PROPERTIES OF THE ROBINSON-SCHENSTED CORRESPONDENCE FOR OSCILLATING AND SKEW OSCILLATING TABLEAUX (EXTENDED ABSTRACT[1])

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ABSTRACT. In this paper, we consider the Robinson-Schensted correspondence for oscillating tableaux and skew oscillating tableaux defined in [15] and [4]. First we give an analogue, for the oscillating tableaux, of the classical geometric construction of Viennot for standard tableaux ([16]). Then, we extend a construction of Sagan and Stanley ([10]), dealing with standard tableaux and skew tableaux, to deduce a property about the number of odd height columns of a skew oscillating tableau. Finally, we give analogues for skew oscillating tableaux of two classical constructions about this correspondence, Knuth classes ([7]) and Beissinger algorithm ([1]).

RÉSUMÉ. Dans cet article, nous considérons la correspondance de Robinson-Schensted pour les tableaux oscillants et les tableaux oscillants gauches définie dans [15] et [4]. Premièrement, nous donnons, pour les tableaux oscillants, une construction analogue à la construction géométrique classique de Viennot pour les tableaux standard ([16]). Ensuite, nous étendons une construction de Sagan et Stanley ([10]), pour en déduire une propriété sur le nombre de colonnes de hauteur impaire d'un tableau oscillant gauche. Finalement, nous donnons des analogues pour les tableaux oscillants gauches de deux constructions classiques liées a cette correspondance, les classes de Knuth ([7]) et l'algorithme de Beissinger ([1]).

1. INTRODUCTION

The Robinson-Schensted correspondence is a classical bijection between permutations and pairs of standard tableaux of the same shape. It was defined in [11], and followed by numerous papers dealing with the combinatorial properties of this correspondence, like [12], [7], [5], [13], [16] or [1]. More recently, this correspondence was extended to various kinds of tableaux that are generalizations, in the Young lattice, of the standard tableaux: semi-standard tableaux ([7]), skew tableaux ([10]), oscillating tableaux (first by Sundaram in [14, 15], then independtly Delest, Dulucq and Favreau in [3]) and skew oscillating tableaux ([4]).

In this article, we extend classical properties and constructions related to the Robinson-Schensted correspondence to the correspondences for oscillating tableaux and skew oscillating tableaux. In the sections 2 and 3, we give basic definitions on biwords and tableaux and we present the correspondence for skew oscillating tableaux. Then we extend a geometric version of the Robinson-Schensted correspondence due to Viennot ([16]) to the case of oscillating tableaux, and, following [10], we define a construction allowing extension of properties of oscillating tableaux to skew oscillating tableaux. Finally, in sections 6 and 7, we extend to skew oscillating tableaux a result of Knuth ([7]) and an algorithm of Beissinger ([1]).

2. DEFINITIONS AND NOTATIONS

We assume that the reader is familiar with the combinatorics of the Young tableaux in general and, in particular, with the Robinson-Schensted correspondence (see [11], or [9] for a survey).

We use the notation $\lambda = (\lambda_1, \dots, \lambda_k)$ for both a partition and the corresponding Ferrers diagram displayed in "French" notation (the smallest part λ_k in the top row). The conjugate of λ is the partition $\lambda' = (\lambda'_1, \dots, \lambda'_m)$, where $\lambda'_j = Card\{i \ \lambda_i \geq j\}$. If $\mu \subseteq \lambda$ then the corresponding skew shape λ/μ is the set $\{c \mid c \in \lambda,\ c \notin \mu\}$. If $|\lambda/\mu| = n$ then we write $\lambda/\mu \vdash n$ and say that λ/μ is a skew partition of n.

A Young tableau T of shape λ/μ is a labeling of the cells of λ/μ with an ordered alphabet (of positive integers here) so that the rows and columns are weakly increasing. We denote by $T(i, j)$ the label of the cell

[1] A complete version of this abstract is avalaible in [2].

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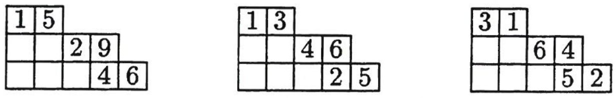
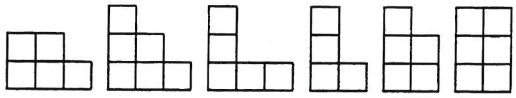
in the i^{th} row and j^{th} column, so that $k \in T$ means $k = T(i,j)$ for some i,j. A Young tableau is partial if its elements are distinct. Further it is standard if it is partial and the labels are 1 through $n = |\lambda/\mu|$. The sets of partial and (skew) standard tableaux of shape λ/μ will respectively be denoted by $PT(\lambda/\mu)$ and $ST(\lambda/\mu)$. Analogously, the set of tableaux of shape λ/μ with rows and columns strictly decreasing will be denoted by $\overline{PT}(\lambda/\mu)$. For example, when $\lambda = (5,4,2)$ and $\mu = (3,2)$, the three following tableaux belong respectively to $PT(\lambda/\mu)$, $ST(\lambda/\mu)$ and $\overline{PT}(\lambda/\mu)$.



A skew oscillating tableau of length n, initial shape α, and final shape β is a sequence of Ferrers diagrams $T = (\alpha = \lambda_0, \lambda_1, \dots, \lambda_n = \beta)$ where λ_k is obtained from λ_{k-1} by insertion or deletion of a cell. If $\alpha = \emptyset$, T is an oscillating tableau. We denote by $SOT_n(\alpha \to \beta)$ the set of skew oscillating tableaux of length n, initial shape α and final shape β and by $OT_n(\beta)$ the set of oscillating tableaux of length n and final shape β. For example, if $\alpha = (3,2)$ and $\beta = (2,2,2)$, the following tableau belongs to $SOT_5(\alpha \to \beta)$.

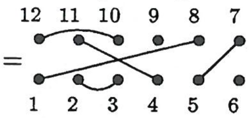


Of course, a skew oscillating tableau of $SOT_n(\mu \to \lambda)$ having only insertion steps, is a (skew) standard tableau of $ST(\lambda/\mu)$, the label of a cell being given by the step of creation of this cell.

Classicaly, we denote by S_n the set of the permutations of $[n]$ and by INV_n the set of the involutions of $[n]$, and, for a permutation σ, we denote by σ^{-1} its inverse. Sometimes, we will write an involution as a product of cycles in increasing order of their greatest element and with $a < b$ for each cycle (a,b), as for example, $\sigma = (1)(2,3)(5)(4,6)$. A sequence of at most n integers, pairwise distinct and less than or equal to n, will be called a partial permutation of $[n]$. For example, 4 1 5 is a partial permutation on $[n]$ for $n \geq 5$.

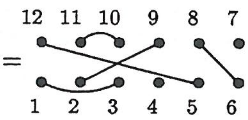
A biword π on $[n] = \{1,2,\dots,n\}$ is a sequence of vertical pairs of positive integers of $[n]$, pairwise distinct, $\pi = \binom{i_1\ i_2\ \dots\ i_k}{j_1\ j_2\ \dots\ j_k}$. In this paper we consider biwords such that all the i_l and j_l are distinct in π and $i_l > j_l$, for $l = 1, \dots k$. We denote by $\hat{\pi}$ the top row of π and by $\check{\pi}$ its bottom row, and we write such biwords with the convention $i_1 > i_2 > \dots > i_k$. We denote the set of these biwords on $[n]$ by BW_n. Moreover, we often represent an element π of BW_{2n} with a graph $G(\pi)$, as in the following example.

$$\pi = \begin{pmatrix} 12 & 11 & 8 & 7 & 3 \\ 10 & 4 & 1 & 5 & 2 \end{pmatrix} \qquad G(\pi) =$$



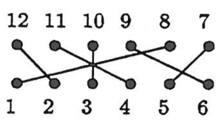
In such a graph, we call diagonal edges the edges between the two rows of vertices. If $\pi \in BW_n$, its inverse π^{-1} is obtained by changing every pair (i_l, j_l) to $(n+1-j_l, n+1-i_l)$. It follows that, for $\pi \in BW_{2n}$, $G(\pi^{-1})$ is obtained from $G(\pi)$ by an horizontal symmetry.

$$\pi^{-1} = \begin{pmatrix} 12 & 11 & 9 & 8 & 3 \\ 5 & 10 & 2 & 6 & 1 \end{pmatrix} \qquad G(\pi^{-1}) =$$



Finally, there is an immediate bijection between elements of BW_{2n} whose graph has exactly n diagonal edges and permutations of S_n.

$$\pi = \begin{pmatrix} 12 & 11 & 10 & 9 & 8 & 7 \\ 2 & 4 & 3 & 6 & 1 & 5 \end{pmatrix} \ G(\pi) =$$



$\longleftrightarrow$ 2 4 3 6 1 5

Hence, if π corresponds to a permutation σ, then π^{-1} corresponds to σ^{-1}.

3. Robinson-Schensted correspondence

In [11], Schensted gives an algorithm which associates to a permutation σ of S_n a pair of standard tableaux (P, Q) with the same shape λ such that $\lambda \vdash n$. The tableau P is called the *insertion tableau* of σ and the tableau Q its *recording tableau*. This construction is the limiting case of one for partial permutations and partial tableaux.

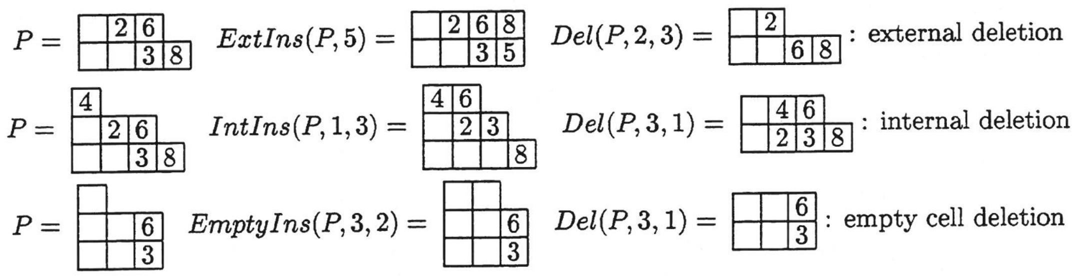
In [4], Dulucq and Sagan extend this algorithm to deal with skew oscillating tableaux. We recall their result, keeping their notations. This algorithm relies on three kinds of insertion in a skew tableau (see Example 3.1).

Let P be a skew tableau of shape λ/μ.

1. The *external insertion* is the insertion defined by Schensted ([11]) and inserts an integer x in P. We denote the new tableau obtained after this insertion by $ExtIns(P, x)$.
2. The *internal insertion* was introduced in [10]. Let (u, v) be a cell of P such that $(u, v) \notin \mu$ or $v = 1$, $(u-1, v) \in \mu$ or $u = 1$, and $(u, v-1) \in \mu$. The internal insertion of the cell (u, v) inserts the integer x contained in $P(u, v)$ from the row $(u + 1)$ using the external insertion algorithm. This process doesn't add a label in the tableau P. We denote the new tableau by $IntIns(P, u, v)$.
3. The *empty insertion* adds an empty cell in $P(u, v)$, where $(u - 1, v) \in \mu$ or $u = 1$, $(u, v - 1) \in \mu$ or $v = 1$, and $(u, v) \notin \lambda$. We denote the new tableau obtained after this insertion by $EmptyIns(P, u, v)$.

Conversely, the deletion of the cell $P(u, v)$, denoted by $Del(P, u, v)$, can be an *empty deletion* if the cell is an empty cell, an *internal deletion* if the process (this is the classical process of deletion defined by Schensted) ends in filling a cell of μ, or an *external deletion* if the process ends with the expulsion of an integer out of P.

Example 3.1.



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Remark 3.1. We call the elements displaced during an insertion process the "bumped" elements.

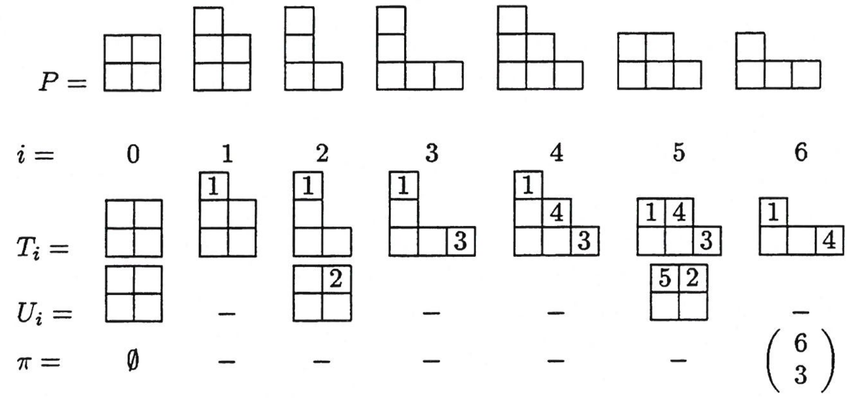
We now give a description and a detailed example (Example 3.2) of this algorithm and of its inverse (cf [4]). The first, denoted by Φ_{SO}, has for input a triple (π, T, U) of $BW_n \times \bigcup_{\mu \subseteq \alpha \cap \beta} [PT(\beta/\mu) \times \overline{PT}(\alpha/\mu)]$ and for output a tableau P of $SOT_n(\alpha \to \beta)$.

Algorithm 1. $\Phi_{SO}(\pi, T, U)$ *- The output is a tableau P.*
Let $P_n = T$.
For i from n to 1:
(a) if there is a cell $P_i(u, v) = i$, then erase this cell to obtain P_{i-1},
(b) else if the pair (i, x) belongs to π, then $P_{i-1} = ExtIns(P_i, x)$,
(c) else if $U(u, v) = i$ and $P_i(u, v)$ exists (with label x), then $P_{i-1} = IntIns(P_i, u, v)$,
(d) else $P_{i-1} = EmptyIns(P_i, u, v)$.
Finally, the tableaux P_i have respective shapes λ_i/μ_i and $P = (\lambda_0, \ldots, \lambda_n)$.

Algorithm 2. $\Phi_{SO}^{-1}(P)$ *- The output is a triple (π, T, U).*
Suppose $P = (\alpha = \lambda_0, \ldots, \lambda_n = \beta)$.
Let $\pi = \emptyset$, $T_0 = \alpha$ and $U_0 = \alpha$.
For i from 1 to n:
(a) if $\lambda_i = \lambda_{i-1} + (u, v)$, then add in T_{i-1} a cell in position (u, v) with label i to obtain T_i, $U_i = U_{i-1}$,
(b) else ($\lambda_i = \lambda_{i-1} - (u, v)$) $T_i = Del(T_{i-1}, u, v)$:
(b.1) if this deletion is external (x ejected out of T_{i-1}), then add the pair (i, x) to π, $U_i = U_{i-1}$.
(b.2) else if it is internal (the cell $T_{i-1}(u', v')$ is filled), then label the cell $U_{i-1}(u', v')$ with i to obtain U_i,
(b.3) else label the cell $U_{i-1}(u, v)$ with i to obtain U_i.
Finally, $T = T_n$ and $U = U_n$.

Example 3.2. Let P be the following skew oscillating tableau.

Then:



P is in bijection with (π, T_6, U_6). $\diamond$

Theorem 3.1. ([4]) *Let α and β be fixed partitions and n a fixed integer. Φ_{SO} is a bijection from triples (π, T, U) of $BW_n \times \bigcup_{\mu \subseteq \alpha \cap \beta}[PT(\beta/\mu) \times \overline{PT}(\alpha/\mu)]$, such that $\pi \dot{\cup} T \dot{\cup} U = [n]$, to tableaux P of $SOT_n(\alpha \to \beta)$.*

Remark 3.2. If the skew oscillating tableau P has only insertion steps (it is a skew standard tableau P' of shape β/α), the bijection Φ_{SO} is such that $\Phi_{SO}^{-1}(P) = (\emptyset, P', \alpha)$.

Theorem 3.2. ([4]) *Let α be a fixed partition and n a fixed integer. There is a bijection RS_{SO} from triples (π, T, U) of $BW_{2n} \times \bigcup_{\mu \subseteq \alpha \cap \beta}[PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)]$ such that $\pi \dot{\cup} T \dot{\cup} U = [2n]$ to pairs of tableaux (P, Q) of $\bigcup_{\beta}[SOT_n(\alpha \to \beta) \times SOT_n(\alpha \to \beta)]$.*

This result follows immediately from the previous theorem and:

$$(\lambda_0(=\alpha), \dots, \lambda_n, \dots, \lambda_{2n}(=\alpha)) \longleftrightarrow (P = (\lambda_0, \lambda_1, \dots, \lambda_n(=\beta)),\ Q = (\lambda_{2n}, \lambda_{2n-1}, \dots, \lambda_n(=\beta))).$$

Using skew oscillating tableaux with empty initial and final shapes, we have a Robinson-Schensted correspondence for oscillating tableaux, as stated in the following results.

Theorem 3.3. ([15]) *Let β be a fixed partition and n a fixed integer. There is a bijection Φ_O from pairs (π, T) of $BW_n \times PT(\beta)$ such that $\pi \dot{\cup} T = [n]$ to tableaux P of $OT_n(\beta)$.*

Theorem 3.4. ([15]) *Let n be a fixed integer. There is a bijection RS_O from biwords π of BW_{2n} of size $2n$ to pairs of tableaux (P, Q) of $\bigcup_{\beta}[OT_n(\beta) \times OT_n(\beta)]$.*

It is straightforward to verify that if P and Q have only insertion steps (i.e. P and Q are standard tableaux), $G(\pi)$ has exactly n diagonal edges (then π corresponds to a permutation) and we have the original Robinson-Schensted correspondence for standard tableaux.

4. Analogue of the geometric construction of Viennot

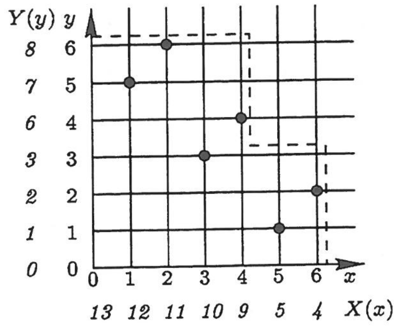
In this section, we give an analogue for oscillating tableaux of a beautiful construction of Viennot for standard tableaux ([16]). We follow the presentation of the construction of Viennot given in [9].

First, we explain how we represent a biword on $[2n]$, $\pi = \binom{i_1\ i_2\ \cdots\ i_n}{j_1\ j_2\ \cdots\ j_n}$ of $[n] \times [n]$ in the part $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$ of the Cartesian plane (see Example 4.1).

1. We define a map X from abscissas x ($x = 0, 1, \dots, n$) into $\{2n+1\} \cup \tilde{\pi}$ such that $X(0) = 2n+1$ and $X(x)$ is the x^{th} greatest element of $\tilde{\pi}$, for $x > 0$.
2. We define a map Y from ordinates y ($y = 0, 1, \dots, n$) into $\{0\} \cup \tilde{\pi}$ such that $Y(0) = 0$ and $Y(y)$ is the y^{th} lowest element of $\tilde{\pi}$, for $y > 0$.
3. We define a *valid domain* which is the set of points (x, y) such that $X(x) > Y(y)$.
4. For each pair (i_l, j_l) of π, we set up the point $(X^{-1}(i_l), Y^{-1}(j_l))$ (which is in the valid domain).

Example 4.1. Here we give a biword π of BW_{12} and its representation (for readability, the limit of the valid domain, the dashed line, is slightly extended on the figure).

$$\pi = \begin{pmatrix} 12 & 11 & 10 & 9 & 5 & 4 \\ 7 & 8 & 3 & 6 & 1 & 2 \end{pmatrix}$$



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We introduce definitions intuitively related to a lighting of the representation of a biword from $(0,0)$ (illustrated in Example 4.2).

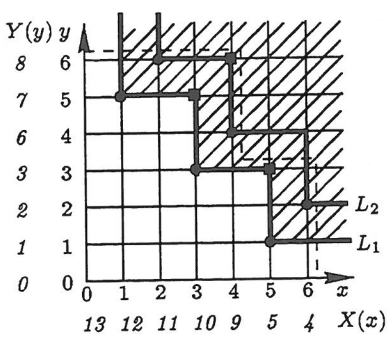
Definition 4.1. *The shadow $S(\pi)$ of a biword π is the set of points (x,y) such that there is a point (x',y') of the representation of π with $x' \leq x$ and $y' \leq y$.*
Shadow lines of π are defined recursively. The first shadow line L_1 of π is the boundary of $S(\pi)$. To construct the shadow line L_{i+1} of π, remove the points of the representation of π lying on L_i and construct the shadow line of the remaining points. This procedure ends when there is no remaining point on the plane.
The SW-corners of a shadow line are the points of the representation of π located on this line.
The NE-corners of a shadow line are the points (x,y) of the shadow line such that (x,y) is in the valid domain and $(x+1,y)$ and $(x,y+1)$ don't belong to this shadow line.

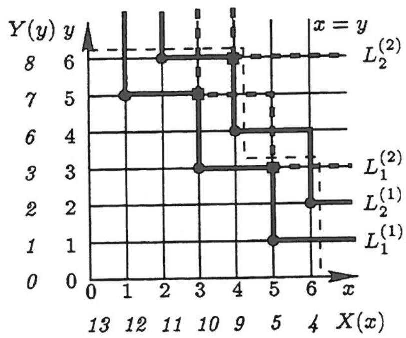
Definition 4.2. *The k^{th} skeleton $\pi^{(k)}$ of a biword π is a biword defined recursively by*

1. $\pi^{(1)} = \pi$
2. $\pi^{(k+1)} = \binom{X(i_1)\ X(i_2)\ \dots\ X(i_m)}{Y(j_1)\ Y(j_2)\ \dots\ Y(j_m)}$ *where $(i_1,j_1),\dots,(i_m,j_m)$ are the NE-corners of $\pi^{(k)}$.*

The shadow diagram of π is the set of shadow lines of all the skeletons $\pi^{(k)}$ of π. The shadow lines of $\pi^{(k)}$ are denoted $L_i^{(k)}$.

Example 4.2. Let π be the biword of Example 4.1. The figure on the left gives the shadow of π (which has two shadow lines). The points marked with a circle (i.e. the points of the biword) are the SW-corners, and the points marked with a square are the NE-corners. The figure on the right gives the shadow diagram of π.





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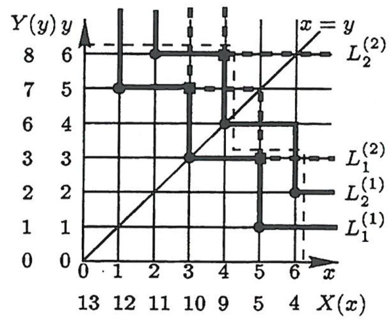
In the following result, we show that, for a biword π, the behaviour of the algorithm $\Phi_O(\pi,\emptyset)$, can be described by the shadow diagram of π.

Theorem 4.1. *Let π be a biword of BW_{2n} and T an oscillating tableau of OT_{2n} such that $\Phi_O(\pi,\emptyset) = T$. Following, from the left to the right, the shadow line $L_j^{(i)}$ describes the behaviour of the j^{th} cell of the i^{th} row of the tableaux $T_{2n},\dots,T_0$ in the following way:*

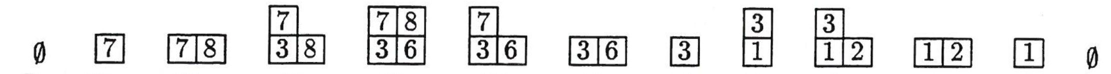
1. *a SW-corner (x,y) indicates that during step $X(x)$, the label $Y(y)$ fills this cell, which is created if not present,*
2. *when the line leaves the valid domain at (x,y), this cell is deleted during the step $Y(y)$,*

3. *otherwise, the cell remains unchanged.*

Example 4.3. Let π be the biword of BW_{12} defined above (see Example 4.1). We recall its shadow diagram.

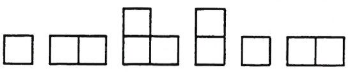
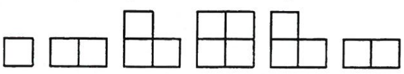


The execution of $\Phi_O(\pi, \emptyset)$ produces the following partial tableaux $T_{12}, \dots, T_0$:



T_{12} T_{11} T_{10} T_9 T_8 T_7 T_6 T_5 T_4 T_3 T_2 T_1 T_0

Now, we observe, for example, the second cell of the first row and the shadow line $L_2^{(1)}$.
The line has a first SW-corner at $(2, 6)$ with $X(2) = 11$ and $Y(6) = 8$, followed by the SW-corner $(4, 4)$ with $X(4) = 9$ and $Y(4) = 6$, leaves the valid domain at the same point $(4, 4)$, enters the valid domain again to go to the SW-corner $(6, 2)$ with $X(6) = 4$ and $Y(2) = 2$ and finally leaves the valid domain at the same point $(6, 2)$.
On the other hand, during the execution of Φ_O, (the tableaux are constructed from T_{12} to T_0), the second cell of the first row is created during step 11 (passing from T_{11} to T_{10}) with label 8, this label is replaced during step 9 by the label 6, which remains unchanged until the first deletion of the cell during step 6. The cell is created again during step 4 with label 2 and is finally deleted during step 2.
Finally we can say that π is in bijection with the following pair (P, Q) of oscillating tableaux.

$P = \emptyset$  , $Q = \emptyset$ 

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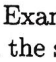
Remark 4.1. If π is a biword of BW_{2n} such that $G(\pi)$ has exactly n diagonal edges, our construction is equivalent to the original construction of Viennot (in particular, the valid domain is $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$ and there is no need of the maps X and Y).

Now, we show how we can use the shadow diagram of a biword to deduce combinatorial properties of the Robinson-Schensted correspondence for oscillating tableaux.

Theorem 4.2. *Let π be a biword of BW_{2n} and P, Q two oscillating tableaux of $OT_n(\beta)$ such that $RS_O(\pi) = (P, Q)$. Then, the i^{th} row of β has k cells if and only if k shadow lines $L_j^{(i)}$ intersect the line $x = y$ in the valid domain.*

Example 4.4. The biword π of Example 4.1 is in bijection with a couple (P, Q) of oscillating tableaux of $OT_6(\beta)$ such that $\beta =$  . On the shadow diagram of π (see Example 4.3), the line $x = y$ intersects in the valid domain only $L_1^{(1)}$ and $L_2^{(1)}$ and we verify the result. ◇

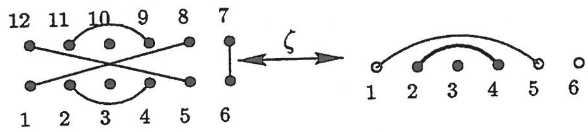
One of the beautiful properties of the Robinson-Schensted correspondence relies on the inversion of the standard tableaux P and Q. Schützenberger showed, in [12], that if (P, Q) is in bijection with a permutation σ, then (Q, P) is in bijection with σ^{-1}, and so, the Robinson-Schensted correspondence is a bijection between involutions and standard tableaux. Later, these results were again demonstrated by Viennot in [16] using symmetry properties of its geometric construction. In the same way, with our construction we can prove similar results for oscillating tableaux.

Property 4.1. *If π is a biword of BW_{2n}, then the valid domain and the shadow lines of π^{-1} are the reflection in the line $x = y$ of those of the representation of π.*

Definition 4.3. *We denote INV_n^c the set of involutions on $[n]$ such that every cycle (a, b) $(a \neq b)$ and every fixed point (a) can be of two types (colors), called bold or normal.*

Proposition 4.1. *There is a bijection ζ: $\{\pi \in BW_{2n} | \pi = \pi^{-1}\} \to INV_n^c$.*

Example 4.5.



We have then $\zeta(\pi) = (\mathbf{3})(\mathbf{2},\mathbf{4})(1, 5)(6)$: the cycle $(\mathbf{2},\mathbf{4})$ and the fixed point $(\mathbf{3})$ are bold, the cycle $(1, 5)$ and the fixed point (6) are normal.

Using Property 4.1 and the previous Proposition, we have a geometric proof of the following results.

Theorem 4.3. ([3]) *Let π be a biword of BW_{2n} and P, Q oscillating tableaux such that $RS_O(\pi) = (P, Q)$. Then $RS_O(\pi^{-1}) = (Q, P)$.*

Corollary 4.1. ([3]) *RS_O induces a bijection between $OT_n(\beta)$ and the involutions of INV_n^c having no bold fixed point.*

Furthermore, with the bijection between involutions and standard tableaux, Schützenberger ([12]) showed that the number of columns of this standard tableau with odd height is the number of fixed points of the involution, a result which has a direct geometric proof using the construction of Viennot. Similarly, using our construction, we have a geometric proof of an analogous result for oscillating tableaux.

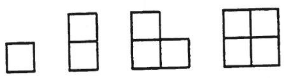
Theorem 4.4. *Let σ be an involution of INV_n^c without bold fixed point and P an oscillating tableau of $OT_n(\beta)$ such that σ is in bijection with P as in Corollary 4.1. Then $odd(\beta) = fix_N(\sigma)$, where $odd(\beta)$ is the number of columns in β with odd height and $fix_N(\sigma)$ is the number of normal fixed points of σ.*

5. Properties of skew oscillating tableaux

In this section, we extend some of the previous properties of the Robinson-Schensted correspondence for oscillating tableaux to the correspondence for skew oscillating tableaux. We define an extension of a construction of Sagan and Stanley (see [10]), in order to treat a pair of skew oscillating tableaux as a pair of oscillating tableaux.

Definition 5.1. *The column word of a partial tableau P is the sequence of integers corresponding to the columns of P read from left to right, each column being read from top to bottom.*
Let α be a Ferrers diagram. We denote C_α the oscillating tableau corresponding to the construction of α column by column, from left to right, each column being constructed from top to bottom. P_α is the standard tableau corresponding to C_α (which has only insertion steps), and π_α the column word of P_α.

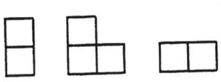
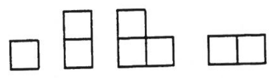
Example 5.1. $\alpha = (2, 2)$

$C_\alpha = \emptyset$ , $P_\alpha = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$ and $\pi_\alpha = 2\ 1\ 4\ 3$

◇

Now we introduce the notion of completion of a skew oscillating tableau. Let α be a partition such that $\alpha \vdash m$. We define a map *cpl* from $SOT_n(\alpha \to \beta)$ into $OT_{n+m}(\beta)$, that associates to a skew oscillating tableau P the oscillating tableau *cpl*(P) that is the concatenation of C_α and P. We say that C_α is the completion of P.

Example 5.2.

$P =$  $\Rightarrow cpl(P) = \emptyset$ 

◇

Furthermore, we use a new alphabet for the labels of partial tableaux, $\mathbb{N} \cup \{\underline{j}^k : j,k \in \mathbb{N}\} \cup \{j^k : j,k \in \mathbb{N}\}$, such that, for x, j, k and k' positive integers we have

$$j^k < j^{k+1} < (j+1)^{k'} < x < \underline{j}^k < \underline{j}^{k+1} < \underline{(j-1)}^{k'}.$$

Next, let α, μ and λ be partitions such that $\mu \subseteq \alpha$ and $\mu \subseteq \lambda$. We use a map $[.]_\lambda$ (defined in [10]) from $PT(\alpha/\mu)$ into $PT(\alpha)$, that associates to a partial tableau P the tableau $[P]_\lambda$, obtained in labeling (from top to bottom) the empty cells of the j^{th} column of P with $j^{\lambda'_j}, j^{\lambda'_j-1}, \dots, j^{\lambda'_j-\mu'_j+1}$. If $\lambda = \mu$, we write $[P]$ for $[P]_\lambda$. For example, let $\lambda = (4,3,1)$ ($\lambda' = (3,2,2,1)$).

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 \\ \cline{1-1} & & 2 & 7 \\ \cline{1-4} & & & 5 & 8 \\ \hline\end{array}, \quad [P]_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1 \\ \cline{1-1} 1^3 & 2^2 & 2 & 7 \\ \cline{1-4} 1^2 & 2^1 & 3^2 & 5 & 8 \\ \hline\end{array}, \quad [P] = \begin{array}{|c|c|c|c|c|}\hline 1 \\ \cline{1-1} 1^2 & 2^2 & 2 & 7 \\ \cline{1-4} 1^1 & 2^1 & 3^1 & 5 & 8 \\ \hline\end{array}$$

Now, following [10], we define a transformation of a triple (π, T, U) of Theorem 3.2 ($(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$ with $\alpha \vdash m$ and α' has l rows) into a biword of $BW_{2n'}$ with $n' = n + m$. We denote this biword $[\pi, T, U]$ or, when there is no possible confusion, $[\pi]$.

Algorithm 3. $[.](\pi, T, U)$ - *The output is a biword* $[\pi]$ *of* $BW_{2n'}$.
Let $[\hat{\pi'}] = \hat{\pi} \cup U \cup \underline{l^1 \dots l^{\alpha'_l}}\ \underline{(l-1)^1 \dots (l-1)^{\alpha'_{l-1}}} \dots \underline{1^1 \dots 1^{\alpha'_1}}$.
For each element a *of* $[\hat{\pi'}]$:
(a) *if* $a \in \hat{\pi}$, *then* $[\pi'](a) = \pi(a)$,
(b) *else if* $a = \underline{j}^k$, *then* $[\pi'](a) = [T]_\alpha(k, j)$,
(c) *else* ($a = U(k, j)$), $[\pi'](a) = j^{k'}$, *with* $k' = \alpha'_j - k + 1$.
Finally, $[\pi]$ *is given by the normalization of* $[\pi']$ *on* $[2n']$.

Example 5.3. With the following triple (π, T, U)

$$\pi = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad T = \begin{array}{|c|c|}\hline 3 & 6 \\ \hline & \\ \hline\end{array} \quad U = \begin{array}{|c|c|}\hline 5 & 2 \\ \hline & \\ \hline\end{array}$$

we have

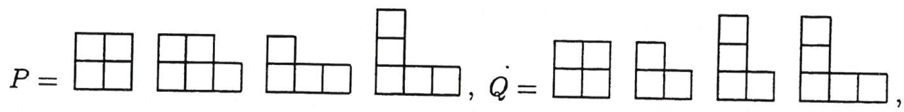
$$[\pi'] = \begin{pmatrix} \underline{1}^2 & \underline{1}^1 & \underline{2}^2 & \underline{2}^1 & 5 & 4 & 2 \\ 3 & 1^2 & 6 & 2^2 & 1^1 & 1 & 2^1 \end{pmatrix}, \quad [\pi] = \begin{pmatrix} 14 & 13 & 12 & 11 & 9 & 8 & 6 \\ 7 & 2 & 10 & 4 & 1 & 5 & 3 \end{pmatrix}.$$

$\diamond$

We now give the result that relates the Robinson-Schensted correspondence for skew oscillating tableaux to the Robinson-Schensted correspondence for oscillating tableaux. This is an extension of a similar result of [10].

Lemma 5.1. *Let* $(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$, P *and* Q *be tableaux of* $SOT_n(\alpha \to \beta)$. *If* $RS_{SO}(\pi, T, U) = (P, Q)$ *then* $RS_O([\pi]) = (cpl(P), cpl(Q))$.

Example 5.4. The triple (π, T, U) of the Example 5.3 is in bijection by RS_{SO} with the couple of skew oscillating tableaux (P, Q).



$[\pi]$ is in bijection by RS_O with the oscillating tableaux $(P', Q') = (cpl(P), cpl(Q))$. $\diamond$

Using this construction, we have new proofs of properties of the Robinson-Schensted correspondence for skew oscillating tableaux.

Definition 5.2. *Let* n *be an integer and* P *a tableau of* $PT(\lambda/\mu)$ *(respectively* $\overline{PT}(\lambda/\mu)$*), such that all these labels are less than or equal to* n. *We define the tableau* P^c *of* $\overline{PT}(\lambda/\mu)$ *(respectively* $PT(\lambda/\mu)$*) by* $P^c(u,v) = n + 1 - P(u,v)$ *for each cell* $(u,v) \in \lambda/\mu$.

Example 5.5. Let $n = 8$.

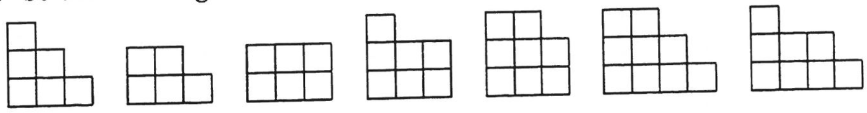
$$P = \begin{array}{llll} 6 & & & \\ & 2 & & \\ & 1 & 3 & 5 \end{array} \Rightarrow P^c = \begin{array}{llll} 3 & & & \\ & 7 & & \\ & 8 & 6 & 4 \end{array}$$

◇

Theorem 5.1. ([4]) *Let $(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$ and P, Q two skew oscillating tableaux of $SOT_n(\alpha \to \beta)$, such that $RS_{SO}(\pi, T, U) = (P, Q)$. Then $RS_{SO}(\pi^{-1}, U^c, T^c) = (Q, P)$.*

Corollary 5.1. ([4]) *RS_{SO} induces a bijection between $SOT_n(\alpha \to \beta)$ and the pairs (σ, T) such that $\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$, $\zeta^{-1}(\sigma)\dot{\cup}T\dot{\cup}T^c = [2n]$.*

Example 5.6. Let P be the following skew oscillating tableau.

$P =$ 

The pair (P, P) is in bijection with (π, T, U):

$$\pi = \begin{pmatrix} 11 & 9 & 8 \\ 4 & 2 & 5 \end{pmatrix} = \pi^{-1},\ T = \begin{array}{lll} 12 & & \\ & 3 & \\ & & 7 \end{array},\ U = \begin{array}{lll} 1 & & \\ & 10 & \\ & & 6 \end{array} = T^c$$

and, as $\pi = \pi^{-1}$ and $U = T^c$, we have P in bijection with (σ, T):

$$\sigma = (\mathbf{1})(\mathbf{3})(2,4)(5)(\mathbf{6}),\ T = \begin{array}{lll} 12 & & \\ & 3 & \\ & & 7 \end{array}.$$

◇

Moreover, using Theorem 4.4 and Lemma 5.1 we can prove a new result on the number of odd height columns of the final shape of a skew oscillating tableau.

Theorem 5.2. *Let σ be an involution of INV_n^c, $T \in PT(\alpha/\mu)$ and $P \in SOT_n(\alpha \to \beta)$, such that (σ, T) is in bijection with P following Corollary 5.1. Then $odd(\beta) = odd(\mu) + fix_N(\sigma)$.*

We can verify this result on the previous example (Example 5.6), where the final shape β of P has two odd columns, μ has one odd column and σ has one normal fixed point.

6. Knuth classes for oscillating and skew oscillating tableaux

Another classical property of the Robinson-Schensted correspondence is due to Knuth ([7]). He gives a characterization of permutations having the same insertion tableau (this result was later extended to give a characterization of permutations having the same recording tableau).

Definition 6.1. *Two partial permutations on $[n]$, σ and σ', differ by a Knuth relation if and only if there are three integers a, b and c $(1 \le a < b < c \le n)$ such that*

$$\sigma = j_1 \dots j_k\ b\ a\ c\ j_{k+4} \dots j_p \text{ and } \sigma' = j_1 \dots j_k\ b\ c\ a\ j_{k+4} \dots j_p, \text{ or}$$
$$\sigma = j_1 \dots j_k\ a\ c\ b\ j_{k+4} \dots j_p \text{ and } \sigma' = j_1 \dots j_k\ c\ a\ b\ j_{k+4} \dots j_p.$$

Definition 6.2. *Two partial permutations on $[n]$, σ and σ', such that the elements of σ are $j_1 < j_2 < \dots < j_p$ and those of σ' are $j'_1 < j'_2 < \dots < j'_p$, differ by a dual Knuth relation if and only if there is an integer k such that*

$$\sigma = j_{i_1} \cdots j_{k+1} \cdots j_k \cdots j_{k+2} \cdots j_{i_p} \text{ and } \sigma' = j'_{i_1} \cdots j'_{k+2} \cdots j'_k \cdots j'_{k+1} \cdots j'_{i_p}, \text{ or}$$
$$\sigma = j_{i_1} \cdots j_{k+1} \cdots j_{k+2} \cdots j_k \cdots j_{i_p} \text{ and } \sigma' = j'_{i_1} \cdots j'_k \cdots j'_{k+2} \cdots j'_{k+1} \cdots j'_{i_p}.$$

Example 6.1. 5 3 4 1 7 6 and 5 3 4 7 1 6 differ by a Knuth relation of the form $(b\ a\ c \to b\ c\ a)$.
$\underline{3}$ 10 $\underline{1}$ 5 $\underline{2}$ and $\underline{4}$ 9 $\underline{2}$ 8 $\underline{7}$ differ by a dual Knuth relation of the form $(j_{k+2} \cdots j_k \cdots j_{k+1} \to j'_{k+1} \cdots j'_k \cdots j'_{k+2})$ (the elements involved by this transformation are underlined). ◇

Definition 6.3. *Two partial permutations on $[n]$, σ and σ', are K-equivalent, $\sigma \simeq_K \sigma'$ (respectively K'-equivalent, $\sigma \simeq_{K'} \sigma'$), if and only if there are $\sigma_1, \ldots, \sigma_p$, partial permutations on $[n]$, such that $\sigma_1 = \sigma$, $\sigma_p = \sigma'$, σ_i and σ_{i+1} differ only by a Knuth relation (respectively a dual Knuth relation).*

The result of Knuth is the following.

Theorem 6.1. ([7]) *Let σ and σ' be two permutations of S_n in bijection, by the Robinson-Schensted correspondence, repectively with (P, Q) and (P', Q').*

1. *$P = P'$ if and only if $\sigma \simeq_K \sigma'$.*
2. *$Q = Q'$ if and only if $\sigma \simeq_{K'} \sigma'$.*

Now, we extend this result to oscillating tableaux.

Definition 6.4. *Two biwords π and π' of BW_n are K-equivalent, $\pi \simeq_K \pi'$ (respectively, K'-equivalent, $\pi \simeq_{K'} \pi'$), if and only if there are $\pi_1, \ldots, \pi_p$, biwords of BW_n, such that $\pi_1 = \pi$, $\pi_p = \pi'$, and $\breve{\pi}_i$ and $\breve{\pi}_{i+1}$ differ only by a Knuth relation (respectively $\hat{\pi}_i = \hat{\pi}_{i+1}$, $\breve{\pi}_i$ and $\breve{\pi}_{i+1}$ differ by a dual Knuth relation).*

Definition 6.5. *Let π be a biword of BW_{2n}. The three biwords π_h, π_l and π_c are defined by*

$$\pi_h = \{(i_l, j_l) | i_l > n,\ j_l > n\},\ \pi_l = \{(i_l, j_l) | i_l \leq n,\ j_l \leq n\},\ \pi_c = \{(i_l, j_l) | i_l > n,\ j_l \leq n\}.$$

Theorem 6.2. *Let π and π' be two biwords of BW_{2n} such that $RS_O(\pi) = (P, Q)$ and $RS_O(\pi') = (P', Q')$.*

1. *$P = P'$ if and only if $\pi_l = \pi'_l$ and $\pi_c \simeq_K \pi'_c$.*
2. *$Q = Q'$ if and only if $\pi_h = \pi'_h$ and $\pi_c \simeq_{K'} \pi'_c$.*

Example 6.2. Let π and π' be the following biwords.

$$\pi = \begin{pmatrix} 14 & 13 & 12 & 11 & 10 & 7 & 4 \\ 9 & 5 & 3 & 8 & 6 & 1 & 2 \end{pmatrix} \pi' = \begin{pmatrix} 14 & 12 & 11 & 10 & 9 & 7 & 4 \\ 13 & 5 & 6 & 8 & 3 & 1 & 2 \end{pmatrix}.$$

If $RS_O(\pi) = (P, Q)$ and $RS_O(\pi') = (P', Q')$, then $P = P'$ because

$$\pi_l = \pi'_l = \begin{pmatrix} 7 & 4 \\ 1 & 2 \end{pmatrix} \text{ and } \pi_c = \begin{pmatrix} 13 & 12 & 10 \\ 5 & 3 & 6 \end{pmatrix} \pi'_c = \begin{pmatrix} 12 & 11 & 9 \\ 5 & 6 & 3 \end{pmatrix} \Rightarrow \breve{\pi}_c = 5\ 3\ 6 \simeq_K 5\ 6\ 3 = \breve{\pi}'_c.$$

◇

Using this result and Lemma 5.1, we have a characterization of Knuth classes and dual Knuth classes for skew oscillating tableaux. We have no simple necessary and sufficient conditions on the triples (π, T, U) of Theorem 3.2 to express these notions of Knuth classes and dual Knuth classes, but we have simple necessary conditions.

Definition 6.6. *Let P be a tableau of $PT(\lambda/\mu)$ or $\overline{PT}(\lambda/\mu)$ and k a fixed integer.*
We denote $P_{>k}$ the restriction of P to cells $P(i,j) > k$ obtained by emptying (suppressing if $P \in \overline{PT}(\lambda/\mu)$) the cells $P(i,j) \leq k$. We define analogously $P_{\geq k}$, $P_{<k}$ and $P_{\leq k}$.

Example 6.3.

$$P = \begin{array}{|c|c|c} \hline 3 & 4 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \Rightarrow P_{<4} = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array},\quad P = \begin{array}{|c|c|c} \hline 3 & 2 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \Rightarrow P_{<4} = \begin{array}{|c|c|c} \hline 3 & 2 & \\ \hline & & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

◇

Corollary 6.1. *Let (π, T, U) and (π', T', U') with $RS_{SO}(\pi, T, U) = (P, Q)$ and $RS_{SO}(\pi', T', U') = (P', Q')$.*

1. *If $P = P'$, then $\pi_l = \pi'_l$ and $U_{\leq n} = U'_{\leq n}$.*
2. *If $Q = Q'$ then $\pi_h = \pi'_h$ and $T_{\geq n} = T'_{\geq n}$.*

7. Beissinger algorithm

The Robinson-Schensted correspondence implies a bijection between involutions and standard tableaux (an involution σ is in bijection with a pair of standard tableaux (P, P)). In [1], Beissinger gives a simple algorithm to directly obtain the standard tableau P in bijection with σ without using the Robinson-Schensted correspondence. In this section, we extend this result to the case skew oscillating tableaux (see Corollary 5.1).

First, we recall briefly the algorithm of Beissinger. Let σ be an involution written as in Section 2.

Algorithm 4. *$Beiss(\sigma)$ - The output is a tableau P.*
Let $P_0 = \emptyset$.
For i from 1 to k, the number of cycles of σ:
(a) if the i^{th} cycle c_i of σ is a fixed point (a), then $P_i = ExtIns(P_{i-1}, a)$.
(b) else ($c_i = (a, b)$), $P_i' = ExtIns(P_{i-1}, a)$, this insertion ending in row u, and add b at the end of row $u+1$ of P_i' to obtain P_i.
Finally, $P = P_k$.

Example 7.1. Let $\sigma = (3)(1,5)(4,6)(2,7)(8)(9)$. The execution of $Beiss(\sigma)$ gives the following result.

i	0	1	2	3	4	5	6
P_i	$\emptyset$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 5\\\hline 3\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c}\hline 5\\\cline{1-2} 3&\multicolumn{1}{c\|}{6}\\\hline 1&\multicolumn{1}{c\|}{4}\\\hline\end{array}$	$\begin{array}{\|c\|c}\hline 7\\\cline{1-2} 5&\multicolumn{1}{c\|}{6}\\\hline 3&\multicolumn{1}{c\|}{4}\\\hline 1&\multicolumn{1}{c\|}{2}\\\hline\end{array}$	$\begin{array}{\|c\|c\|c}\hline 7\\\cline{1-2} 5&6\\\hline 3&4\\\hline 1&2&\multicolumn{1}{c\|}{8}\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 7\\\cline{1-2} 5&6\\\cline{1-2} 3&4\\\hline 1&2&8&9\\\hline\end{array} = P$

$\diamond$

Theorem 7.1. ([1]) *Let $\sigma \in INV_n$ and $P = Beiss(\pi)$. Then σ is in bijection, by the Robinson-Schensted correspondence for standard tableaux, with (P, P).*

Now, we propose a similar algorithm related to the Robinson-Schensted correspondence for skew oscillating tableaux. This algorithm, denoted $Beiss_{SO}$, has for input a pair (σ, T) satisfying the conditions of Corollary 5.1, for output a triple (π', T', U') where π is a biword, T and P two partial tableaux and has the following property, that is a generalization of Theorem 7.1.

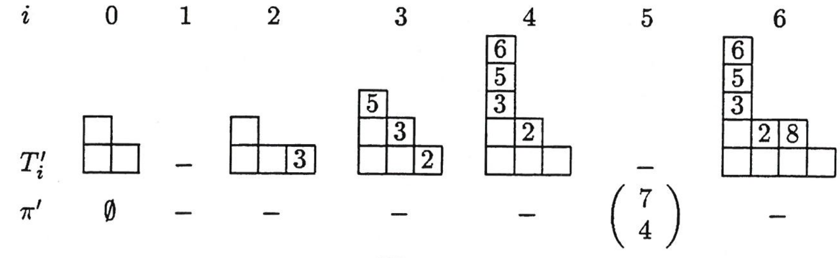
Theorem 7.2. *Let $P \in SOT_n(\alpha \to \beta)$, $\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$ and $\pi = \zeta^{-1}(\sigma)$ such that $RS_{SO}^{-1}(P, P) = (\pi, T, T^c)$. Then $\Phi_{SO}(Beiss_{SO}(\sigma, T)) = P$.*

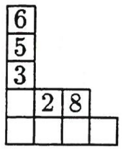
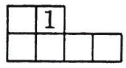
Algorithm 5. *$Beiss_{SO}(\sigma, T)$ - The output is a triple (π', T', U').*
($\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$) $\pi' = \emptyset$, $T_0' = \mu$, $U' = T^c_{<n+1}$.
For i from 1 to the number of cycles of σ:
(a) if the i^{th} cycle c_i of σ is such that $c_i = (\mathbf{a}, \mathbf{b})$, then add (a, b) to π', $T_i' = T_{i-1}'$.
(b) else if $c_i = (a, b)$, then $T_i'' = ExtIns(T_{i-1}', a)$, this insertion ending in row q, and adding b at the end of row $q+1$ of T_i'' gives T_i'.
(c) else if $c_i = (a)$, then $T_i' = ExtIns(T_{i-1}', a)$.
(d) else ($c_i = (\mathbf{a})$),
(d.1) if $a = T_{<n+1}(u, v)$ and $T'_{i-1}(u, v)$ exists, then $T_i'' = IntIns(T_{i-1}', u, v)$, this insertion ending in row q, and adding a at the end of row $q+1$ of T_i'' gives T_i'.
(d.2) else if $a = T_{<n+1}(u, v)$ and the cell $T'_{i-1}(u, v)$ doesn't exist, then $T_i'' = EmptyIns(T_{i-1}', u, v)$, and adding a at the end of row $u+1$ of T_i'' gives T_i'.
(d.3) else $a = T^c_{<n+1}(u, v)$, $T_i' = T_{i-1}'$.
Finally, $T' = T_k$.

Example 7.2. Let (σ, T) be the following pair,

$$\left(\sigma = (\mathbf{1})(3)(2,5)(\mathbf{6})(\mathbf{4},\mathbf{7})(\mathbf{8}),\ T = \begin{array}{|c|c|cc}\hline & 16 \\ \hline & & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{8} \\ \hline\end{array}\right) \Rightarrow T_{<9} = \begin{array}{|c|ccc}\cline{1-1} \\ \hline & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{8} \\ \hline\end{array},\ T^c_{<9} = \begin{array}{|c|c|cc}\hline & 1 \\ \hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline\end{array}$$

$Beiss_{SO}(\sigma, T)$ has the following behaviour



$$\pi' = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \; T' =$$  $$, \; U' =$$ .

◇

Remark 7.1. Using Theorem 7.2, we have a new and simple proof of Theorem 5.2.

8. Conclusion

As stated in [4], the Robinson-Schensted correspondence for the skew oscillating tableaux defined by Dulucq and Sagan is a very natural extension of the Robinson-Schensted correspondence, and, following the ideas of Sagan and Stanley in [10], it seems reasonable to hope find another properties of this correspondence, in particular, with the notions of generalized insertion and generalized deletion (see [6]). On the other hand, it could be interesting to study the class of shifted skew oscillating tableaux (see [8, 17]). Finally, there are few other classical properties of the Robinson-Schensted correspondence we didn't extend, like the "Jeu de taquin" of Schützenberger ([13]) or the "vidage-remplissage" ([12, 5], extended in the case of oscillating tableaux in [3]).

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