ON THE ENUMERATION OF INDEXED MONOMIALS AND THE COMPUTATION OF HILBERT FUNCTIONS OF LADDER DETERMINANTAL VARIETIES

SUDHIR R. GHORPADE

ABSTRACT. We outline the computation of an explicit formula for the Hilbert function of the ladder determinantal varieties defined by the vanishing of all minors of a fixed size of a rectangular matrix with indeterminate entries such that the indeterminates in these minors are restricted to lie in some ladder shaped region of the rectangular array. Finding such a formula is equivalent to enumerating the set of monomials of a fixed degree such that the support of these monomials is a subset of a 'ladder' and satisfies a certain "index condition".

1. INTRODUCTION

Typically, a ladder is a subset of a rectangle which looks as follows.

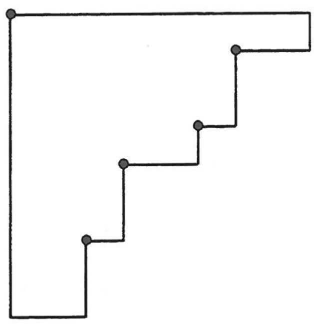

FIGURE 1

Given a rectangular matrix with indeterminate entries, the algebraic variety defined by all minors of a fixed size within a ladder shaped subset of the rectangle, is usually called a *ladder determinantal variety*. These varieties are intimately related to Schubert varieties in flag manifolds (cf. [21], [22]). Motivated partly by this connection, the ladder determinantal varieties (and, in fact, more general varieties of this kind) were introduced by Abhyankar [1]. It is known that such varieties are irreducible (cf. [1], [23]), Cohen-Macaulay (cf. [12]) and normal (cf. [5], [19]). Some of these properties can also be deduced using the connection with Schubert varieties from the work of Ramanathan [25] (see also [3], [17], [18], [24] [26]). Using [25] or otherwise (cf. [7], [11]) one knows also that most of these varieties have rational

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singularities. Other properties such as Gorensteinness as well as the determination of the divisor class group have also been investigated (cf. [5], [6], [20]).

We consider in this paper the problem of finding an explicit formula for the Hilbert function of ladder determinantal varieties. More precisely, we consider an $m(1) \times m(2)$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over a field K, and a subset $\mathcal{L}$ of the rectangle $\{(i,j) : 1 \leq i \leq m(1),\ 1 \leq j \leq m(2)\}$ such that $\mathcal{L}$ is like a (one-sided) ladder as in Fig. 1 or, more generally, a 'biladder' or a two-sided ladder (See Section 2 for precise definitions). Let $K[\mathcal{L}]$ denote the ring of polynomials in the indeterminates $\{X_{ij} : (i,j) \in \mathcal{L}\}$ with coefficients in K. Let $I_p(\mathcal{L})$ denote the ideal of $K[\mathcal{L}]$ generated by all $p \times p$ minors of X in $K[\mathcal{L}]$.

From a combinatorial viewpoint, the problem of finding the Hilbert function of $I_p(\mathcal{L})$ is equivalent to enumerating a set of monomials in $K[\mathcal{L}]$ satisfying a certain 'index condition' and of a fixed degree. Indeed, from the work of Abhyankar [1], we know that such "indexed monomials" form a K-basis for the graded components of the residue class ring $K[\mathcal{L}]/I_p(\mathcal{L})$. In fact, this equivalence is a basic starting point for the arguments in [2], [15], [16] and in this paper as well. The connection with indexed monomials is explained in details in Section 2.

The problem of finding explicitly the Hilbert function of the homogeneous ideal $I_p(\mathcal{L})$ or of the corresponding projective variety $\mathcal{V}_p(\mathcal{L})$ was first studied by Kulkarni in his 1985 thesis [15] (see also [16]). There he obtained a nice formula in the first nontrivial case of $p = 2$. It may be noted that in the degenerate case when $\mathcal{L}$ is the entire rectangle $[1, m(1)] \times [1, m(2)]$, the ideal $I_p(\mathcal{L})$ reduces to the classical determinantal ideal $I_p(X)$ that arises frequently in Algebraic geometry and Invariant Theory. In the case of $I_p(X)$, the Hilbert function is explicitly known from the work of Abhyankar [1]. In particular, the Hilbert function of $I_p(X)$ coincides with the Hilbert polynomial of $I_p(X)$ for all nonnegative integers; ideals with this property are called ***hilbertian.*** For a survey of Abhyankar's work, see [8] and for a short proof of a formula for the Hilbert function of $I_p(X)$, see [4] or [9]. Returning to ladders, it was shown in 1989 by Abhyankar and Kulkarni [2] that the ideals $I_p(\mathcal{L})$ are also hilbertian for any $p \geq 1$ and any biladder $\mathcal{L}$; in fact, this result is applicable to sets more general than biladders, called generalized ladders or saturated sets (see Section 2 for details). Ladder determinantal ideals such as $I_p(\mathcal{L})$ were considered from the viewpoint of Gröbner bases and lattice paths by Herzog and Trung [12]. They showed that one can describe the Hilbert function of $I_p(\mathcal{L})$ in terms of the f-vector of the associated simplicial complex. While this would also prove the Hilbertianness of $I_p(\mathcal{L})$, there still remains the problem of finding explicitly the Hilbert function of $I_p(\mathcal{L})$. To this end, Conca and Herzog [4] conjectured a 'remarkable formula' for the Hilbert series (that is, the generating function for the sequence of values of the Hilbert function) in the case of one-sided ladders. Recently, Krattenthaler [14] has established this Conjecture using the so called 'two-rowed arrays'.

Our main result is an explicit (albeit, complicated!) formula for the Hilbert function of $I_p(\mathcal{L})$ for any biladder $\mathcal{L}$ and any $p \geq 1$. This may be viewed as a natural extension of the results of Kulkarni [16] and a refinement of the technique used by Abhyankar and Kulkarni [2] to prove that $I_p(\mathcal{L})$ is hilbertian. A detailed proof of this result shall appear in a forthcoming paper [10]. In this paper, we shall only try to outline some of the main ideas involved in the proof and give the statements of the main lemmas and theorems. It is hoped that this would make [10], which appears to be a rather long and technical paper, a little more accessible.

This paper is organized as follows. The next section sets up some notation and preliminary notions that we shall use. We consider the case of $I_2(\mathcal{L})$ in Section 3 and the general case in Section 4.

2. Preliminaries

By $\mathbb{Z}, \mathbb{N}$, and $\mathbb{N}^+$ we denote the sets of all integers, nonnegative integers, and positive integers respectively. Given any $a, b \in \mathbb{Z}$, we define the closed and semi-closed integral intervals $[a,b]$, $[a,b)$, $(a,b]$ in the obvious way; for example,

$$[a,b) = \{c \in \mathbb{Z} : a \leq c < b\}.$$

Fix a pair $m = (m(1), m(2))$ of positive integers, a field K and an $m(1) \times m(2)$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over K. Given any subset Y of the rectangle

$$[1, m(1)] \times [1, m(2)] = \{(i,j) : 1 \leq i \leq m(1),\ 1 \leq j \leq m(2)\},$$

let $K[Y]$ denote the polynomial ring in the indeterminates $\{X_{ij} : (i,j) \in Y\}$ with coefficients in K. Given any $p \in \mathbb{N}$, we let $I_p(Y)$ denote the ideal of $K[Y]$ generated by all $p \times p$ minors of X in $K[Y]$.

Given any $h \in \mathbb{N}^+$, by a *ladder generating bisequence* (LGB) of *length* h, we mean a map $S : [1,2] \times [0,h] \to \mathbb{N}$ such that

$$1 = S(1,0) \leq S(1,1) < S(1,2) < \cdots < S(1,h) = m(1)$$

and

$$m(2) = S(2,0) > S(2,1) > \cdots > S(2,h-1) \geq S(2,h) = 1.$$

The positive integer h may be denoted by $\operatorname{len}(S)$. We shall find it convenient to also consider the empty bisequence, which we declare to be the unique LGB of length 0. Given any LGB S of length h, we define

$$L(S) = \bigcup_{k=1}^{\operatorname{len}(S)} [S(1,k-1), S(1,k)] \times [1, S(2,k-1)],$$

and

$$L(S)^o = \bigcup_{k=1}^{\operatorname{len}(S)} [S(1,k-1), S(1,k)) \times [1, S(2,k-1)).$$

We call $L(S)$ to be the *ladder* corresponding to S and $L(S)^o$ to be the *interior* of $L(S)$. Note that if $h > 0$, then

$$L(S)^o \underset{\neq}{\subset} L(S) \subseteq [1, m(1)] \times [1, m(2)].$$

In case $h = 0$, we have $L(S) = L(S)^0 = \emptyset$, whereas if $h = 1$, then $L(S) = [1, m(1)] \times [1, m(2)]$. We shall denote by ∂S or by $\partial L(S)$ the *boundary* of $L(S)$, which is defined by $\partial S = L(S) \backslash L(S)^o$. Points $(S(1,k), S(2,k))$, where $1 \leq k \leq h-1$, are called the *nodes* of S or of the ladder $L(S)$ and we denote by $\mathcal{N}(S)$ the set of all nodes of S.

Given any LGB's S' and S such that $\operatorname{len}(S) \neq 0$ and $L(S') \subseteq L(S)$, we define

$$\mathcal{L}(S', S) = L(S) \setminus L(S') \quad \text{and} \quad \mathcal{L}(S', S)^o = L(S)^o \setminus L(S').$$

We call $\mathcal{L}(S', S)$ to be the *biladder* corresponding to S' and S while $\mathcal{L}(S', S)^o$ to be the *interior* of $\mathcal{L}(S', S)$. Note that since we allow $L(S') = \emptyset$, a ladder is a special case of a biladder. Pictorially, a ladder looks as in Fig. 1 above and a biladder looks as in Fig. 2 (a) or, more generally, as in Fig. 2 (b) below.

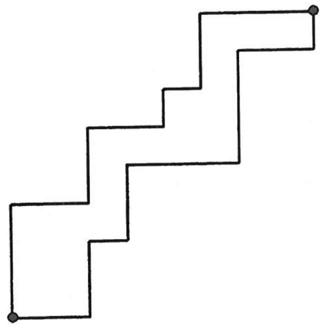

FIGURE 2 (a)

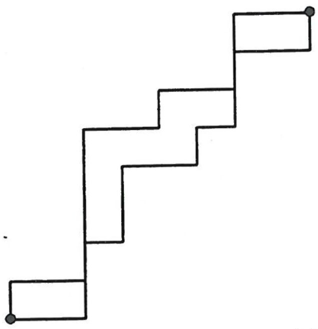

FIGURE 2 (b)

It may be remarked that in these pictures, we adopt the 'matrix notation' rather than that of Coordinate Geometry to represent points. Thus in Fig. 1, the bullet on the top left hand corner indicates the point $(1,1)$ while the other bullets indicate the 'nodes' $(S(1,k), S(2,k))$, $1 \leq k < h$. In Fig. 2 (a) and Fig. 2 (b), we have only marked the points $(1, m(2))$ and $(m(1), 1)$ corresponding to $h = 0$ and $h = 1$.

Given any biladder $\mathcal{L} = \mathcal{L}(S', S)$, we shall denote by $\Delta(S', S)$ the intersection of the boundaries of $L(S)$ and $L(S')$, and by $\mathcal{N}(S', S)$ the set of common nodes of $\mathcal{N}(S')$ and $\mathcal{N}(S)$, that is,

$$\Delta(S', S) = \partial S' \cap \partial S \quad \text{and} \quad \mathcal{N}(S', S) = \mathcal{N}(S') \cap \mathcal{N}(S).$$

It may be noted that $\Delta(S', S) = \partial S \cap L(S')$.

Observe that ladders as well as biladders are subsets Y of $[1, m(1)] \times [1, m(2)]$ with the property that whenever (i_1, i_2), $(j_1, j_2) \in Y$ with $i_1 < i_2$ and $j_1 < j_2$, we have that $(i_1, j_2) \in Y$ and $(j_1, i_2) \in Y$. Sets Y with this property may be called ***generalized ladders*** or ***saturated sets***. Some authors simply refer to them as ladders. It is not difficult to see that if a generalized ladder is 'connected', then it must be a biladder.

Given any $Y \subseteq [1, m(1)] \times [1, m(2)]$, we let $\text{mon}(Y)$ denote the set of all maps of $Y \to \mathbb{N}$. Given any $\theta \in \text{mon}(Y)$, we let

$$\text{supp}(\theta) = \{(i,j) \in Y : \theta(i,j) \neq 0\}$$

denote the ***support*** of θ and

$$X^\theta = \prod_{(i,j) \in Y} X_{ij}^{\theta(i,j)}$$

denote the corresponding monomial in $K[Y]$. Following Abhyankar [1], we define the ***index*** of any subset $M \subseteq [1, m(1)] \times [1, m(2)]$ by

$$\begin{aligned} \text{ind}(M) = \max\{p \in \mathbb{N} \ : \ & \exists\ (i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p) \text{ in } M \text{ with} \\ & i_1 < i_2 < \cdots < i_p \text{ and } j_1 < j_2 < \cdots < j_p\}. \end{aligned}$$

For a monomial $\theta \in \text{mon}(Y)$, the index is defined by putting

$$\text{ind}(\theta) = \text{ind}(\text{supp}(\theta)).$$

For every $p \in \mathbb{N}$ we let

$$\text{mon}(Y, p) = \{\theta \in \text{mon}(Y) : \text{ind}(\theta) \leq p\}$$

and, restricting attention to monomials of a specified degree, for every $p \in \mathbb{N}$ and $V \in \mathbb{N}$ we let

$$\text{mon}(Y, p, V) = \{\theta \in \text{mon}(Y, p) : \sum_{y \in Y} \theta(y) = V\}.$$

We now recall a basic result of Abhyankar [1, Thm. 20.10] (see also [8, Thm. 6.7]), which was alluded to in the Introduction.

2.1. Theorem. *Let $Y \subseteq [1,m(1)] \times [1,m(2)]$ be any generalized ladder and let $p \in \mathbb{N}$. Given any $V \in \mathbb{N}$, the set $\{X^\theta : \theta \in \text{mon}(Y,p,V)\}$ forms a free K-basis of the V-th homogeneous component $K[Y]_V/I_{p+1}(Y)_V$ of the residue class ring $K[Y]/I_{p+1}(Y)$. Consequently, the Hilbert function of $I_{p+1}(Y)$ is given by*

$$\mathcal{H}(V) = |\text{mon}(Y,p,V)|, \qquad (V \in \mathbb{N}).$$

Following Kulkarni [15], we consider the so called radicals and skeletons, which are defined as follows. Fix some $Y \subseteq [1,m(1)] \times [1,m(2)]$. A subset $R \subseteq Y$ is called a *radical* if $\text{ind}(R) \le 1$, and it is called a *skeleton* if for any two distinct elements (i_1,i_2) and (j_1,j_2) of R, we have

$$\text{either: } i_1 < j_1 \text{ and } i_2 > j_2 \quad \text{or: } i_1 > j_1 \text{ and } i_2 < j_2.$$

The set of all radicals (resp: skeletons) in Y is denoted by $\text{rad}(Y)$ (resp: $\text{skel}(Y)$). Note that $\text{skel}(Y) \subseteq \text{rad}(Y)$. More generally, given any $p \in N$, we let $\text{rad}^p(Y)$ denote the set of all $R \subseteq Y$ such that $\text{ind}(R) \le p$. Elements of $\text{rad}^p(Y)$ may be called *p-fold radicals*. Finally, we set for any $p \in \mathbb{N}$ and $d \in \mathbb{N}$,

$$\text{rad}^p(Y,d) = \{R \in \text{rad}^p(Y) : |R| = d\} \quad \text{and} \quad \text{skel}(Y,d) = \{R \in \text{skel}(Y) : |R| = d\}.$$

3. Radicals and Skeletons

It is easy to see that the problem of counting the desired set of monomials can be reduced to the problem of enumerating the p-fold radicals of a given size.

3.1. Lemma. *Given any $Y \subseteq [1,m(1)] \times [1,m(2)]$, $p \in \mathbb{N}$ and $V \in N$, we have*

$$|\text{mon}(Y,p,V)| = \sum_{d \ge 0} \binom{V-1}{V-d} |\text{rad}^p(Y,d)|,$$

where the summation on the right is essentially finite (that is, all except finitely many summands are zero).

Fix any biladder $\mathcal{L} = \mathcal{L}(S',S)$, and let $\mathcal{L}^o$ denote its interior. Let $h = \text{len}(S)$ and $h' = \text{len}(S')$. Also let

$$\delta = |\Delta(S',S)| \quad \text{and} \quad \nu = |\mathcal{N}(S',S)|.$$

Given any $(k,k') \in [1,h] \times [1,h']$, we let

$$\mu_{\mathcal{L}}(k,k') = \text{card}([S(1,k-1), S(1,k)) \cap [S'(1,k'-1), S'(1,k')))$$

and

$$\nu_{\mathcal{L}}(k,k') = \text{card}([S(2,k), S(2,k-1)) \cap (S'(2,k'), S'(2,k'-1)]).$$

Note that these numbers are completely (and easily) determined by $\mathcal{L}$.

As a preliminary step towards calculating $|\text{rad}^p(\mathcal{L},d)|$, we shall restrict our attention to the case of $p = 1$ so as to determine $|\text{rad}(\mathcal{L},d)|$. To this end, we use techniques similar to [16] except that now instead of ladders we consider the more general biladders, and so one has to be a little more careful, especially since we are allowing overlaps (as in Fig. 2 (b)) of smaller ladder $L(S')$ with the bigger ladder $L(S)$. As in Kulkarni [15], we reduce the problem to skeletons by constructing two maps

$$\lambda : \text{rad}(\mathcal{L}) \to \text{skel}(\mathcal{L}^o) \quad \text{and} \quad \mu : \text{skel}(\mathcal{L}^o) \to \text{rad}(\mathcal{L})$$

such that λ is surjective, μ is injective and moreover, μ is the inverse of the restriction of λ to maximal subsets of $\text{rad}(\mathcal{L})$. This leads to the following.

3.2. Theorem. *Let $M = m(1) + m(2) - 1 - \delta$. Given any $d \in \mathbb{N}$, we have*

$$|\mathrm{rad}(\mathcal{L}, d)| = \sum_{\ell \geq 0} \binom{M-\ell}{d-\ell} |\mathrm{skel}(\mathcal{L}^o, \ell)|,$$

where the summation on the right is essentially finite (that is, all except finitely many summands are zero).

To describe an explicit formula for the number of skeletons in the interior of a biladder, we need some notation.

Given any $\ell \in \mathbb{N}$, let $M_{h,h'}(\mathbb{N}, \ell)$ denote the set of all $h \times h'$ matrices with integral entries such that the sum of all the entries is ℓ. Note that this is a finite set. Given any $\alpha = (\alpha_{kk'}) \in M_{h,h'}(\mathbb{N}, \ell)$ and any $(i,j) \in [1,h] \times [1,h']$, we let

$$\sigma_i(\alpha) = \sum_{k=1}^{i} \sum_{k'=1}^{h'} \alpha_{kk'} \quad \text{and} \quad \tau_j(\alpha) = \sum_{k=1}^{h} \sum_{k'=j}^{h'} \alpha_{kk'}.$$

Given any $\ell \in \mathbb{N}$ and $\alpha, \beta \in M_{h,h'}(\mathbb{N}, \ell)$, we define

$$\sigma(\beta) \leq \sigma(\alpha) \text{ to mean that } \sigma_i(\beta) \leq \sigma_i(\alpha) \text{ for all } i \in [1,h]$$

and

$$\tau(\beta) \leq \tau(\alpha) \text{ to mean that } \tau_j(\beta) \leq \tau_j(\alpha) \text{ for all } j \in [1,h'].$$

Finally, for any $\ell \in \mathbb{N}$, we define

$$\mathcal{S}(\mathcal{L}^o, \ell) = \sum_{\substack{\alpha, \beta \in M_{h,h'}(\mathbb{N}, l) \\ \sigma(\beta) \leq \sigma(\alpha),\ \tau(\beta) \leq \tau(\alpha)}} \prod_{\substack{1 \leq k \leq h \\ 1 \leq k' \leq h'}} \binom{\mu_{\mathcal{L}}(k,k')}{\alpha_{kk'}} \binom{\nu_{\mathcal{L}}(k,k')}{\beta_{kk'}}.$$

3.3. Theorem. *Given any $\ell \in \mathbb{N}$, we have*

$$|\mathrm{skel}(\mathcal{L}^o, \ell)| = \mathcal{S}(\mathcal{L}^o, \ell).$$

As a consequence of the above results, we obtain the following formula, which may be viewed as an extension of Kulkarni's formula [16, Thm. 11]

3.4. Theorem. *The Hilbert function as well as the Hilbert polynomial of $I_2(\mathcal{L})$ in $K[\mathcal{L}]$ is given by*

$$F(V) = \sum_{\ell \geq 0} \binom{V + M - 1 - \ell}{M - 1} \mathcal{S}(\mathcal{L}^o, \ell)$$

where $M = m(1) + m(2) - 1 - \delta$ and $\mathcal{S}(\mathcal{L}^o, \ell)$ is given by the formula above.

4. General Case

As in Section 3, we fix a biladder $\mathcal{L} = \mathcal{L}(S', S)$, and let $\mathcal{L}^o$ denote its interior. Let h, h', δ and ν be as defined in Section 3.

Given any LGB S^*, we shall write $S^* \leq S$ to mean that $L(S^*) \subseteq L(S)$. Further, given any LGB's S_1, S_2 such that $S_i \leq S$ for $i = 1, 2$, we shall write $S_1 < S_2$ to mean that $L(S_1) \subseteq L(S_2)$, $\mathrm{len}(S_2) \neq 0$ and $\Delta(S_1, S_2) = \Delta(S_1, S)$.

The following basic result allows us to tackle the general case recursively by applying the results of Section 3. The map Γ mentioned in the theorem below can be described quite explicitly and it yields a decomposition of $\mathrm{rad}(\mathcal{L})$, which may be viewed as a refinement of the superskeleton decomposition of [2, Thm. 10].

4.1. Theorem. *Let $p \in \mathbb{N}^+$ and $\mathcal{L}$ be as above. Then there exists a LGB S^* such that $S' < S^* \leq S$ and there exists an injective map*

$$\Gamma : \mathrm{rad}^p(\mathcal{L}) \to \mathrm{rad}(\mathcal{L}) \times \mathrm{rad}^{p-1}(\mathcal{L}^*),$$

where $\mathcal{L}^*$ denotes the biladder $\mathcal{L}(S^*, S)$.

This leads to the following enumerative result.

4.2. Theorem. *Given any $d \in \mathbb{N}$ and any $p \in \mathbb{N}^+$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{S' < S^* \leq S} \sum_{d_1 + d_2 = d} \binom{M - \nu^*}{d_1 - \nu^*} |\mathrm{rad}^{p-1}(\mathcal{L}^*, d_2)|,$$

where the first sum is taken over all LGB's S^* such that $S' < S^* \leq S$ and the second sum is over all nonnegative integer pairs (d_1, d_2) such that $d_1 + d_2 = d$, and where $M = m(1) + m(2) - 1 - \delta$, $\nu^* = |\mathcal{N}(S) \cap \mathcal{N}(S^*)|$, and $\mathcal{L}^*$ denotes the biladder $\mathcal{L}(S^*, S)$.

Successive applications of the above result yields the following.

4.3. Theorem. *Given any $d \in \mathbb{N}$ and any $p \in \mathbb{N}^+$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{S' < S_1 < \cdots < S_{p-1} \leq S} \sum_{\ell \geq 0} \binom{M_p - \nu_1 - \cdots - \nu_{p-1} - \ell}{d - \nu_1 - \cdots - \nu_{p-1} - \ell} S(\mathcal{L}_{p-1}, \ell)$$

where the first sum is taken over all $(p-1)$-tuples $(S_1, \ldots, S_{p-1})$ of LGB's such that $S' = S_0 < S_1 < S_2 < \cdots < S_{p-1} \leq S$, and

$$M_p = p(m(1) + m(2) - 1) - \delta_1 - \cdots - \delta_{p-1},$$

where $\delta_i = |\partial S_{i-1} \cap \partial S|$ and $\nu_i = |\mathcal{N}(S_i) \setminus (\mathcal{N}(S_i) \cap \mathcal{N}(S))|$ for $1 \leq i \leq p-1$, and $\mathcal{L}_{p-1} = \mathcal{L}(S_{p-1}, S)$.

If for a given $(S_1, \ldots, S_{p-1})$ as in Theorem 4.3, and any nonnegative integers u and ℓ, we let

$$F_u(\ell) = \binom{\nu_1 + \cdots + \nu_{p-1} + \ell}{u} S(\mathcal{L}_{p-1}, \ell),$$

where ν_i and $\mathcal{L}_{p-1}$ are as in Theorem 4.3, then we can state the main result as follows.

4.4. Theorem. *Let $p \in \mathbb{N}$ and $\mathcal{L}$ be as above. The Hilbert function as well as the Hilbert polynomial of $I_{p+1}(\mathcal{L})$ is given by*

$$F(V) = \sum_{u \geq 0} \sum_{S' < S_1 < \cdots < S_{p-1} \leq S} (-1)^u F_u(\ell) \binom{V + M_p - 1 - u}{M_p - 1 - u}$$

where M_p is as in Theorem 4.3. In particular, $I_{p+1}(\mathcal{L})$ is a Hilbertian ideal.

4.5. *Remarks.* 1. The first two theorems in this section may motivate the use of biladders although one may only be interested in (one-sided) ladders. Indeed, even if $\mathcal{L}$ were a ladder to begin with, the $\mathcal{L}^*$ that one obtains in Theorem 4.1 is necessarily a biladder. Thus it makes sense to have the results of Section 3 in the general case of biladders.

2. The formulae in Theorem 4.3 and Theorem 4.4 are no doubt complicated and perhaps they may seem unworthy of being called 'explicit', in view of the rather unwieldy summation over the tuples $(S_1, \ldots, S_{p-1})$. Nevertheless, they can be used to deduce some interesting information about the variety associated to $I_{p+1}(\mathcal{L})$. For example, one can derive fairly simple expressions for the degree of the Hilbert polynomial. Also, as Krattenthaler [14, Sec. 7] seems to suggest, it appears unlikely that an elegant and simple formula for the Hilbert function of $I_{p+1}(\mathcal{L})$ can be found.

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY
POWAI, MUMBAI 400076 - INDIA
E-mail address: `srg@math.iitb.ernet.in`