A q-Enumeration of Directed Diagonally Convex Polyominoes

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Abstract

We q-enumerate directed diagonally convex (*ddc-*) polyominoes by an approach which partly goes column by column, and partly goes row by row. In the end we obtain fairly nice formulas.

1. Introduction

Ddc-polyominoes originate from (and are, in fact, equivalent to) so-called *fully directed compact (fdc-) lattice animals.* And fdc-animals are young: physicists Bhat, Bhan and Singh introduced them in 1986 [1].

The early publications [1] and Privman & Švrakić [15] are focused on the number (say p_n) of fdc-animals with cardinality n. In [1], the authors argue that p_n is asymptotically equal to λ^n, where $\lambda = 2.66185 \pm 0.00005$. In [15], the function $D = \sum_{n\geq 1} p_n q^n$ is derived exactly for the first time. (This D is, in fact, the area gf for ddc-polyominoes.)

Counting ddc-polyominoes by perimeter was first undertaken by Delest and Fédou [5]. Let r_k be the number of ddc-polyominoes with *site perimeter* $k+1$ (that is to say, with k diagonals). One of the results of [5] is that r_k equals the number of ternary trees with k internal nodes. That is,

$$r_k = \frac{1}{3k+1} \cdot \binom{3k+1}{k} \quad . \tag{1}$$

By now, this nice fact about r_k has been established in several different ways: in [5], there are the original algebraic-language proof as well as a bijective proof. Then there are two other bijective proofs: an earlier one by Penaud [13], and a more recent one by Svrtan and Feretić [9]. (In [9], the bijection is relatively simple, and is no longer defined recursively.) And then there is– also in [9]– a proof based on Raney's generalized lemma [11, p. 348].

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Further, when ddc-polyominoes with k diagonals are enumerated, then one of the probabilistic *percolation models* can be solved with (a bit) greater accuracy. Having realized this fact, Bousquet-Mélou [3] and Inui *et al.* [12] gave yet two different derivations of (1).

Ddc-polyominoes have an interesting "companion". It is the generating function (*gf*) D, whose variables are the following: $d =$ diagonals, $x =$ horizontal semiperimeter, and $y =$ vertical semiperimeter. The function D is algebraic, and satisfies the equation

$$D = d(D+1)(D+x)(D+y) \quad . \tag{2}$$

Equation (2) first appeared in Svrtan and Feretić [9]. Then, on pp. 61–62 of her habilitation thesis [4], Bousquet-Mélou derived (2) in a new way. Namely, she took the approach called *object grammars* [6].

Let us now return to *q-enumeration* (that is, to enumeration by both area and some– or none– other properties). It was shown in [9], and was integrated with some corollaries in Feretić [7], that the q-enumeration of ddc-polyominoes may be done by applying Gessel's q-Lagrange inversion formula [10]. The resulting formula for the gf then involves both positive and negative powers of q. In that respect, the formula in question is unique.

Moreover, in [4, pp. 66–67], Bousquet-Mélou q-enumerated ddc-polyominoes by a certain method coming from her previous paper [2].

In the present paper we, too, shall q-enumerate ddc-polyominoes by the method of [2]. This does not mean, however, that we are going to make a copy of [4, pp. 66–67]. Indeed, our way of applying the method will be different, and our final result (*i.e.*, expression for the gf) will look simpler than those of [15] and [4, pp. 66–67].

Incidentally, our planned enumerations might as well be performed by Svrtan's method [8] for solving the Temperley recurrences [16]. In fact, although the methods of [2] and [8] were developed independently, they are pretty close to each other.

This paper now continues as follows. In Section 2, we state the necessary definitions and conventions. In Sections 3 and 4, we q-enumerate so-called escalier polyominoes, as well as certain close relatives of theirs; the name of those relatives is floorsitters. We are then able to state and solve our new functional equation for ddc-polyominoes, and we do so in Section 5.

2. Definitions and conventions

If c is a closed unit square in the Cartesian plane, and if the vertices of c have integer coordinates, then c is called a *cell*.

Imagine one or more cells which all lie in the same vertical strip of width one. If connected, the union of those cells is called a *column*.

A *row* is a column rotated by 90 degrees.

Let $K_1, \ldots, K_r$ $(r \in N)$ be columns. Suppose that, for $i{=}2, \ldots, r$, the following holds:

- the bottom cell of K_i is the right neighbor of the bottom cell of K_{i-1},
- compared with K_{i-1}, the column K_i is lower by one unit, or equally high, or higher by ≥ 1 units.

The union $\bigcup_{i=1}^{r} K_i$ is then an *escalier polyomino.*

Incidentally, the term "polyomino escalier"– coined by the Bordeaux group [14, p. *xi*]– is a bit problematical, because the English word for *escalier* is *staircase*, and the name *staircase polyomino* is commonly used for another object.

Let $c_1, \ldots, c_j$ $(j \in N)$ be cells, and let c_i $(i = 2, \ldots, j)$ be the lower neighbor of the right neighbor of c_{i-1}. Further, let s_0 be the upper neighbor of c_1, and let s_i $(i = 1, \ldots, j)$ be the right neighbor of c_i. Then the union $\bigcup_{i=1}^{j} c_i$ is a *diagonal*, and the union $\bigcup_{i=0}^{j} s_i$ is the *shadow* of that diagonal.

Let $D_1, \ldots, D_k$ $(k \in N)$ be diagonals such that D_1 has just one cell, and such that D_i $(i = 2, \ldots, k)$ lies in the shadow of D_{i-1}. Then

- the union $P = \bigcup_{i=1}^{k} D_i$ is a *directed diagonally convex polyomino* (a *ddc--polyomino*),
- the only cell of D_1 is the *source cell* of P,
- the cells of D_k are *target cells* of P,
- the diagonals $D_1, D_2, \ldots, D_k$ are the *first, second, ...,kth* diagonals of P.

See Figure 1.

By the *floor* of a ddc-polyomino P we mean the horizontal line containing the lower side of P's source cell.

Let P be a ddc-polyomino. If every diagonal of P touches the floor of P, then P is a *floorsitter* [1].

Let $m \in N$. An *m-floorsitter* is a floorsitter with exactly m target cells.

So far we have said what are an escalier polyomino and a ddc-polyomino, but we have not said what is a *polyomino.* So let us say it: a *polyomino* is a union of cells which is finite and possesses connected interior. It is easy to verify that our escalier polyominoes and ddc-polyominoes are indeed polyominoes.

Finally, let us state that in this paper we again count polyominoes up to translations.

[1] As a matter of fact, floorsitters are nothing but escaliers with one-cell last columns.

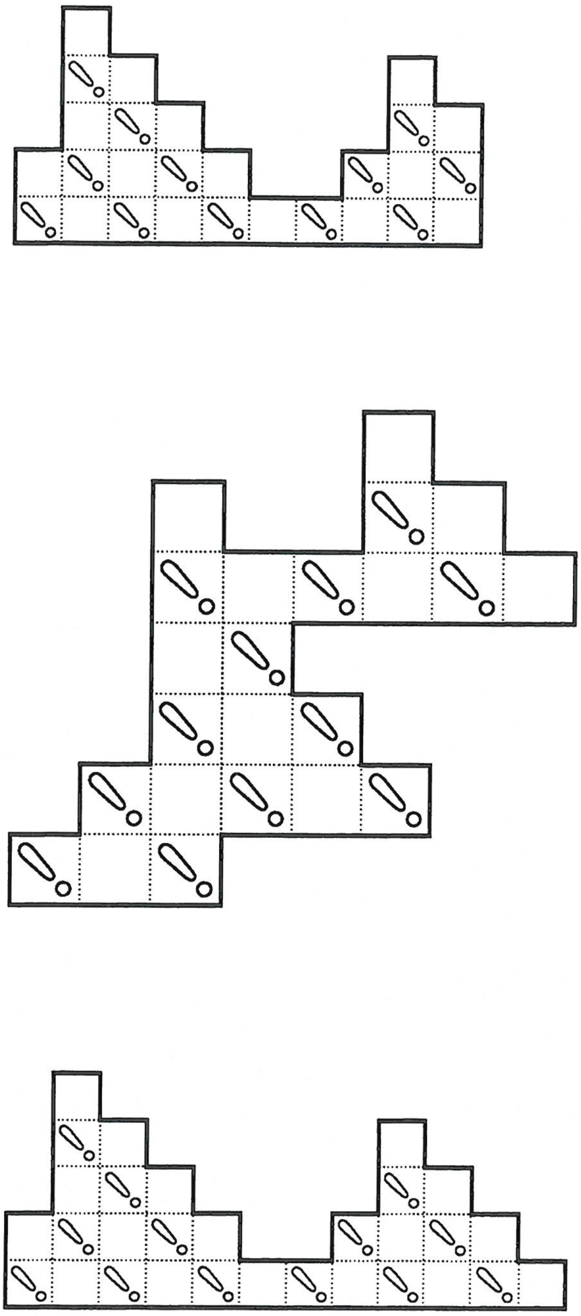

Figure 1: From top: An escalier polyomino, a directed diagonally convex (*ddc*-) polyomino, and a floorsitter.

3. Escalier polyominoes

In what follows, the "position" of the gf for escaliers will be occupied by the power series $E(s)$, which actually has five variables: $x =$ horizontal semiperimeter, $y =$ vertical semiperimeter, $q =$ area, $s =$ the height of the first column, and $u =$ the height of the last column.

For those of us who have read [2], the following two propositions will be a simple matter. For the rest of us, some related explanations are given in Section 5 of this paper.

Proposition 1. *The gf $E(s)$ satisfies the equation*

$$E(s) = \frac{xyqsu}{1-yqsu} + \frac{xqs}{1-qs} \cdot E(1) - \frac{xqs \cdot (1-y+yqs)}{1-qs} \cdot E(qs) \tag{3}$$

□

Proposition 2. *The gf for escaliers is given by*

$$E(1) = \frac{\sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i y q^{\binom{i+1}{2}} u \cdot \prod_{\ell=1}^{i-1} (1-y+yq^{\ell})}{(q)_{i-1}(1-yq^i u)}}{\sum_{i=0}^{\infty} \frac{(-1)^i x^i q^{\binom{i+1}{2}} \cdot \prod_{\ell=1}^{i-1} (1-y+yq^{\ell})}{(q)_i}}, \tag{4}$$

where every empty product is assumed to be one, and where $(q)_0 = 1$, $(q)_1 =$ $= 1-q$, $(q)_2 = (1-q)(1-q^2)$ *etc.*

□

4. Floorsitters

As we told in Section 2, a floorsitter is just an escalier with one-cell last column. So, to find the gf for floorsitters, it is enough to read off the coefficient of u^1 on the right-hand side of (4). (And that is easy.)

According to our program, however, counting *all* floorsitters is not the first thing to do here. Instead, we should count the j-floorsitters (*i.e.*, the floorsitters with j target cells).

So, let P be an escalier with $j \in N$ cells in the last column, and let S be the escalier produced by continuing P with $j-1$ new columns, whose heights are $j-1, j-2, \ldots, 1$ in that order.

What can we say about S? First, S is a floorsitter. Second, as witnessed by the top and bottom creatures in Figure 1, there is no guarantee that S has *exactly* j target cells. However, S has *at least* j such cells. And third, S has $j-1$ columns more than P, as well as $\frac{(j-1)j}{2}$ cells more than P.

Are we now able to write down some gf for floorsitters with $\geq j$ target cells? Yes, certainly: one such gf is given by

$$x^{j-1}q^{\binom{j}{2}} < u^j > E(1) \quad , \tag{5}$$

where $< u^j > E(1)$ denotes the coefficient of u^j in $E(1)$.

And what is more, we are able to add that

$$\left[x^{j-1}q^{\binom{j}{2}} < u^j > E(1)\right] - \left[x^j q^{\binom{j+1}{2}} < u^{j+1} > E(1)\right] \tag{6}$$

is a gf for floorsitters with exactly j target cells.

In (5) and (6), the gf's for floorsitters have three variables: $x =$ horizontal semiperimeter, $y =$ vertical semiperimeter and $q =$ area. But in what follows, by "the gf for floorsitters with j target cells" we shall mean the power series $f_j(s)$, which, in addition to the just mentioned variables x, y and q, also has the variables $d =$ diagonals and $s =$ floor-touching diagonals.

Proposition 3. *The gf for floorsitters with j target cells is given by*

$$f_j(s) = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i (dsx)^{i+j} y^j q^{\binom{i+j+1}{2}} \cdot \prod_{\ell=1}^{i-1}(1-y+yq^\ell)}{(q)_i}}{\sum_{i=0}^{\infty} \frac{(-1)^i (dsx)^i q^{\binom{i+1}{2}} \cdot \prod_{\ell=1}^{i-1}(1-y+yq^\ell)}{(q)_i}} \tag{7}$$

Proof. Formula (7) can be derived by combining (4) and (6), and then making the substitution $x = dsx$. The substitution works because, if P is a floorsitter, then bottoms of P's columns are also bottoms of P's diagonals and *vice versa.* Hence

the number of columns of P
=the number of diagonals of P
=the number of floor-touching diagonals of P

□

Our next proposition will show that the gf $f_j(s)$ admits of an interesting factorization. But let us first prepare the ground for that.

Let $\mathcal{D}$ stand for the set of all ddc-polyominoes, and let $\mathcal{F}_j$ stand for the set of j-floorsitters. For $P \in \mathcal{D}$, we shall use the following notations:

$di(P)$:= number of diagonals of P,
$ft(P)$:= number of floor-touching diagonals of P,
$h(P)$:= horizontal semiperimeter of P,
$v(P)$:= vertical semiperimeter of P,
$ce(P)$:= number of cells of P.

For $j \in N$, by $f_1^{[j]}(s)$ we shall mean the product $f_1(s) \cdot f_1(qs) \cdots f_1(q^{j-1}s)$.

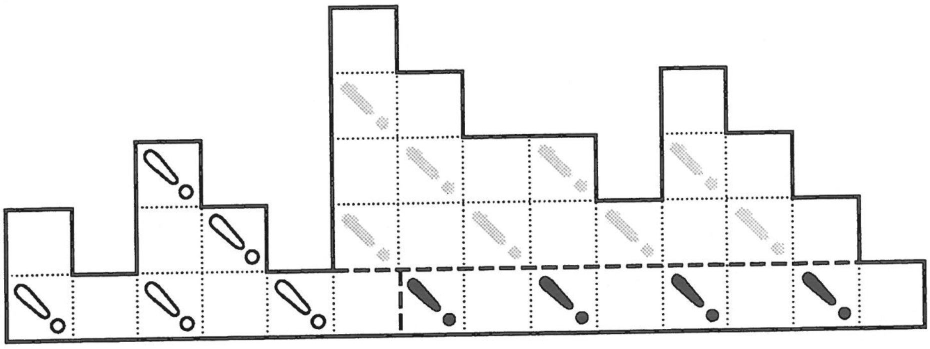

Figure 2: A 4-floorsitter decomposed into a 1-floorsitter (white !'s), a 3-floorsitter (gray !'s), and a row of cells (black !'s).

Proposition 4. *For every* $j \in N$, $f_j(s) = f_1^{[j]}(s)$.

Proof. For $j=1$ there is nothing to prove.

Suppose the assertion holds for $j=m$.

Induction step. Let a *big diagonal* be a diagonal consisting of at least two cells.

Let $P \in \mathcal{F}_{m+1}$. Let D_- be the last one-celled diagonal of P, and let D_+ be the diagonal immediately following D_- . The diagonal D_+ is big, but is anyway contained in the shadow of D_-. Accordingly, D_+ has exactly two cells.

Let S be the figure formed by those diagonals of P which occur not later than D_-. Let T be the figure formed by those diagonals of P which occur not earlier than D_+ .

The figure S is no doubt a 1-floorsitter.

Next, consider the horizontal line situated one unit above the floor of P. That line divides the figure T into two parts. The upper part (say U) is an element of $\mathcal{F}_m$, while the lower part (say V) is just a row of cells. See Figure 2. We have $di(P) = di(S) + di(U)$ together with similar decompositions for $ft(P)$, $h(P)$ and $v(P)$. On the other hand, since $ce(V) = ft(U)$, we have $ce(P) = ce(S) + ce(U) + ft(U)$.

In addition, the mapping $P \mapsto (S, U)$ is a bijection between the set $\mathcal{F}_{m+1}$ and the Cartesian product $\mathcal{F}_1 \times \mathcal{F}_m$.

Now it only remains to collect information together. Thus we obtain $f_{m+1}(s)$ $= f_1(s) \cdot f_m(qs) = f_1(s) \cdot f_1^{[m]}(qs) = f_1^{[m+1]}(s)$.

□

5. All ddc-polyominoes

Our gf for all ddc-polyominoes is denoted $D(s)$. In $D(s)$, the variables have the same names and roles as in $f_j(s)$.

The next proposition is something like the heart of this paper.

Proposition 5. *The gf $D(s)$ satisfies the equation*

$$D(s) = f_1(s) + \frac{x^{-1}}{1-qs} \cdot f_1(s)D(1) - \frac{x^{-1}(1-x+xqs)}{1-qs} \cdot f_1(s)D(qs) \ . \qquad (8)$$

Proof. For the matter of generality, no harm will be done if we only retain the essential variables. Hence we set $d = x = y = 1$. (What survives is s and q.) Instead of (8), we now have the equation

$$D(s) = f_1(s) + \frac{1}{1-qs} \cdot f_1(s)D(1) - \frac{qs}{1-qs} \cdot f_1(s)D(qs) \ . \qquad (8-)$$

Consider the right-hand side (rhs) of (8–). The first term being self-explanatory, we proceed to the second term. Now it is handy to write down an algorithm.

Algorithm A. Input an ordered triple (L, P, R) such that L lies in $\mathcal{F}_1$, P lies in $\mathcal{D}$, and R is either the empty set or a finite row of cells. Then:

1. place P so that its source cell be the upper neighbor of the target cell of L, and
2. if R is not empty, place R so that its leftmost cell be the right neighbor of the target cell of L.

Finally output the union $L \cup P \cup R$.

See Figure 3.

Suppose that Algorithm A transforms an ordered triple (L, P, R) into a figure V. The last diagonal of L can then be recognized as the last among those diagonals of V which are (at the same time) one-celled, floor-touching, and neighbored from above by a cell which also belongs to V. And R is, of course, the part of the bottom row of V which is not contained in L.

Further, we have $ft(V) = ft(L) + ft(R)$ and $ce(V) = ce(L) + ce(P) + ce(R)$.

The remarks just made amount to the following: Algorithm A is an injection and the gf for its image is nothing but the second term on the rhs of (8–).

Now, what is the image of Algorithm A? It is a set composed of two blocks: one block is $\mathcal{D} \setminus \mathcal{F}_1$, the set of ddc-polyominoes which are not 1-floorsitters, and the second block is made up of certain– so to speak– useless objects. (To be specific, a part of those useless objects are 1-floorsitters, and the other part are not even ddc-polyominoes.)

From what triples does Algorithm A produce useless objects? As a look at Figure 3 reveals, the answer is: from precisely those triples (L, P, R) in which $ce(R)$ is strictly greater than $ft(P)$. And thence it quickly follows that the gf, say $UL(s)$, for useless objects is $f_1(s)D(qs) \cdot \frac{qs}{1-qs}$. In other words, the third term on the rhs of (8–) is $-UL(s)$.

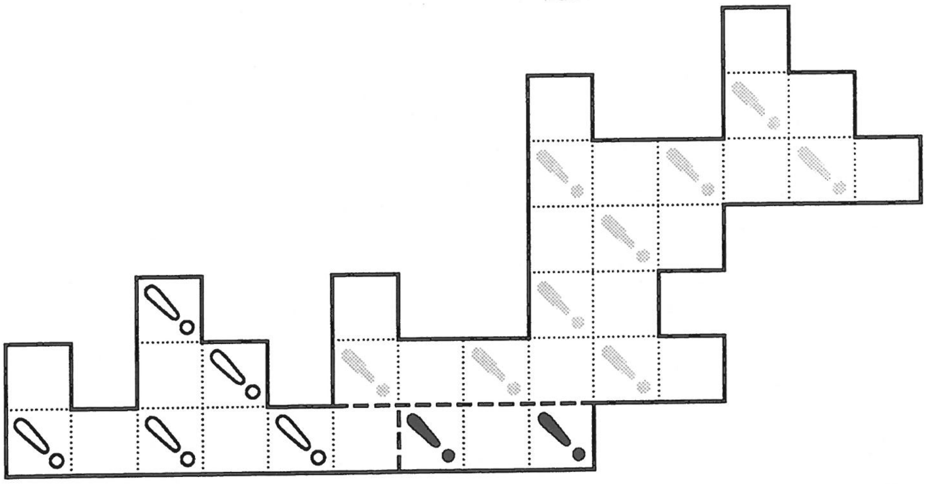

Figure 3: A fruit of Algorithm A. The corresponding input triple is unique, and is indicated as follows: the 1-floorsitter is sprinkled with white !'s, the ddc-polyomino with gray !'s, and the row of cells with black !'s.

Putting the pieces together, we now find that after $f_1(s)$, the gf for 1-floor--sitters, on the rhs of (8–) we have the gf for use*ful* objects, *i.e.*, for ddc-poly--ominoes which are not 1-floorsitters. Equation (8–) is thus justified. □

Let $D(d,x,y,q)$ be another name for $D(1)$.

Theorem 1. *The gf for all ddc-polyominoes is given by*

$$D(d,x,y,q) =$$

$$dxy \cdot \frac{\sum_{i=0}^{\infty}(-1)^i d^i q^{\binom{i+2}{2}} \sum_{j=0}^{i} \frac{x^{i-j}\left[\prod_{k=1}^{j}(1-x+xq^k)\right]\left[\prod_{\ell=1}^{i-j-1}(1-y+yq^\ell)\right]y^j}{(q)_{i-j}(q)_j}}{\sum_{i=0}^{\infty}(-1)^i d^i q^{\binom{i+1}{2}} \sum_{j=0}^{i} \frac{x^{i-j}\left[\prod_{k=1}^{j-1}(1-x+xq^k)\right]\left[\prod_{\ell=1}^{i-j-1}(1-y+yq^\ell)\right]y^j}{(q)_{i-j}(q)_j}} \tag{9}$$

Proof. We first iterate (8) in the usual way. For wider audience, this essentially means that we make a copy, say (C), of equation (8),

then we replace the term $D(qs)$, which equation (8) involves, with the case $s = qs$ of the rhs of (C),

then we replace the term $D(q^2s)$, which the equation obtained in the previous step involves, with the case $s = q^2s$ of the rhs of (C), and so on.

The iteration leaves us with

$$D(1) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j x^{-j} \cdot \prod_{k=1}^{j}(1-x+xq^k)}{(q)_j} \cdot f_1^{[j+1]}(1)}{1 - \sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^{-j} \cdot \prod_{k=1}^{j-1}(1-x+xq^k)}{(q)_j} \cdot f_1^{[j]}(1)} \tag{10}$$

But, owing to Propositions 3 and 4, we know that $f_1^{[j]}(s)$ is equal to $f_j(s)$, and we have a formula for $f_j(s)$. Substituting that formula (with s set to 1) into (10), we *ipso facto* obtain an expression for $D(1)$. However, to put this latter expression in simpler form, we then multiply both its numerator and its denominator by the denominator of $f_j(1)$. (This can be done because– fortunately– the denominator of $f_j(1)$ does not depend on j.) At this stage, the formula for $D(1)$ has a denominator of the form (the denominator of $f_j(1)$) minus (a certain double sum) . But those two items readily merge into one. In fact, in the denominator of (9), the denominator of $f_j(1)$ is just the part with $j=0$. □

We knew it all along (because it is geometrically obvious) that the function D is symmetric in x and y. But the following fact is nevertheless worth pointing out.

Fact 1. *The relation* $D(d,x,y,q) = D(d,y,x,q)$ *may readily be seen from formula (9).*

Proof. Here it is profitable to remark that, in the numerator of (9), the sum over j can be written as

$$\sum_{j=0}^{i} \frac{x^{i-j}\left[\prod_{k=1}^{j}(1-x+xq^k)\right]\left[\prod_{\ell=1}^{i-j}(1-y+yq^\ell)\right]y^j}{(q)_{i-j}(q)_j} +$$

$$+xy \cdot \sum_{j=0}^{i-1} \frac{x^{i-j-1}\left[\prod_{k=1}^{j}(1-x+xq^k)\right]\left[\prod_{\ell=1}^{i-j-1}(1-y+yq^\ell)\right]y^j}{(q)_{i-j-1}(q)_j}$$

Let (11) be the version of (9) produced by the above rewrite. The swap of x and y converts (11) into a certain different-looking formula (12). But (12) may be obtained from (11) in yet one way, *viz.* by letting each sum over j pass through the following procedure: redefine the index j (*e.g.*, new $j = i -$ old j), swap the indices k and l, and commute factors as situation requires. Now, being reachable from (11) both by this procedure and by the swap of x and y, (12) is at the same time a formula for $D(d,x,y,q)$ and a formula for $D(d,y,x,q)$. □

With x and y set equal to 1, formula (9) looks a good deal simpler.

Corollary 2. *We have*

$$D(d,1,1,q) = d \cdot \frac{\sum_{i=0}^{\infty}(-1)^i d^i \sum_{j=0}^{i} \frac{q^{(i-j)^2+(i+1)(j+1)}}{(q)_{i-j}(q)_j}}{\sum_{i=0}^{\infty}(-1)^i d^i \sum_{j=0}^{i} \frac{q^{(i-j)^2+ij}}{(q)_{i-j}(q)_j}} \tag{13}$$

□

The less standard the derivation, the more important it is to check the answer (cf. [11, p. 175]). Hence we checked (and found to be correct) formula (9) up to the terms in d^8, and formula (13) up to the terms in d^{10}. To do so, we resorted to *Maple* and *BASIC*, and we also recalled our [9] bijection between ddc-polyominoes and $\frac{1}{2}$-*good paths*.

Note. *The referees gave us several hints on how to make this a better paper. Being in a hurry, here we took just a part of those hints (and taking the others is continuing in real time). However, the benefit from this first round of revisal seems us rather visible.*

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