# Cartographic Generation of Mathieu Groups

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6th April 1999

The Mathieu groups have now been known for a century and a quarter, but are still capable of surprising us.

([12], p. 330)

**Abstract**

This is an experimental work of constructing planar maps and hypermaps that "represent", in a certain way, Mathieu groups. Among numerous well known relations between group theory and combinatorics of maps, the notion of a cartographic group provides yet a new one. The main motivation for this study comes from the theory of "Dessins d'Enfants" ("Children's Drawings") [32], that is, the Galois group action on maps and hypermaps.

**Résumé**

Nous présentons un travail expérimental de construction de cartes et hypercartes qui "représentent", d'une certaine manière, les groupes de Mathieu. Parmi les nombreuses relations bien connues entre la théorie des groupes et la combinatoires des cartes, la notion de groupe cartographique en fournit une nouvelle. La principale motivation de cette étude vient de la théorie des "Dessins d'Enfants" [32], à savoir de l'action du groupe de Galois sur les cartes et hypercartes.

## 1 Introduction

It seems incredible that the "little mouse" shown in Figure 1 may represent, in a certain way, the Mathieu group $M_{12}$ (a permutation group of degree 12 and of order 95040). In Section 2 we explain in detail what we mean by the term "represent". But the idea is simple: it has now become classic to represent combinatorial maps as pairs of permutations (see [13]). These same permutations also generate a group: we call it a *cartographic group* of the map in question. Of course, in the majority of cases the group we obtain in this way is either the symmetric group $S_n$ or the alternating group $A_n$ [6]; this case is not very interesting. The majority of the remaining maps represent imprimitive groups (the last assertion is based not on rigorous results but rather on experience). We call a map *special* if it represents a primitive group different from $S_n$ and $A_n$. Special maps are rare: for example, there are 7 457 847 082 plane trees with 23 edges [22]; but only 4 of them are special: they represent the Mathieu group $M_{23}$ [5].

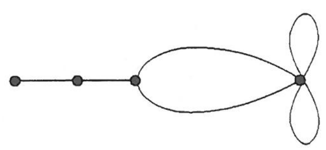

Figure 1: A planar map that represents the group $M_{12}$

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An exhaustive list of special plane *trees* can be found in [4]. The conjecture of [16] implies that the number of special plane maps is finite. The project of compiling an exhaustive list of such maps is probably too ambitious; but it can be carried out for small maps, what we are partially trying to do. The search of these maps is both difficult and enriching. The main motivation comes from the theory of "dessins d'enfants" (see, for example, [32], [31]), that is, from the study of the Galois action on maps. It turns out that the cartographic group is an invariant of this action [20], and a very powerful one. The *rigidity method* in the Inverse Galois Problem is largely based on the search for special maps ([1], [33]).

The problem of topological classification of the ramified coverings of the Riemann sphere ([21], [14]) leads to a braid group action on combinatorial objects which are generalized maps (we call them "constellations"). Once again, the cartographic group is a very powerful invariant that permits to distinguish different orbits of the braid group action.

Besides purely scientific motivations there is one of a rather esthetic, or psychological nature. In the paper [10] M. Conder discusses the methods of generating the Mathieu groups that would be easy to remember and to reconstruct "even in a desert". We find that some geometric images, such as the one in Figure 1, are very easy to remember; at least, much easier than the corresponding pair of permutations. Even more so, a map encodes only a pertinent structural information, without presenting the details that make sense only "up to a relabelling". (There is, however, an aspect that remains not visible in the picture. Two different labellings of the same map may represent the same permutation group (as a particular subgroup of $S_n$), but they may be not conjugate to each other inside this group; that is, one of them may be obtained from the other via an *outer* automorphism.) In all the statistical data below "different" means "non conjugate".

Our work should be considered in the context of the experimantal trend in mathematics, which consists, among other things, in compiling various catalogues, atlases, and similar lists of examples. Sometimes making such a list may be a goal in itself; more often it serves as a raw material for future research. Of many examples, we would like to cite here [7] and [19], for which our work provides a kind of a bridge.

We have limited our research to the Mathieu groups just in order to make the paper concise. Many other groups are equally interesting.

# 2 Maps, hypermaps, and constellations

**Definition 1 (Constellation; cartographic group)** A *k-constellation* $C = [g_1, g_2, \dots, g_k]$ of degree $n$ is a $k$-tuple of permutations $g_i \in S_n$ which satisfies two conditions:

1) $\prod_{i=1}^{k} g_i = \mathrm{id}$;

2) The group $G = <g_1, g_2, \dots, g_k>$ generated by the permutations $g_i$, $i = 1, 2, \dots, k$ acts transitively on the set $\{1, 2, \dots, n\}$.

The permutation group $G \leq S_n$ is called the *cartographic group* of the constellation $C$.

It is clear that non-trivial examples of constellations exist only for $k \geq 3$.

Two constellations $C = [g_1, \dots, g_k]$ and $C' = [g_1', \dots, g_k']$ are *isomorphic* if there exists an $h \in S_n$ such that $g_i' = h^{-1} g_i h$ for $i = 1, \dots, k$; they are *conjugate* if $h \in G$.

We say that a permutation $g \in S_n$ is *of type* $\lambda \vdash n$, if the parts of $\lambda$ are equal to the cycle lengths of $g$. The number of parts of $\lambda$, which is also the number of cycles of $g$, is denoted by $z(\lambda)$.

**Definition 2 (Passport)** The list $\pi = [\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}]$ of $k$ partitions of $n$ that represent the types of the permutations of a constellation is called the *passport* of this constellation.

If $G = S_n$, the passport provides complete information about conjugacy classes of $g_i$. For other groups such information must often be added in order to refine the information given in the passport. A list $[K_1, K_2, \dots, K_k]$ of conjugacy classes in $G$ such that $g_i \in K_i$ is called a *refined passport* of the constellation $[g_1, g_2, \dots, g_k]$.

If we consider a constellation as a ramification data of a covering of the Riemann sphere, then the group $G$ becomes the *monodromy group* of the covering, and the corresponding Riemann surface is of genus $g$, where $g$ satisfies the Riemann–Hurwitz equation

$$\sum_{i=1}^{k} z(\lambda^{(i)}) - (k-2)n = 2 - 2g.$$

We may find this formula as follows: draw a map on the Riemann sphere, with the vertices at the branch points; take the preimage of this map under the covering projection. Then the Euler formula for the resulting preimage map gives the Riemann–Hurwitz formula. (Some variations of this procedure are possible. For example, some branch point may be placed inside the faces of the map on the Riemann sphere, but not more than one branch point per face.) Some basic theory of this relation is set out, for example, in [30]. In fact, our notion of constellation is very close to that of "marked finite transitive permutation group" in [30].

This construction, besides giving a geometric representations of a constellation, leads to profound relations of Riemann surfaces to combinatorics of maps, to their enumeration and to some algorithmic questions, such as explicit computation of coverings [24].

The most interesting case is that of $k = 3$. First, it is for $k = 3$ that the absolute Galois group (the automorphism group of the field $\overline{\mathbb{Q}}$ of algebraic numbers) acts on the constellations. Second, this case is the closest one to the classical combinatorics. We may take, as an underlying map on the Riemann sphere, a segment that joins two out of three branch points. Then its preimage is a bipartite map (or, equivalently, a hypermap [13]): its black and white vertices are preimages of one or the other of the segment ends, and inside each face there is exactly one preimage of the third branch point. The permutations of the constellation may be considered as acting on the edges of this map. If all the white vertices are of degree 2, we may "erase" them from the picture and consider the former edges of the bipartite map as the half-edges of the resulting map. In this case the permutations act on the half-edges.

**Example 1** Let us label the half-edges of the map of Figure 1 as is shown in Figure 2.

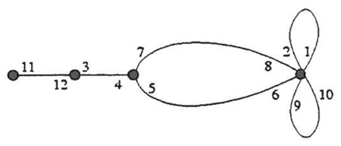


Figure 2: The "mouse map" with labelled half-edges

The permutation

$$g_1 = (1,2,8,6,9,10)(3,12)(4,5,7)$$

shows the cyclic order of the half-edges around the vertices of the map, and in this way it "describes" the vertices. The permutation

$$g_2 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$

describes the edges of the map: to each half-edge it assigns the opposite half-edge of the same edge. Finally, the permutation

$$g_3 = (g_1 g_2)^{-1} = (2,10,6,4,12,11,3,7)(5,8)$$

describes the faces: it shows the order in which we encounter the half-edges (one half-edge for each edge) when we go around a face.

The passport of this constellation is $\pi = [6^1 3^1 2^1 1^1, 2^6, 8^1 2^1 1^2]$: the first partition gives the vertex degrees, the second one, the edge degrees (all of them are equal to 2 because it is a usual map and not a hypermap), and the third one, the face degrees. And, as we have already said at the beginning of the introduction, the group $<g_1, g_2, g_3>$ generated by $g_1, g_2, g_3$ is the Mathieu group $M_{12}$. The conjugacy classes of $g_1, g_2, g_3$ are, respectively, $6B, 2A, 8B$ (we use the notation of the Atlas [11]). This triple $[6B, 2A, 8B]$ is the refined passport of the constellation.

Let $K_1, K_2, \dots, K_k$ be conjugacy classes in a group $G$. Denote by $\Sigma(K_1, K_2, \dots, K_k)$ the number of the $k$-tuples of permutations $(g_1, g_2, \dots, g_k)$ such that $g_i \in K_i$ and $\prod_{i=1}^{k} g_i = \mathrm{id}$. The following formula of Frobenius is one of our most important tools:

$$\Sigma(K_1, K_2, \dots, K_k) = \frac{|K_1||K_2|\dots|K_k|}{|G|} \sum_{\chi} \frac{\chi(K_1)\chi(K_2)\dots\chi(K_k)}{\chi(\mathrm{id})^{k-2}}, \tag{1}$$

where the sum is taken over all the irreducible complex characters of the group $G$.

The formula looks somewhat scary, but in fact it is very easy to use provided we have the character table of the group in question. Quite often the computations may be carried out by hand.

**Remark 1** The objects counted by the Frobenius formula do not necessarily satisfy the second condition of the constellation: they may be not transitive. Even if they are, they may generate not the group $G$ itself but some of its subgroups $H \leq G$. The following lemma is evident:

**Lemma 1** *If the subgroup $H = <g_1, g_2, \dots, g_k>$ has a trivial centralizer in $G$, then the tuple $(g_1, g_2, \dots, g_k)$ has exactly $|G|$ conjugate copies in $G$, counted by the number $\Sigma(K_1, K_2, \dots, K_k)$.*

Therefore in order to have an idea of the number of constellations *up to a conjugacy* it is often convenient to compute $\frac{\Sigma(K_1, K_2, \dots, K_k)}{|G|}$. If the center of $G$ is trivial, this number provides an upper bound for the number of constellations generating $G$, up to a conjugacy.

# 3 Strategies

## 3.1 Limiting conditions

To look for all possible generating sets $[g_1, g_2, \dots, g_k]$ such that $<g_1, g_2, \dots, g_k> = G$ would be completely useless, as they are abundant. We limit our research by the following conditions:

- We consider only the constellations with $k = 3$.
- We consider only the planar case: $g = 0$.
- We take as cartographic groups only five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$.
- We consider only the minimal presentations of these groups.
- Whenever possible, we look for *maps* and *trees*.
- We try to single out constellations with nice visual properties, or those for which the computation of Belyi functions [24] would seem interesting and/or not exceedingly difficult.

## 3.2 Generation strategies

### 3.2.1 Probabilistic algorithm

Our main algorithm consists of the following steps. Let a refined passport $\pi = [A, B, C]$ be given, $A, B, C$ being conjugacy classes in $G$.

1. Find an $f \in A$ and a $g_2 \in B$.
2. Conjugate $f$ by a random permutation $h \in G$, thus obtaining a random element $g_1 \in A$.

3. Verify if $g_3 = (g_1 g_2)^{-1} \in C$.
4. If yes, verify if $<g_1, g_2, g_3> = G$; to do that we may just compute the order of the group generated by $g_1, g_2, g_3$: if it is equal to $|G|$, then $g_1, g_2, g_3$ generate the whole group $G$.

It goes without saying that we actively use systems of symbolic computations, namely, Maple, GAP and Magma. For the details on group theory algorithms see [8].

### 3.2.2 Verification of isomorphism

Each newly found triple $[g_1, g_2, g_3]$ satisfying the condition 3 above must be tested on an isomorphism with the objects previously found. The algorithm of the isomorphism testing is quadratic, though comparing non transitive triples presents some technical difficulties.

The triples that do not satisfy the condition 4 must not be thrown away: we need them in order to be sure that the bound given by the Frobenius formula is attained and we may stop the search.

### 3.2.3 Manual methods

Paradoxically enough, there are no efficient algorithms to generate maps or hypermaps with a given passport. But for some special cases this operation may easily be carried out by hand.

**Example 2** The passport $\pi = [3^6 1^6, 2^{12}, 23^1 1^1]$ is one of the possibilities to be considered for generating $M_{24}$, as all the three partitions are cyclic structures of some elements of $M_{24}$. It is easy to see that all maps having this passport contain a loop (a face of degree 1), with a binary rooted trees with 5 internal vertices attached to it. It is well-known that there are Catalan$(n) = C_n = \frac{1}{n+1}\binom{2n}{n}$ binary rooted trees of $n$ internal vertices. Thus we get $C_5 = 42$ "candidate" maps to be tested. Only two of them do represent $M_{24}$: the map given in Figure 3, and its mirror image. This result (the existence of two maps) also corresponds to the Frobenius formula. This example was first found by Adrianov [3].

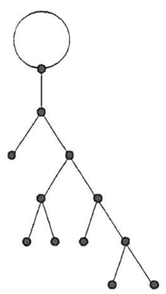

Figure 3: A map of passport $[3^6 1^6, 2^{12}, 23^1 1^1]$ representing $M_{24}$

### 3.2.4 Using the braid group action

The following trick may sometimes be useful. Define an operation $\sigma_i$ on the set of $k$-constellations:

$$\sigma_i : [g_1, \ldots, g_k] \mapsto [g_1^{'}, \ldots, g_k^{'}],$$

where

$$g_i^{'} = g_{i+1}, \quad g_{i+1}^{'} = g_{i+1}^{-1} g_i g_{i+1}, \quad g_j^{'} = g_j \text{ for } j \neq i, i+1.$$

One may easily verify that these operations generate an action of the braid group $B_k$ on $k$-constellations: namely,

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

(see [21], [14]). This action is nontrivial for $k \geq 4$. It is clear that the cartographic group is invariant under this action. Therefore we may proceed as follows:

1. Take a 3-constellation $C = [g_1, g_2, g_3]$ representing a group $G$.
2. Take a random element $h \in G$, and replace $C$ by a 4-constellation $[g_1, g_2, g_3 h^{-1}, h]$.
3. Act several times on this constellation by the braid group (or, rather, by the pure braid group, which does not permute the conjugacy classes).
4. Having obtained a 4-constellation $[f_1, f_2, f_3, f_4]$, reduce it to the 3-constellation $[f_1, f_2, f_3 f_4]$. (This last operation may give a subgroup of $G$ instead of the whole group $G$.)

In this way we may hope to get some constellations that would be difficult to find otherwise. The general question of connectedness of the set of all constellations generating $G$ under this operation is interesting.

### 3.3 Elimination strategies

Some of the condidates to represent the Mathieu groups may be eliminated by very simple geometric consideration.

**Example 3** One of the a priori possible passports for the group $M_{24}$ is $\pi = [3^8, 2^{12}, 4^6]$. But we know the (only) corresponding map pretty well! It is the cube, and its cartographic group is isomorphic to $S_4$.

A more general observation is given in the following

**Proposition 1** *If all face degrees of a map are even, then its cartographic group is imprimitive.*

Indeed, the map in question is bipartite, and the half-edges may be grouped into two blocks: those adjacent to the black vertices and those adjacent to the white ones.

It is clear that a map and its dual have the same cartographic group; thus all the maps with even vertex degrees are also eliminated.

### 3.4 How to verify our results

Some of the results presented below are easy to verify; the others are not.

For any map (resp. hypermap) one may label the half-edges (resp. edges), write down the corresponding permutations and verify (using the Maple "group package", for example) that the order of the cartographic group is what it should be. This operation takes several seconds of computer time. Taking into account that the Mathieu groups are the only groups of given order and degree, we may conlude that the (hyper)map indeed generates the group in question.

The computations with character tables are even easier to verify. The tables themselves may be found in [11].

What is not at all easy to verify is the fact that our lists are complete. In order to do that one must have the complete list of all the triples of permutations representing the number $\Sigma(K_1, K_2, K_3)$, and for each element of this list, the full information on the group generated by it and on the number of its conjugate copies inside $G$. We don't see any way of presenting these results in a concise form. Therefore this statement remains the authors' responsiblity.

(An alternative way would be to use the so-called "local analysis" inside the group: that is, the inclusion-exclusion method on the lattice of subgroups of $G$, together with character table computations for every subgroup. This method was used (for another reason) in [34]. To use it in our context would be also very complicated.)

# 4 Results for the Mathieu groups

The five Mathieu groups, $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$, have been discovered last century by Émile Mathieu [25], [26], [27]. For about 100 years they were the only known sporadic simple groups (the term "sporadic groups" itself was first applied to these groups by Burnside in 1911). They have many remarkable properties. For example, besides $S_n$ which is $n$-transitive, and $A_n$ which is $(n-2)$-transitive, the groups $M_{12}$ and $M_{24}$ are the only 5-transitive groups, and $M_{11}$ and $M_{23}$ are the only 4-transitive groups. For their other (and numerous) properties the reader may consult papers and books cited in our bibliography.

We do not give an independent definition of these groups, as any one of the (hyper)maps given below may serve as a definition.

## 4.1 $M_{11}$

This is a group of degree 11 (in its minimal presentation) and of order 7 920. The group obviously does not contain an involution without fixed points and therefore cannot be represented by a map.

**Theorem 1** *There exist* 318 *planar hypermaps (among them* 76 *essentially different ones) with the cartographic group* $M_{11}$ *in its minimal presentation.*

| | Passport | Refined passport | Nb of hypermaps | Coeff |
|---|---|---|---|---|
| 1 | $[3^3 1^2; 3^3 1^2; 8^1 2^1 1^1]$ | $[3A, 3A, 8A$ or $8A^{-1}]$ | $2 \times 4$ | 3 |
| 2 | $[3^3 1^2, 4^2 1^3, 5^2 1^2]$ | $[3A, 4A, 5A]$ | 5 | 6 |
| 3 | $[3^3 1^2, 4^2 1^3, 6^1 3^1 2^1]$ | $[3A, 4A, 6A]$ | 5 | 6 |
| 4 | $[3^3 1^2; 4^2 1^3; 8^1 2^1 1^1]$ | $[3A, 4A, 8A$ or $8A^{-1}]$ | $2 \times 3$ | 6 |
| 5 | $[4^2 1^3; 4^2 1^3; 5^2 1^1]$ | $[4A, 4A, 5A]$ | 6 | 3 |
| 6 | $[4^2 1^3; 4^2 1^3; 6^1 3^1 2^1]$ | $[4A, 4A, 6A]$ | 12 | 3 |
| 7 | $[4^2 1^3; 4^2 1^3; 8^1 2^1 1^1]$ | $[4A, 4A, 8A$ or $8A^{-1}]$ | $2 \times 8$ | 3 |
| 8 | $[2^4 1^3; 4^2 1^3; 11^1]$ | $[2A, 4A, 11A$ or $11A^{-1}]$ | $2 \times 1$ | 6 |
| 9 | $[2^4 1^3; 5^2 1^1; 8^1 2^1 1^1]$ | $[2A, 5A, 8A$ or $8A^{-1}]$ | $2 \times 3$ | 6 |
| 10 | $[2^4 1^3; 6^1 3^1 2^1; 8^1 2^1 1^1]$ | $[2A, 6A, 8A$ or $8A^{-1}]$ | $2 \times 3$ | 6 |
| 11 | $[2^4 1^3; 8^1 2^1 1^1; 8^1 2^1 1^1]$ | $[2A, 8A, 8A^{-1}]$ or $[2A, 8A^{-1}, 8A]$ | $2 \times 2$ | 3 |
| | TOTAL | | 76 (or 318) | |

Table 1: Statistics of the planar hypermaps representing $M_{11}$

Some comments concerning Table 1 are due.

(1) The column "Nb of hypermaps" gives this number for the corresponding *ordered* passport. If all three conjugacy classes in a refined passport $[A, B, C]$ are different, then they may be permuted in 6 ways; thus, the above number must be multiplied by 6. If two of the classes are equal, the coefficient is 3 instead of 6. This coefficient is given in the column "Coeff".

(2) Cyclic structures $8^1 2^1 1^1$ and $11^1$ correspond to *two* (mutually inverse) conjugacy classes. This explains the notation $2 \times 4$ etc. for the number of hypermaps. In such a case only one half of the pictures in question is given in Figure 4.

(3) If $[g_1, g_2, g_3]$ is a hypermap, its mirror image may be labelled in such a way that one gets $[g_1^{-1}, g_2^{-1}, g_2 g_1]$, and the third permutation $g_2 g_1$ is conjugate to $g_3^{-1} = g_1 g_2$. Therefore the mirror image of a hypermap corresponding to $[A, B, C]$ should have the refined passport $[A, B, C^{-1}]$. However, it may also appear inside the $[A, B, C]$ family, as one may observe by closely scrutinizing Figure 4. Here we are confronted to the "isomorphism vs. conjugacy" phenomenon briefly mentioned in the Introduction.

The geometric form of the hypermaps in Figure 4 may seem strange at first sight. The explanation is that the first author has designed a special software for drawing maps using permutations

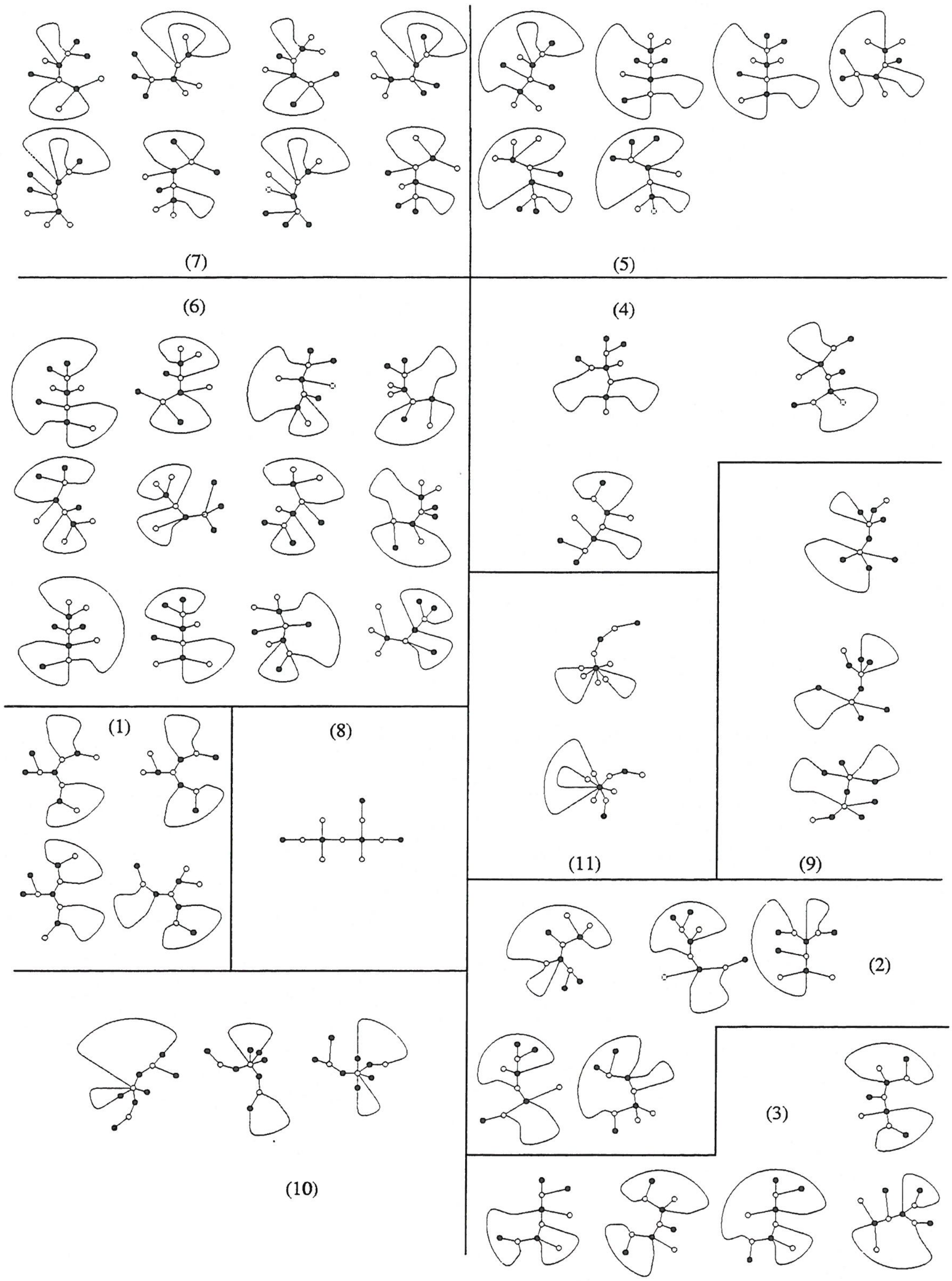


Figure 4: Planar hypermaps with cartographic group $M_{11}$

as an input data [17], and all the pictures in Figures 4 and 5 are drawn automatically using this software.

## 4.2 $M_{12}$

This is a group of degree 12 (in its minimal presentation) and of order 95 040.

**Theorem 2** *There exist* 50 *planar maps (among them* 31 *essentially different ones), and, besides them,* 1287 *planar hypermaps (among them* 272 *essentially different ones) with cartographic group* $M_{12}$ *in its minimal presentation.*

Figure 5 contains all the 50 planar maps that represent $M_{12}$.

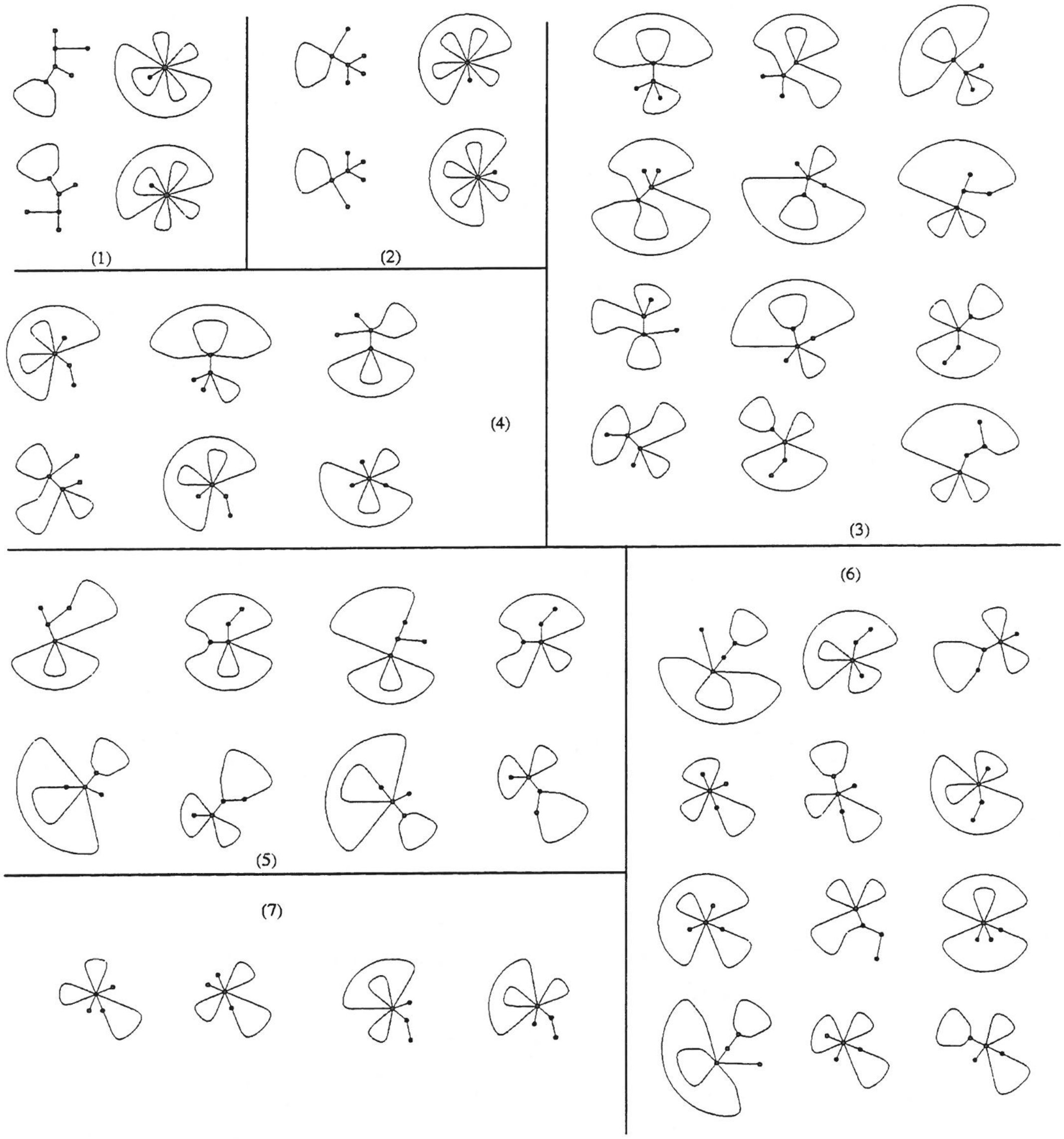


Figure 5: Planar maps with cartographic group $M_{12}$

Table 2 summarizes the information about these 50 maps. As it concerns only maps, we don't repeat every time the cyclic structure $2^6$ (and the corresponding conjugacy class $2A$) for the

| | Passport | Refined passport | Nb of maps | Coeff |
|---|---|---|---|---|
| 1 | $[3^3 1^3; 11^1 1^1]$ | $[3A, 11A \text{ or } 11A^{-1}]$ | $2 \times 1$ | 2 |
| 2 | $[4^2 1^4; 11^1 1^1]$ | $[4B, 11A \text{ or } 11A^{-1}]$ | $2 \times 1$ | 2 |
| 3 | $[5^2 1^2; 6^1 3^1 2^1 1^1]$ | $[5A, 6B]$ | 6 | 2 |
| 4 | $[5^2 1^2; 8^1 2^1 1^2]$ | $[5A, 8B]$ | 3 | 2 |
| 5 | $[6^1 3^1 2^1 1^1; 6^1 3^1 2^1 1^1]$ | $[6B, 6B]$ | 8 | 1 |
| 6 | $[6^1 3^1 2^1 1^1; 8^1 2^1 1^2]$ | $[6B, 8B]$ | 6 | 2 |
| 7 | $[8^1 2^1 1^2; 8^1 2^1 1^2]$ | $[8B, 8B]$ | 4 | 1 |
| | TOTAL | | 31 (or 50) | |

Table 2: Statistics of the planar maps representing $M_{12}$

edges. We also consider the position of this class inside the passport fixed, and thus the value of the "coefficient" is reduced to 2 or 1. Similar details concerning hypermaps representing $M_{12}$ may be found in [17].

**Remark 2** Using the same methods we obtained maps of higher genera representing $M_{12}$. There are 124 maps of genus $g = 1$, 96 maps of genus $g = 2$, and there are no maps of genus $g > 2$ (in the minimal presentation).

### 4.3 $M_{22}$

This is a group of degree 22 (in its minimal presentation) and of order 443 520. It does not contain an involution without fixed points, therefore it is impossible to represent this group by a map.

**Theorem 3** *There exist* 657 *planar hypermaps (among them* 163 *essentially different ones) with the cartographic group* $M_{22}$ *in its minimal presentation.*

More detailed information concerning this group may be found in [17]. Here we only comment on a very interesting case presented below. There are two conjugacy classes in $M_{22}$ with cyclic structure $4^4 2^2 1^2$, namely, $4A$ and $4B$ (see [11]). These classes are not inverse to one another; they even have different size. There are 45 hypermaps with the passport $\pi = [4^4 2^2 1^2, 4^4 2^2 1^2, 4^4 2^2 1^2]$. Among them, there are 6 hypermaps with the refined passport $[4B, 4B, 4B]$, and $3 \times 13 = 39$ ones with $[4A, 4B, 4B]$. As N. Adrianov explained to us [3], these sets form (at least) four different orbits of the Galois group action, the latter statement being one of the consequences of the Ihara–Drinfeld theory. This is the only known example of this nature up to now.

Figure 6 presents one of the hypermaps with the refined passport $[4B, 4B, 4B]$.

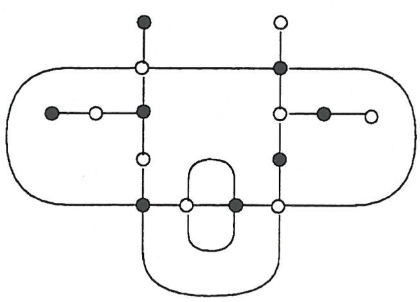

Figure 6: A nice butterfly representing $M_{22}$

## 4.4 $M_{23}$

This is a group of degree 23 (in its minimal presentation) and of order 10 200 960. Evidently there is no involution without fixed points, and therefore the group cannot be represented by a map.

This case could in principle be the most interesting one, as $M_{23}$ is the only sporadic simple group for which it is still unknown if it can be realized as Galois group over $\mathbb{Q}$ (see [33]). Unfortunately, the abundance of the hypermaps representing this group did not permit us to find anything specifically interesting.

**Theorem 4** *There exist* 6348 *planar hypermaps (among them* 1596 *essentially different ones) with the cartographic group $M_{23}$ in its minimal presentation.*

Four of the above hypermaps are trees: they were found in [5]. The corresponding number field is computed by Yu. Matiyasevich and M. Vsemirnov, see [28].

## 4.5 $M_{24}$

This is a group of degree 24 (in its minimal presentation) and of order 244 823 040. Below we concentrate only on maps. In the last line of Table 3 (marked by an asterisk) our results are not final: we have not attained the bound given by the Frobenius formula. However, not more than one map may be missing, and most probably nothing is missing at all.

**Theorem 5** *There exist* 130 *or* 131 *essentially different planar maps with the cartographic group $M_{24}$ in its minimal presentation.*

| | Passport | Refined passport | Bound | Nb of maps | Coeff |
|---|---|---|---|---|---|
| 1 | $[3^6 1^6, 21^1 3^1]$ | $[3A, 21A$ or $21A^{-1}]$ | $2\times 1$ | $2\times 1$ | 2 |
| 2 | $[3^6 1^6, 23^1 1^1]$ | $[3A, 23A$ or $23A^{-1}]$ | $2\times 1$ | $2\times 1$ | 2 |
| 3 | $[4^4 2^2 1^4, 11^2 1^2]$ | $[4B, 11A]$ | 18 | 11 | 2 |
| 4 | $[4^4 2^2 1^4, 14^1 7^1 2^1 1^1]$ | $[4B, 14A$ or $14A^{-1}]$ | $2\times \frac{43}{2}$ | $2\times 8$ | 2 |
| 5 | $[4^4 2^2 1^4, 15^1 5^1 3^1 1^1]$ | $[4B, 15A$ or $15A^{-1}]$ | $2\times 13$ | $2\times 8$ | 2 |
| 6 | $[5^4 1^4, 7^3 1^3]$ | $[5A, 7A]$ | 8 | 5 | 2 |
| 7 | $[5^4 1^4, 8^2 4^1 2^1 1^2]$ | $[5A, 8A]$ | 35 | 14 | 2 |
| 8 | $[6^2 3^2 2^2 1^2, 7^3 1^3]$ | $[6A, 7A$ or $7A^{-1}]$ | $2\times 32$ | $2\times 11$ | 2 |
| 9 | $[6^2 3^2 2^2 1^2, 8^2 4^1 2^1 1^2]$ | $[6A, 8A]$ | $\frac{153}{2}$ | 42* | 2 |
| | TOTAL | | | 130 | |

Table 3: Statistics of the planar maps representing $M_{24}$

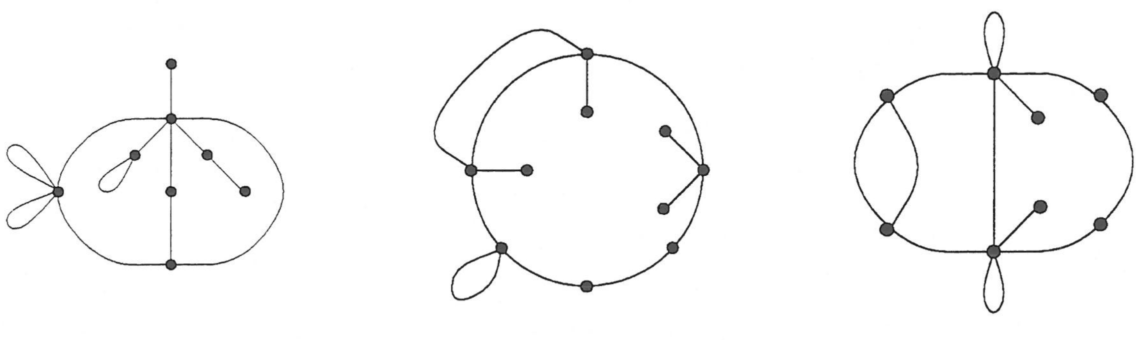

Figure 7: Three maps representing $M_{24}$

In Table 3 we omitted references to the conjugacy class $2B = 2^{12}$ for edges. Some examples of maps representing $M_{24}$ are given in Figures 3 and 7. Other pictures may be found in [35] and [17].

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