Distance-Regular Graphs Related to the Quantum Enveloping Algebra of $sl(2)$

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Summary. We present a connection between distance-regular graphs and the quantum enveloping algebra $U_q(sl(2))$ of the Lie algebra $sl(2)$. Let Γ be a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$ which is not isomorphic to the d-cube. Fix a vertex x and let $\mathcal{T} = \mathcal{T}(x)$ denote the Terwilliger algebra of Γ. Then $\mathcal{T}$ is generated by certain matrices satisfying the defining relations of $U_q(sl(2))$ for some complex number $q \notin \{0, 1, -1\}$ if and only if Γ is bipartite and 2-homogeneous in the sense of Nomura.

Nous présentons une connexion entre des graphes distance-réguliers et l'algèbre enveloppante quantique $U_q(sl(2))$ de l'algèbre de Lie $sl(2)$. Soit Γ un graphe distance-régulier de diamètre $d \geq 3$ et de valence $k \geq 3$ qui n'est pas isomorphe à un hypercube. Soit x un sommet de Γ et soit $\mathcal{T} = \mathcal{T}(x)$ l'algèbre de Terwilliger de Γ. Alhors $\mathcal{T}$ est généré par certaines matrices qui satisfait les relations de la définition de $U_q(sl(2))$ pour un nombre complexe $q \notin \{0, 1, -1\}$ si et seulement si Γ est bipartite et 2-homogne dans le sens de Nomura.

Extended abstract.

This poster is based upon [4]. We present a connection between distance-regular graphs and $U_q(sl(2))$, the quantum enveloping algebra of the Lie algebra $sl(2)$. It is well known that there is a "natural" $sl(2)$ action on the d-cubes (see Proctor [10] or Go [5]). Here we describe the distance-regular graphs with a similar natural $U_q(sl(2))$ action. We show that these graphs are precisely the bipartite distance-regular graphs which are 2-homogeneous in the sense of [8, 9], excluding the d-cubes. To state this precisely, we recall some definitions.

Let $U(sl(2))$ denote the unital associative $\mathbb{C}$-algebra generated by $\mathcal{X}^-$, $\mathcal{X}^+$, and $\mathcal{Z}$ subject to the relations

$$\mathcal{Z}\mathcal{X}^- - \mathcal{X}^-\mathcal{Z} = 2\mathcal{X}^-, \quad \mathcal{Z}\mathcal{X}^+ - \mathcal{X}^+\mathcal{Z} = -2\mathcal{X}^+, \quad \mathcal{X}^-\mathcal{X}^+ - \mathcal{X}^+\mathcal{X}^- = \mathcal{Z}. \qquad (1)$$

$U(sl(2))$ is called the *universal enveloping algebra of* $sl(2)$. For any complex number q satisfying

$$q \neq 1, \qquad q \neq 0, \qquad q \neq -1, \qquad (2)$$

let $U_q(sl(2))$ denote the unital associative $\mathbf{C}$-algebra generated by $\mathcal{X}^-$, $\mathcal{X}^+$, $\mathcal{Y}$, and $\mathcal{Y}^{-1}$ subject to the relations

$$\mathcal{Y}\mathcal{Y}^{-1} = \mathcal{Y}^{-1}\mathcal{Y} = 1, \tag{3}$$

$$\mathcal{Y}\mathcal{X}^- = q^2\mathcal{X}^-\mathcal{Y}, \quad \mathcal{Y}\mathcal{X}^+ = q^{-2}\mathcal{X}^+\mathcal{Y}, \quad \mathcal{X}^-\mathcal{X}^+ - \mathcal{X}^+\mathcal{X}^- = \frac{\mathcal{Y} - \mathcal{Y}^{-1}}{q - q^{-1}}. \tag{4}$$

$U_q(sl(2))$ is called the *quantum enveloping algebra of* $sl(2)$. For more on $U_q(sl(2))$, see [6, 7].

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges and having vertex set X, edge set R, distance function ∂, and diameter d. Γ is said to be *distance-regular* whenever for all integers ℓ, i, j $(0 \le \ell, i, j \le d)$ there exists a scalar p^ℓ_{ij} such that for all x, $y \in X$ with $\partial(x, y) = \ell$, $|\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}| = p^\ell_{ij}$. Assume that Γ is distance-regular. Set $c_0 = 0$, $c_i = p^i_{1i-1}$ $(1 \le i \le d)$, $a_i = p^i_{1i}$ $(0 \le i \le d)$, $b_i = p^i_{1i+1}$ $(0 \le i \le d-1)$, and $b_d = 0$. Γ is regular with valency $k = b_0 = p^0_{11}$, and $c_i + a_i + b_i = k$ $(0 \le i \le d)$. Γ is bipartite precisely when $a_i = 0$ $(0 \le i \le d)$. For more on distance-regular graphs, see [1, 2].

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph. Γ is said to be *2-homogeneous* whenever for all integers i $(1 \le i \le d)$ there exists a scalar γ_i such that for all x, y, $z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i$, $\partial(y, z) = 2$, $|\{w \in X \mid \partial(x, w) = i, \partial(y, w) = 1, \partial(z, w) = 1\}| = \gamma_i$. The *d-cube* is the graph with vertex set $X = \{0, 1\}^d$ (the d-tuples with entries in $\{0, 1\}$) such that two vertices are adjacent if and only if they differ in precisely one coordinate. The d-cube is a 2-homogeneous bipartite distance-regular graph with $\gamma_i = 1$ $(1 \le i \le d-1)$. In the d-cube, there is a unique vertex at distance d from any given vertex, so the d-cube is 2-homogeneous despite the fact that γ_d is not defined. The 2-homogeneous bipartite distance-regular graphs have been studied in [3, 4, 9, 12].

Let $Mat_X(\mathbf{C})$ denote the $\mathbf{C}$-algebra of matrices with rows and columns indexed by X. Let $A \in Mat_X(\mathbf{C})$ denote the adjacency matrix of Γ. For the rest of this section fix $x \in X$. For all i $(0 \le i \le d)$, define $E^*_i = E^*_i(x)$ to be the diagonal matrix in $Mat_X(\mathbf{C})$ such that for all $y \in X$, E^*_i has (y, y)-entry equal to 1 if $\partial(x, y) = i$, and 0 otherwise. Let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $Mat_X(\mathbf{C})$ generated by A, E^*_0, E^*_1, …, E^*_d. $\mathcal{T}$ is called the *Terwilliger algebra of* Γ *with respect to* x. For more on Terwilliger algebras, see [11]. We set $L = \sum_{i=0}^{d-1} E^*_i A E^*_{i+1}$, $F = \sum_{i=0}^{d} E^*_i A E^*_i$, and $R = \sum_{i=1}^{d} E^*_i A E^*_{i-1}$.

Proctor [10] showed that if Γ is isomorphic to the d-cube, then the matrices $X^- = L$, $X^+ = R$, and $Z = \sum_{i=0}^{d}(d-2i)E^*_i$ satisfy the relations (1) (see also Go [5]). In fact, we may consider matrices of a slightly more general form:

$$X^- = \sum_{i=0}^{d-1} x^-_i E^*_i A E^*_{i+1}, \qquad X^+ = \sum_{i=1}^{d} x^+_i E^*_i A E^*_{i-1}, \qquad Z = \sum_{i=0}^{d} z_i E^*_i, \tag{5}$$

where x^-_i, x^+_i, z_i are arbitrary complex scalars.

Theorem 1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Fix $x \in X$, and write $E^*_i = E^*_i(x)$, $\mathcal{T} = \mathcal{T}(x)$. Let X^-, X^+, and Z be any matrices of the form (5). Then the following are equivalent.*

(i) *X^-, X^+, and Z generate $\mathcal{T}$ and satisfy (1).*

(ii) *Γ is isomorphic to the d-cube, and*

$$\begin{aligned} x^-_i x^+_{i+1} &= 1 \qquad (0 \le i \le d-1), \\ z_i &= d - 2i \qquad (0 \le i \le d). \end{aligned}$$

While studying $U_q(sl(2))$ structures, we shall consider matrices of the form:

$$X^- = \sum_{i=0}^{d-1} x_i^- E_i^* A E_{i+1}^*, \qquad X^+ = \sum_{i=1}^{d} x_i^+ E_i^* A E_{i-1}^*, \qquad Y = \sum_{i=0}^{d} y_i E_i^*, \tag{6}$$

where x_i^- $(0 \le i \le d-1)$, x_i^+ $(1 \le i \le d)$, y_i $(0 \le i \le d)$ are arbitrary complex scalars. Observe that Y is invertible if and only if $y_i \ne 0$ $(0 \le i \le d)$, in which case $Y^{-1} = \sum_{i=0}^{d} y_i^{-1} E_i^*$.

Theorem 2 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Assume that Γ is not isomorphic to the d-cube. Fix $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \le i \le d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let X^-, X^+, and Y be any matrices of the form (6), and let q be any complex number. Then the following are equivalent.*

(i) *Y is invertible, X^-, X^+, Y, Y^{-1} generate $\mathcal{T}$, and (2)-(4) hold.*

(ii) *Γ is bipartite and 2-homogeneous, $(q+q^{-1})^2 = c_2^2 b_2^{-1}(k-2)(c_2-1)^{-1}$, and there exists $\epsilon \in \{1, -1\}$ such that*

$$\begin{aligned} y_i &= \epsilon q^{d-2i} && (0 \le i \le d), \\ x_i^- x_{i+1}^+ &= \epsilon q^{-2i+1}(q^d + q^{2i})(q^d + q^{2i+2})(q^d + q^2)^{-2} && (0 \le i \le d-1). \end{aligned}$$

The factor of ϵ appears in (ii) because the defining relations of $U_q(sl(2))$ are invariant under changing the signs of any two of $\mathcal{X}^-$, $\mathcal{X}^+$, and $\mathcal{Y}$.

The proofs of Theorems 1 and 2 are similar. The following combinatorial interpretations of L and R allow us to translate the 2-homogeneous and bipartite conditions into algebraic relations in $\mathcal{T}$. This is the first step in the proof. For the moment, identify each vertex with its characteristic column vector so that $Mat_X(\mathbf{C})$ acts by left multiplication. Then for all i $(0 \le i \le d)$ and all $y \in \Gamma_i(x)$, $Ly = \sum w$, where the sum runs over all $w \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$, $Fy = \sum w$, where the sum runs over all $w \in \Gamma_1(y) \cap \Gamma_i(x)$, $Ry = \sum w$, where the sum runs over all $w \in \Gamma_1(y) \cap \Gamma_{i+1}(x)$, and $E_j^* y = \delta_{ij} y$. Observe that Γ is bipartite if and only if $F = 0$. Moreover, for all i $(0 \le i \le d)$ and for all $y, z \in \Gamma_i(x)$ $(LRE_i^*)(y,z) = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i+1}(x)|$ $(RLE_i^*)(y,z) = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)|$. With this observation it is not hard to show that a bipartite distance-regular graph is 2-homogeneous if and only if LRE_i^*, RLE_i^*, and E_i^* are linearly dependent for all i $(0 \le i \le d)$.

The next step is to demonstrate a $U(sl(2))$ structure on the hypercubes and a $U_q(sl(2))$ structure on the remaining 2-homogeneous bipartite distance-regular graphs. We compute the precise dependence relation for LRE_i^*, RLE_i^*, and E_i^*, after which it is easy to verify that there is a $U(sl(2))$ or $U_q(sl(2))$ structure. To compute the coefficients of the dependence for the hypercubes, we use the fact that the intersection numbers satisfy $c_i = i$, $b_i = d - i$ $(0 \le i \le d)$ and $\gamma_i = 1$ $(1 \le i \le d-1)$. In the case of the remaining 2-homogeneous bipartite distance-regular graphs, we use the following result.

Theorem 3 *([3, Theorem 35]) Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Suppose Γ is not isomorphic to the d-cube. Then Γ is bipartite and 2-homogeneous if and only if there exists a complex number $q \notin \{0, 1, -1\}$ such that*

$$c_i = e_i[\,i\,], \qquad b_i = e_i[\,d-i\,] \qquad (0 \le i \le d),$$

where $e_i = q^{i-1}(q^d + q^2)(q^d + q^{2i})^{-1}$ and $[\,n\,] = (q^n - q^{-n})(q - q^{-1})^{-1}$. Moreover, any such q is real, and

$$\gamma_i = e_2 e_i e_{i+1}^{-1} \qquad (1 \le i \le d-1).$$

Now to prove Theorems 1 and 2 (i)$\Rightarrow$(ii) we use a technical lemma which is applicable to both the $U(sl(2))$ and $U_q(sl(2))$ structures. Most of the work goes into proving the following result.

Lemma 4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, $\mathcal{T} = \mathcal{T}(x)$. Suppose that $\mathcal{T}$ is generated by $\{X^-, X^+, E_0^*, E_1^*, \ldots, E_d^*\}$ and $X^-X^+ - X^+X^- \in \text{span}\{E_0^*, E_1^*, \ldots, E_d^*\}$, where X^- and X^+ are as in (5) or (6). Then Γ is bipartite and 2-homogeneous.*

Once Lemma 4 is proved, we complete the proofs of Theorems 1 and 2 (i)$\Rightarrow$(ii) by comparing the previously demonstrated $U(sl(2))$ or $U_q(sl(2))$ structure on the 2-homogeneous bipartite distance-regular graphs to that which is assumed in (i).

The proofs of Theorems 1 and 2 (ii)$\Rightarrow$(i) require two steps. The first is to show that the assumptions of (ii) imply a $U(sl(2))$ or $U_q(sl(2))$ structure (according to which theorem we are proving). This is straight forward given the already demonstrated structures. The second step is to show that $\mathcal{T}$ has the desired generators. First we show that E_0^*, E_1^*, ..., E_d^* are polynomials in Z or Y, according to which case we are in. Then we express R and L, and thus $A = R + L$, in terms of X^-, X^+, and the E_i^*. This shows that $\mathcal{T}$ has the desired generators and completes the proof of Theorems 1 and 2.

Remark 5 The intersection numbers of 2-homogeneous bipartite distance-regular graphs are determined in [9] as follows. Excluding the hypercubes, there are three infinite families with $d \geq 3$ and $k \geq 3$. Their intersection arrays $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ are

(i) $\{k, k-1, 1; 1, k-1, k\}$, $k \geq 3$.

(ii) $\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\}$ for γ a positive integer.

(iii) $\{k, k-1, k-\mu, \mu, 1; 1, \mu, k-\mu, k-1, k\}$, with $k = \gamma(\gamma^2 + 3\gamma + 1)$, $\mu = \gamma(\gamma + 1)$ for $\gamma \geq 2$, an integer.

These arrays are realized by the following graphs: (i) complement of the $2 \times (k+1)$-grid; (ii) Hadamard graphs of order 4γ; (iii) antipodal 2-cover of the Higman-Sims graph when $\gamma = 2$. No examples of (iii) with $\gamma \geq 3$ are known.

Remark 6 We know of a few other distance-regular graphs related to $U_q(sl(2))$ and $U(sl(2))$. Suppose Γ is a $2d$-cycle ($d \geq 2$). Observe that Γ is vacuously 2-homogeneous. Let q be a primitive $2d^{\text{th}}$ root of unity, and set $X^- = \sum_{i=0}^{d-1}[d-i]E_i^*AE_{i+1}^*$, $X^+ = \sum_{i=1}^{d}[i]E_i^*AE_{i-1}^*$, and $Y = \sum_{i=0}^{d} q^{d-2i}E_i^*$. Then X^-, X^+, and Y satisfy (4). However, these matrices do not generate $\mathcal{T}$. In addition to the $U(sl(2))$ structure of Theorem 1, the 4-cycle has the $U_q(sl(2))$ structure of Theorem 2 whenever $q^4 \neq 1$.

Suppose Γ is the Hamming graph $H(d, n)$, $n \geq 3$. The results of [11, p. 202] can be used to show that $X^- = L$, $X^+ = R$, and $Z = LR - RL$ satisfy (1). However, these matrices do not generate $\mathcal{T}$ and Z is not of the form (5).

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