 2 of the above corollary reduces to Theorem 2.1.

Finally, we show that as far as simplicial complexes are concerned constructing generalized shellings is equivalent to constructing discrete Morse functions. For the following theorem, recall that for any discrete Morse function on a simplicial complex, there will always be a critical vertex which minimizes the Morse function or equivalently, this is the unique vertex which is a source node in the directed graph $G_M(\Sigma)$. We will call all other critical faces of a Morse function *non-trivial.*

Theorem 4.4. *For a subset C of the faces of a simplicial complex Σ, the following are equivalent:*

(i) C is the set of non-trivial critical faces of some discrete Morse function on Σ.

(ii) C is the set of homology faces of some generalized shelling for Σ.

In the next section we show that there are interesting complexes which are not shellable but do admit very structured generalized shellings.

5. On some complexes related to matroids

In this section we use discrete Morse theory to study the topology of a set of simplicial complexes related to matroids called *Steiner complexes* which were introduced by Colbourn and Pulleyblank [10]. We will not present the original definition but rather a simpler reformulation in terms of matroid ports which is shown to be equivalent to the original in [9]. In what follows, we assume familiarity with the basic concepts of matroid theory.

Definition. Given a connected matroid N and an element e of the ground set of N, the *port* of N at the element e is the set

$\mathcal{P} = \{C - \{e\} : e \in C, C \text{ is a circuit of } N\}$.

A Steiner complex on a ground set E is a simplicial complex $\mathcal{S}$ defined by

$$\mathcal{S} = \{E - A : P \subseteq A \text{ for some } P \in \mathcal{P}\}$$

where $\mathcal{P}$ is the port of some connected matroid N on ground set $E \cup \{e\}$ at the element e.

The most important example of ports from the point of view of applications are *Steiner trees* of a graph with respect to fixed subset K of the vertices of a connected graph. The number of faces of the corresponding Steiner complex is of great interest in network reliability applications [10],[9]. A matroid complex is also a Steiner complex, however, it is easily seen that Steiner complexes are neither pure nor shellable in general. We will show that the topology of these complexes is closely related to the topology of broken-circuit complexes [2]. Steiner complexes also have a natural matroid-theoretic duality property that is interesting from a topological perspective. Consider the set $\mathcal{P}^*$ of inclusion-minimal elements of $2^E - \mathcal{S}$. It follows from elementary matroid theory that $\mathcal{P}^*$ is the port of the matroid N^* at the element e. An important consequence of this fact is that if $\mathcal{S}$ is a Steiner complex defined with respect to the matroid N then $\mathcal{S}^b$ is a Steiner complex associated with N^*, where N^* is the the dual of N, and $\mathcal{S}^b = \{E - F : F \in 2^E - \mathcal{S}\}$. It follows that the topology of $\mathcal{S}$ and $\mathcal{S}^b$ are related by the combinatorial version of Alexander duality (see [2],[4]).

If we define M to be the matroid $N - e$ and denote by $\mathcal{I}(M)$ the complex of independent sets of M, then $\mathcal{I}(M^*) \subseteq \mathcal{S}$ and $\mathcal{I}(M) \subseteq \mathcal{S}^b$. For the rest of this section, we will assume a fixed total order on $E \cup \{e\}$ in which e is the smallest element. With respect to this total order, let $RBC(N)$, and $RBC(N^*)$ be the reduced broken circuit complexes associated with N and N^*(see [2] for definitions). It follows from the definition of these complexes that $RBC(N^*) \subseteq \mathcal{I}(M^*) \subseteq \mathcal{S}$ and dually, $RBC(N) \subseteq \mathcal{I}(M) \subseteq \mathcal{S}^b$. The following theorem may seem surprising at first since the complex $RBC(N^*)$ depends heavily on the total order on E while the Steiner complex does not.

Theorem 5.1. *$\mathcal{S}$ is homotopy equivalent to $RBC(N^*)$; in fact, $\mathcal{S}$ collapses to $RBC(N^*)$.*

Remarks. Due to its shellability, $RBC(N^*)$ is homotopy equivalent to a wedge of $\beta(N^*)$ $(|E| - \rho)$-dimensional spheres, where ρ is the rank of N [2],[22]. Therefore, the same result is also true for $\mathcal{S}$. For the connected matroid N and its dual N^* on the ground set $E \cup \{e\}$, we have $\beta(N) = \beta(N^*)$. As observed by Björner, ([2], (7.39)) this implies the following for every i.

$$H_i(RBC(N)) \cong H^{|E|-i-3}(RBC(N^*))$$

To quote Björner [2] - "(this is)...a curious topological duality for reduced broken-circuit complexes that seems to lack a systematic explanation". We have already shown that $\mathcal{S}$ is homotopy equivalent to $RBC(N^*)$. By matroid port duality mentioned earlier, $\mathcal{S}^b$ is homotopy equivalent to $RBC(N)$. Therefore, the topological duality of the reduced broken-circuit complexes observed by Björner follows, via the above theorem, from the "natural" Alexander duality of the appropriate pair of Steiner complexes.

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