HODGE NUMBERS OF PSEUDO SYMMETRIC FANO VARIETIES

NOMBRES DE HODGE DE VARIÉTÉS DE FANO PSEUDOSYMMÉTRIQUES

CHRISTIAN HAASE
JOINT WORK WITH DIMITRIOS I. DAIS AND GÜNTER M. ZIEGLER

ABSTRACT. Motivated by a problem in the algebraic geometry of toric varieties, we investigate the polar polytopes of (pseudo-) symmetric Fano polytopes. We show that they admit (regular) unimodular triangulations and compute the Ehrhart series of all their faces.

RÉSUMÉ. Motivés par un problème de la géométrie algébrique des variétés toriques nous étudions les polytopes polaires aux polytopes de Fano (pseudo-)symmetriques. Nous démontrons l'existence de triangulations unimodulaires (régulières) et nous calculons les séries de Ehrhart de tous les faces.

1. INTRODUCTION

1.1. **The problem.** Toric geometry associates a projective toric variety X_P with any lattice polytope $P \subseteq \mathbb{R}^d$. There are many fascinating interactions between the discrete geometry of lattice polytopes (and lattice cones) on the one hand and the algebraic geometry of toric varieties on the other [5, 7, 9]. As a result there is a whole dictionary that translates between properties on both sides. The definition of certain invariants, the *string theoretic Hodge numbers of Gorenstein toric varieties*, makes heavy use of this dictionary. Their actual computation for a specific example, the pseudo-symmetric Fano varieties, poses interesting geometric/combinatorial problems and at the same time provides an example for the use of the mirror symmetry established by Batyrev and Borisov [1].

On a smooth complex variety Y a differential form is pure of type (p, q) if in local (real) coordinates z_i and $\overline{z}_i$ it involves p of the dz_i's and q

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of the $d\bar{z}_i$'s. This induces a decomposition of the complex cohomology $\mathrm{H}^n(Y, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(Y)$. The ranks $h^{p,q}(Y) = \mathrm{rk}\, \mathrm{H}^{p,q}(Y)$ are the *(ordinary) Hodge numbers* of Y. Batyrev and Dais [2] generalize them to the *string theoretic Hodge numbers* $h^{p,q}_{str}(Y)$ in the more general setting of toroidal Gorenstein singularities $Y = X_P$.

One gets a more subtle invariant if one considers a certain class of hypersurfaces $Z \hookrightarrow Y$ and their string theoretic Hodge numbers. In the case that P is a simple polytope that admits a unimodular triangulation there are formulae by Danilov and Khovanskiĭ for this invariant, involving the Ψ-vector of the faces of P (see below). We treat the case where P is a pseudo-symmetric Fano polytope (see section 1.3 below). These are reflexive polytopes, and for those Batyrev and Borisov [1] proved a mirror duality

$$h^{p,q}_{str}(Z \hookrightarrow X_P) = h^{d-1-p,q}_{str}(Z' \hookrightarrow X_{P^\vee}).$$

We thus can apply Danilov and Khovanskiĭ's formulae to the (simple) polar of our (simplicial) polytopes — provided that the latter admit unimodular triangulations.

1.2. **Notions and notation.** Let $S \subseteq \mathbb{R}^d$. Denote by $\mathrm{aff}(S)$ the affine hull of S and set $\dim(S) := \dim(\mathrm{aff}(S))$. A *lattice polytope* P is the convex hull of finitely many integral points. If the origin $\mathbf{0}$ is an interior point of P, the *polar polytope* is given by $P^\vee = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, P\rangle \leq 1\}$. If both P and $P^\vee$ have integral vertices then they are *reflexive*. A polytope is *simplicial* if all faces are simplices; it is *simple* if every vertex is incident to dimension many facets. A lattice simplex $\mathfrak{s}$ whose vertices form an affine lattice basis for $\mathrm{aff}(\mathfrak{s}) \cap \mathbb{Z}^d$ is called *unimodular* (or *basic*). Two sets $S \subset \mathbb{R}^d$ and $S' \subset \mathbb{R}^{d'}$ are *lattice equivalent* if there is an affine map $\mathrm{aff}(S) \to \mathrm{aff}(S')$ that maps $\mathbb{Z}^d \cap \mathrm{aff}(S)$ bijectively onto $\mathbb{Z}^{d'} \cap \mathrm{aff}(S')$ and which maps S to S'; e.g., all d-dimensional unimodular simplices embedded in $\mathbb{R}^{d'}$ ($d' \geq d$) are lattice equivalent to the *standard simplex* $\mathfrak{s}^{(d)}$ which is defined to be the convex hull of the standard unit vectors $\mathbf{e}_i (1 \leq i \leq d+1)$ in $\mathbb{R}^{d+1}$. A subdivision of a lattice polytope into unimodular simplices is a *unimodular triangulation*.

Let P be a lattice polytope. Then the number of integral points in the dilation $k\,P$ for $k \in \mathbb{Z}_{\geq 0}$ is a polynomial, the *Ehrhart polynomial* $\mathrm{Ehr}(F, k) = \mathrm{card}(k\,F \cap \mathbb{Z}^d) = \sum_{i=0}^{\dim P} a_i(P) k^i$. The corresponding generating function, the *Ehrhart series*

$$\mathfrak{Ehr}(P, t) = \sum_{k \geq 0} \mathrm{Ehr}(P, k) t^k = \frac{\sum_{i=0}^{\dim P} \psi_i(P) t^i}{(1-t)^{\dim P + 1}}$$

gives rise to the Ψ-vector $(\psi_0(P), \ldots, \psi_{\dim P}(P))$ of P.

1.3. **The classification.** The polytopes for which we want to carry out the calculation are the pseudo-symmetric Fano polytopes. They constitute the only non-trivial, infinite class of reflexive polytopes for which a classification is available [5, 6].

A polytope P is *centrally symmetric* if $P = -P$, *pseudo symmetric* if it has two facets F, F' satisfying $F = -F'$, *Fano* if it is reflexive and all its faces are unimodular simplices.

Let $\mathbb{1} = \sum \mathbf{e}_i$ denote the all-one-vector. For even $d \geq 2$ call

- $\mathrm{DP}^d = \mathrm{conv}(\pm\mathfrak{s}^{(d-1)}, \pm\mathbb{1})$ the *del Pezzo* polytope and
- $\mathrm{preDP}^d = \mathrm{conv}(\pm\mathfrak{s}^{(d-1)}, \mathbb{1})$ the *pre del Pezzo* polytope.

Let $P \subseteq \mathbb{R}^d$ and $P' \subseteq \mathbb{R}^{d'}$ be full-dimensional polytopes with $\mathbf{0}$ in their interior. We define

$$P \circ P' = \mathrm{conv}(P \times \{\mathbf{0}\} \cup \{\mathbf{0}\} \times P') \quad \subseteq \mathbb{R}^{d+d'},$$

and say that $P \circ P'$ splits into P and P'. In the sequel we will identify P with $P \times \{\mathbf{0}\}$ and also P' with $\{\mathbf{0}\} \times P'$. The polar operation is the Cartesian product:

$$(P \circ P')^\vee = P^\vee \times P'^\vee.$$

If P and P' are Fano polytopes, then so is $P \circ P'$. One example for this construction is the d-dimensional *crosspolytope*

$$C_d^\vee = \mathrm{conv}(\pm\mathfrak{s}^{(d-1)}) = [-1,1] \circ \ldots \circ [-1,1] \ (d \text{ components}).$$

After all this notation we can finally formulate:

Theorem 1 (Ewald [5, 6]). *Let $P \subseteq \mathbb{R}^d$ be a Fano polytope.*

- *If P is centrally symmetric, then it is equivalent to $C_{d_0}^\vee \circ \mathrm{DP}^{d_1} \circ \ldots \circ \mathrm{DP}^{d_r}$.*
- *If P is pseudo symmetric, then it is equivalent to $P' \circ \mathrm{preDP}^{d_1} \circ \ldots \circ \mathrm{preDP}^{d_r}$ for some centrally symmetric Fano Polytope P'.*

2. Triangulations

It is clear by definition that every Fano polytope has a unimodular triangulation. The fact that the polar polytopes also admit such triangulations is a special case of a slightly more general situation. The tool we use here is the refinement of a given subdivision Σ of a polytope P by pulling lattice points $v \in P$ (cf. [8]):

- $\mathrm{pull}_v(\Sigma)$ contains all $Q \in \Sigma$ for which $v \notin Q$, and
- if $v \in Q \in \Sigma$, then $\mathrm{pull}_v(\Sigma)$ contains all the polytopes having the form $\mathrm{conv}(F \cup v)$, with F a facet of Q such that $v \notin F$,

By pulling successively all the lattice points within a given lattice polytope P one obtains a triangulation into simplices whose vertices are the only lattice points they contain.

Proposition 2 (Paco's Lemma [10]). *If P has facet width 1, i.e., for every facet, P lies between the hyperplane spanned by this facet and the next parallel lattice hyperplane, then every pulling triangulation is unimodular.*

Proof. By decreasing induction on the dimension one sees that every face of P has facet width 1. Furthermore the restriction of a pulling triangulation to a face is a pulling triangulation and thus unimodular (by another induction). But then every simplex in the triangulation of P is the join of two unimodular simplices in adjacent lattice hyperplanes. □

Proposition 3. *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subset \mathbb{Z}^d$ be a collection of vectors that form a unimodular matrix, i.e., such that $\{\det[\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_d}] : 1 \leq i_j \leq r\} = \{-1, 0, 1\}$, then for any choice of integers c_i the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq c_i\}$ admits a unimodular triangulation.*

Proof. The hyperplanes $H_i(k) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{v}_i, \mathbf{x} \rangle = k\}$ for integers k form an arrangement that subdivides P (and all of $\mathbb{R}^d$) into polytopes. The vertices of these polytopes are lattice points by the determinant condition. Moreover these polytopes have facet width 1. □

Corollary 4. *The polars of the (pre) del Pezzo's admit unimodular triangulations.*

Proof. The vertices of the (pre) del Pezzo's satisfy the condition of Proposition 3. The polar polytope is then obtained by choosing all c_i's to be 1. □

Another application of Paco's Lemma 2 is the following.

Corollary 5. *Let P and P' be lattice polytopes. If both admit a unimodular triangulation then so does $P \times P'$.*

Proof. The obvious subdivision of $P \times P'$ into products of unimodular simplices also satisfies the facet width condition. □

Both corollaries together imply:

Corollary 6. *Let $P \subset \mathbb{R}^d$ be a pseudo symmetric Fano polytope. Then its polar $P^\vee$ admits a unimodular triangulation.*

3. Computations

3.1. **(pre)DP$^\vee$.** The polar polytopes of the (pre) del Pezzos have the following description by inequalities.

$$(\mathrm{DP}^d)^\vee = \{\mathbf{x} \in [-1,+1]^d : \sum x_i \in [-1,+1]\} \text{ , respectively}$$
$$(preDP^d)^\vee = \{\mathbf{x} \in [-1,+1]^{d-1} \times (-\infty, 1] : \sum x_i \in [-1,+1]\}.$$

They are lattice equivalent to

$$\{\mathbf{x} \in [-1,+1]^{d+1} : \sum x_i = 0\} \text{ , respectively}$$
$$\{\mathbf{x} \in [-1,+1]^{d} \times (-\infty, 1] : \sum x_i = 0\}.$$

Remember that d is even, the (pre) del Pezzos are simplicial, and the $((\mathrm{pre})\mathrm{DP}^d)^\vee$ are simple polytopes. This means, that every $d-k$ face of the latter is just the intersection of k facets. Consider first the case of $(\mathrm{DP}^d)^\vee$. The facet defining inequalities are either of the form $x_i \leq 1$ or $x_i \geq -1$. A face F fulfills some of these inequalities with equality, say for $i \in I_+ \subseteq \{1, \dots, d+1\}$ the first one and for $i \in I_-$ the second one:

$$F = \{\mathbf{x} \in [-1,+1]^{d+1} : \sum x_i = 0, x_i = 1 (i \in I_+), x_i = -1 (i \in I_-)\}.$$

We call such a face *of type* $(d; s = \mathrm{card}(I_-), t = \mathrm{card}(I_+))$. It has dimension $d' = d - s - t$ and it is lattice equivalent to

$$F(d; s, t) = \{\mathbf{x} \in [-1,+1]^{d'+1} : \sum x_i = s - t\}.$$

If $s, t \leq d/2$ then there are $\binom{d+1}{d'+1, s, t}$ such faces (otherwise $F(d; s, t)$ is empty anyway).

In the case of $(\mathrm{preDP}^d)^\vee$ there are other faces showing up. The facet defining inequalities are the same as in the case of $(\mathrm{DP}^d)^\vee$, but the inequalitiy $x_{d+1} \geq -1$ is missing. So, if $d+1 \in I_+$, the considered face is of type $(d; s, t)$ and there are $\binom{d}{d'+1, s, t-1}$ such faces, provided $s, t \leq d/2$. But if the $d+1^{st}$ coordinate is not fixed, we get a new kind of faces. They are equivalent to

$$F'(d; s, t) = \{\mathbf{x} \in [-1,+1]^{d'} \times (-\infty, 1] : \sum x_i = s - t\}.$$

If $s \leq d/2$, then there are $\binom{d}{d', s, t}$ faces of that kind.

Proposition 7. *The Ehrhart polynomial of* $F(d;s,t)$ *respectively* $F'(d;s,t)$ *are the following.*

$$\mathrm{Ehr}(F(d;s,t),k) = \sum_{r=0}^{d'+1}(-1)^r\binom{d'+1}{r}\binom{(d-2t-2r+1)k+d'-r}{d'},$$

$$\mathrm{Ehr}(F'(d;s,t),k) = \sum_{r=0}^{d'}(-1)^r\binom{d'}{r}\binom{(d-2t-2r+1)k+d'-r}{d'}.$$

Proof. The polytope $F(d;s,t)+\mathbb{1}$ is a subset of the dilated standard simplex $(d-2t)\mathfrak{s}^{(d')}$. We want to count the integral points in $kF(d;s,t)+k\mathbb{1}$. If we denote by M_j the set of those points of $k\cdot(d-2t)\cdot\mathfrak{s}^{(d')}$ whose j^{th} coordinate exceeds $2k+1$:

$$M_j = \{\mathbf{x}\in[0,\infty)^{d'+1} : \sum x_i = k\cdot(d-2t+1), x_j \geq 2k+1\},$$

then these are the same as the integral points in

$$k\cdot(d-2t+1)\cdot\mathfrak{s}^{(d')}\setminus\bigcup_{j=1}^{d'+1}M_j.$$

Thus we have to count integral points in $M_{j_1\dots j_r} = M_{j_1}\cap\ldots\cap M_{j_r}$. But up to a translation by $-(2k+1)\sum e_{j_i}$ this is just the simplex $[k\cdot(d-2t-2r+1)-r]\cdot\mathfrak{s}^{(d')}$. The Ehrhart polynomial of a simplex is a binomial coefficient $\mathrm{Ehr}(\mathfrak{s}^{(d')},k) = \binom{d'+k}{d'}$ and hence the formula follows by inclusion/exclusion. The argument for $F'(d;s,t)$ is analogous. □

In order to deduce the Ehrhart series we have to compute the following kind of sums:

$$\sum_{k\geq 0}\binom{pk+q}{n}\tau^k = \frac{\sum_{i=0}^{n}\begin{bmatrix} n & i \\ p & q\end{bmatrix}\tau^i}{(1-\tau)^{n+1}}.$$

The coefficients $\begin{bmatrix} n & i \\ p & q\end{bmatrix} := \sum_{j=0}^{n}(-1)^j\binom{n+1}{j}\binom{pi-pj+q}{n}$ are the non-central Eulerian numbers [3]. So that we finally obtain

Proposition 8.

$$\psi_k(F(d;s,t)) = \sum_{r=0}^{d'+1}(-1)^r\binom{d'+1}{r}\begin{bmatrix} d' & k \\ d-2t-2r+1 & d'-r\end{bmatrix},$$

$$\psi_k(F'(d;s,t)) = \sum_{r=0}^{d'}(-1)^r\binom{d'}{r}\begin{bmatrix} d' & k \\ d-2t-2r+1 & d'-r\end{bmatrix}.$$

3.2. **Cartesian products.** For a complete treatment of the polytopes we had in mind, we have to observe how our combinatorial data behave with respect to products. The faces of $P \times P'$ are the sets of the form $F \times F'$ for faces F, F' of P, P'. Clearly

$$\mathrm{Ehr}(F \times F', n) = \mathrm{Ehr}(F, n) \cdot \mathrm{Ehr}(F', n),$$

and thus

$$a_k(F \times F') = \sum_{i+j=k} a_i(F) a_j(F').$$

In order to calculate the Hodge numbers in the general case

$$F = C_d^\vee \times F(d_1; s_1, t_1) \times \cdots \times F(d_p; s_p, t_p) \\ \times F'(d_1'; s_1', t_1') \times \cdots \times F'(d_q'; s_q', t_q'),$$

one determines the coefficients of the Ehrhart polynomial

$$a_i(F) = \sum_{|\alpha|=i} 2^{\alpha_0} \binom{d}{\alpha_0} \prod_{\nu=1}^{p} a_{\alpha_\nu}(F(d_\nu; s_\nu, t_\nu)) \prod_{\nu=1}^{q} a_{\alpha_{p+\nu}}(F'(d_\nu'; s_\nu', t_\nu'))$$

($\alpha \in \mathbb{Z}_{\geq 0}^{p+q+1}$, $|\alpha| = \alpha_0 + \cdots + \alpha_{p+q}$) and then applies the following linear transformation to get the Ψ-vector:

$$\Psi(F) = \mathbb{M}\ \mathbf{a}(F) \quad , \quad \mathbb{M}_{i,j} = \sum_{\nu=0}^{i} \binom{\dim F + 1}{\nu} (i - \nu)^j.$$

For example, the faces of $(\mathrm{DP}^6)^\vee$ have the following Ψ-vectors:

$$\begin{aligned}
\Psi(F(6;0,0) = (\mathrm{DP}^6)^\vee) &= (1, 386, 5405, 11964, 5405, 386, 1)\\
\Psi(F(6;1,0) = F(6;0,1)) &= (0, 90, 706, 765, 120, 1)\\
\Psi(F(6;1,1) = (\mathrm{DP}^4)^\vee) &= (1, 46, 136, 46, 1)\\
\Psi(F(6;2,0) = F(6;0,2)) &= (0, 5, 45, 25, 1)\\
\Psi(F(6;2,1) = F(6;1,2)) &= (0, 10, 12, 1)\\
\Psi(F(6;3,0) = F(6;0,3)) &= (0, 0, 0, 1)\\
\Psi(F(6;3,1) = F(6;1,3)) &= (0, 0, 1)\\
\Psi(F(6;2,2) = (\mathrm{DP}^2)^\vee) &= (1, 4, 1)\\
\Psi(F(6;3,2) = F(6;2,3)) &= (0, 1)\\
\Psi(F(6;3,3)) &= (1)
\end{aligned}$$

(All the faces of $(\mathrm{DP}^2)^\vee$ and $(\mathrm{DP}^4)^\vee$ appear in this list, e.g., $F(2;1,0) = F(4;2,1) = F(6;3,2)$.)

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Fachbereich Mathematik, TU-Berlin, Sekr. MA 7-1, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: Haase@Math.TU-Berlin.DE