Complexes of not 3-connected graphs

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Abstract

Using techniques from the discrete Morse theory developed by Robin Forman [F], we prove that the simplicial complex Δ_n^3 of not 3-connected graphs on n vertices is homotopy equivalent to a wedge of $(n-3)\frac{(n-2)!}{2}$ spheres of dimension $2n-4$, thereby verifying a conjecture by E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker [BBLSW]. In particular, the reduced Euler characteristic

$$\sum_{G \in \Delta_n^3} (-1)^{|E(G)|-1}$$

is equal to $(n-3)\frac{(n-2)!}{2}$; $E(G)$ is the edge set of G.

1 Introduction

Given a family of graphs on a fixed vertex set, we may identify the graphs in the family with their edge sets. If the family is closed under deletion of edges, then this identification makes it possible to interpret the family as a simplicial complex. The purpose of this paper is to study simplicial complexes of not 3-connected graphs. The main tool will be the discrete Morse theory developed by Robin Forman [F].

For topological spaces X and Y,

$$X \vee Y := (X \times \{y\}) \cup (\{x\} \times Y) \subseteq X \times Y$$

is the *wedge* of X and Y (with respect to the points $x \in X$ and $y \in Y$). For an abstract simplicial complex Δ, let $||\Delta||$ denote the geometric realization of Δ. Let Δ_n^2 be the complex of not 2-connected graphs on n vertices. E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker [BBLSW] have proved that $||\Delta_n^2||$ is homotopy equivalent to

$$\bigvee_{(n-2)!} S^{2n-5},$$

that is, a wedge of $(n-2)!$ spheres of dimension $2n-5$. A homology version was proved independently by V. Turchin [T]. We will verify a conjecture in [BBLSW] about the complex Δ_n^3 of not 3-connected graphs on n vertices; $||\Delta_n^3||$ is homotopy equivalent to

$$\bigvee_{(n-3)\frac{(n-2)!}{2}} S^{2n-4}.$$

Each of the $(n-3)\frac{(n-2)!}{2}$ spheres corresponds to a graph with vertex set

$$\{a_1, a_2, \ldots, a_n\} = \{1, \ldots, n\}$$

and with edge set

$$\{a_ia_{i+1} : 1 \leq i \leq n-1\} \cup \{a_ia_{i+2} : 1 \leq i \leq k\} \cup \{a_ka_i : k+3 \leq i \leq n\},$$

where $2 \leq k \leq n-2$, $a_1 = 1$, $a_2 < a_3$, and $a_n = n$. An example for $n = 9$ is illustrated in Figure 4.1.

Acknowledgment

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2 Discrete Morse theory

In this section we will discuss some consequences of Robin Forman's discrete Morse theory in the special case of simplicial complexes. We will define the concept of an acyclic matching on a family of sets and interpret some of the basic theorems in [F]. Note that combinatorial interpretations of discrete Morse theory have been made earlier by M. K. Chari [Ch] and by J. Shareshian [Sh].

Let $\mathcal{B}_n$ be the family of all subsets of a set X_n with n elements and let $\mathcal{A}$ be a sub-family to $\mathcal{B}_n$. A *matching* on $\mathcal{A}$ is a family $\mathcal{P}$ of pairs $\{A, B\}$ $(A, B \in \mathcal{A})$ such that no $A \in \mathcal{A}$ is contained in more than one pair in $\mathcal{P}$. We say that a matching $\mathcal{P}$ on $\mathcal{A}$ is an *element matching* if every pair in $\mathcal{P}$ is of the form $\{A \backslash \{x\}, A \cup \{x\}\}$ for some $A \subseteq X_n, x \in X_n$. A set $A \in \mathcal{A}$ is *critical* or *unmatched* with respect to $\mathcal{P}$ if A is not contained in any pair in $\mathcal{P}$. Let $\mathcal{U} = \mathcal{U}(\mathcal{A}, \mathcal{P})$ be the family of critical sets in $\mathcal{A}$ with respect to $\mathcal{P}$.

We let $D = D(\mathcal{A}, \mathcal{P})$ be a digraph with vertex set $\mathcal{A}$ and with an arc (A, B) if and only if either

$$\{A, B\} \in \mathcal{P} \text{ and } B = A \cup \{x\} \text{ for some } x \notin A$$

or

$$\{A, B\} \notin \mathcal{P} \text{ and } A = B \cup \{x\} \text{ for some } x \notin B.$$

Thus every arc in D corresponds to an edge in the Hasse diagram of $\mathcal{A}$. We write $A \rightsquigarrow B$ if there is a directed path from A to B in D and $\mathcal{V} \rightsquigarrow B$ if there is some $V \in \mathcal{V}$ such that $V \rightsquigarrow B$.

The element matching $\mathcal{P}$ is an *acyclic matching* if D is cycle-free, that is, $A \rightsquigarrow B$ and $B \rightsquigarrow A$ implies that $A = B$. One easily proves that if there are cycles in a digraph D corresponding to an element matching, then they are of the form $(A_1, B_1, \ldots, A_r, B_r)$, where

$$A_i \subset B_i, A_i \subset B_{i-1} \ (B_0 = B_r), \text{ and } \{A_i, B_i\} \in \mathcal{P} \tag{1}$$

(the doubtful reader may consult [Sh]). The following two lemmas are very easy, but they will simplify the proofs in later sections.

Lemma 2.1 *Let $\mathcal{A} \subseteq \mathcal{B}_n$ and $x \in X_n$. Put*

$$\mathcal{P}_x(\mathcal{A}) = \{\{A \setminus \{x\}, A \cup \{x\}\} : A \setminus \{x\}, A \cup \{x\} \in \mathcal{A}\}$$

and

$$\mathcal{A}_x = \{A : A \setminus \{x\}, A \cup \{x\} \in \mathcal{A}\}.$$

Let $\mathcal{P}_0$ be an acyclic matching on $\mathcal{A}_0 := \mathcal{A} \setminus \mathcal{A}_x$. Then $\mathcal{P} := \mathcal{P}_x(\mathcal{A}) \cup \mathcal{P}_0$ is an acyclic matching on $\mathcal{A}$.

Proof. Let $(A_1, B_1, \ldots, A_r, B_r)$ be a cycle in $D = D(\mathcal{A}, \mathcal{P})$ satisfying (1). Since $\mathcal{P}_0$ is an acyclic matching on $\mathcal{A}_0$, there must be some pair $\{A_i, B_i\} \in \mathcal{P}_x(\mathcal{A})$, where $B_i = A_i \cup \{x\}$); we may assume that $i = 1$. (A_1, B_r) is not an arc in D, which implies that $x \notin B_r$. Thus there is a $j > 1$ such that $x \notin A_j$ but $x \in B_{j-1}$. However, this means that $B_{j-1} = A_j \cup \{x\}$ and $\{A_j, B_{j-1}\} \in \mathcal{P}_x(\mathcal{A})$, which is certainly a contradiction. □

Lemma 2.2 *(Concatenation Lemma) Let $\mathcal{A} \subseteq \mathcal{B}_n$ and $\mathcal{A} = \dot{\bigcup}_{i=1}^{r} \mathcal{A}_i$ (disjoint union). Let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be acyclic matchings on $\mathcal{A}_i, \ldots, \mathcal{A}_r$, respectively, and put*

$$\mathcal{P} = \bigcup_{i=1}^{r} \mathcal{P}_i.$$

Define the relation $\dashv$ on $\{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$ by

$$\mathcal{A}_i \dashv \mathcal{A}_j \iff A \subseteq B \textit{ for some } A \in \mathcal{A}_i, B \in \mathcal{A}_j.$$

Suppose that $\dashv$ gives a partial order on $\{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, that is, if $\mathcal{A}_i \dashv \mathcal{A}_j$ and $\mathcal{A}_j \dashv \mathcal{A}_i$, then $i = j$. Then $\mathcal{P}$ is an acyclic matching.

Proof. By assumption, a cycle in $D = D(\mathcal{A}, \mathcal{P})$ cannot be completely contained in any $\mathcal{A}_i$. If a set in $\mathcal{A}_j$ is followed by a set in $\mathcal{A}_i$, then $\mathcal{A}_i \dashv \mathcal{A}_j$. Thus each time we go from one family to another, we go down in the poset defined by $\dashv$; hence we cannot find a cycle in D. □

For the rest of this section, $\mathcal{A}$ is a simplicial complex, that is, $\mathcal{A} \supsetneq \{\phi\}$ and $\mathcal{A}$ is closed under deletion of elements. Given an acyclic matching $\mathcal{P}$ on $\mathcal{A}$, there is no loss of generality assuming that the empty set is contained in a pair in $\mathcal{P}$. Namely, if all 1-sets are matched with larger sets, then there is a cycle in $D(\mathcal{A}, \mathcal{P})$. Unless otherwise stated, we will always assume that $\{\phi, \{x\}\} \in \mathcal{P}$ for some $x \in X_n$. The following two results are interpretations of two of the basic theorems in discrete Morse theory.

Theorem 2.3 ([F], Theorem 3.3) *Let $\mathcal{P}$ be an acyclic matching on the complex $\mathcal{A}$. If $\mathcal{A}_0$ is a subcomplex of $\mathcal{A}$ such that $\mathcal{A}_0 \not\rightarrow \mathcal{A} \setminus \mathcal{A}_0$ and $\mathcal{U}(\mathcal{A}, \mathcal{P}) \subseteq \mathcal{A}_0$, then $||\mathcal{A}||$ and $||\mathcal{A}_0||$ are homotopy equivalent.* □

Theorem 2.4 ([F], Theorem 3.4) *If $\mathcal{U}(\mathcal{A}, \mathcal{P})$ consists of one single set of size $p + 1$, ($p \geq 0$), then $||\mathcal{A}||$ is homotopy equivalent to a sphere of dimension p.* □

Example. Consider the simplicial complex Σ on the set $\{1, 2, 3, 4, 5, 6\}$ consisting of all subsets of 124, 245, 23, 35, and 36; 124 denotes the set $\{1, 2, 4\}$, and so on. In Figure 2.1, a geometric realization of Σ is illustrated. The figure indicates an acyclic matching on Σ with the only critical set 35; an arrow from the set A to the set B means that A and B are matched. We also match 2 and the empty set; however, since the empty set has no obvious geometric interpretation, it is customary to consider 2 as a critical point in the geometric realization. Note that $||\Sigma||$ is homotopy equivalent to a CW complex consisting of a 1-cell corresponding to 35 and a 0-cell corresponding to 2. □

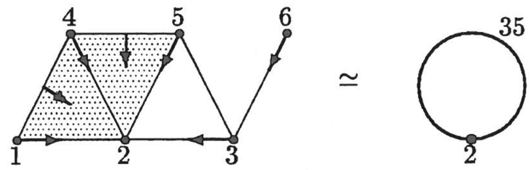

Figure 2.1: $\|\Sigma\|$ is homotopy equivalent to a circle.

For a (possibly empty) family $\mathcal{V} \subseteq \mathcal{U}$, put

$$\mathcal{A}_{\mathcal{V}} = \{A \in \mathcal{A} : \mathcal{V} \rightsquigarrow A\} \cup \{\phi, \{x\}\},$$

where $\{x\}$ is the set matched with the empty set in $\mathcal{P}$; if $\mathcal{V}$ is nonempty, then $\mathcal{A}_{\mathcal{V}} = \{A \in \mathcal{A} : \mathcal{V} \rightsquigarrow A\}$. The next result implies that

$$\mathcal{U}(\mathcal{A}_{\mathcal{V}}, \mathcal{P}_{\mathcal{V}}) = \mathcal{A}_{\mathcal{V}} \cap \mathcal{U}(\mathcal{A}, \mathcal{P}),$$

where $\mathcal{P}_{\mathcal{V}}$ is the restriction of $\mathcal{P}$ to $\mathcal{A}_{\mathcal{V}}$.

Lemma 2.5 *$\mathcal{A}_{\mathcal{V}}$ is a subcomplex of $\mathcal{A}$.*

Proof. Assume the opposite and let A be a largest set such that $A \notin \mathcal{A}_{\mathcal{V}}$ but there is an $y \in X_n$ such that $A \cup \{y\} \in \mathcal{A}_{\mathcal{V}}$. Since there is a $V \in \mathcal{V}$ such that $V \rightsquigarrow A \cup \{y\}$, $\{A, A \cup \{y\}\} \in \mathcal{P}$. In particular, $A \cup \{y\} \notin \mathcal{U}(\mathcal{A}, \mathcal{P})$. This implies that there must be an arc $(B, A \cup \{y\})$ in D such that $B \in \mathcal{A}_{\mathcal{V}}$. Clearly $A \cup \{y\} \subset B$; thus there is a $z \neq y$ such that $B = A \cup \{y, z\}$. By the maximality of A among sets below $\mathcal{A}_{\mathcal{V}}$, $A \cup \{z\} \in \mathcal{A}_{\mathcal{V}}$. However, $(A \cup \{z\}, A)$ is an arc in D, and a contradiction is obtained. □

Theorem 2.6 *Suppose that $\mathcal{V} \subseteq \mathcal{U} = \mathcal{U}(\mathcal{A}, \mathcal{P})$ has the property that $\mathcal{U} \setminus \mathcal{V} \not\rightsquigarrow \mathcal{V}$ and $\mathcal{V} \not\rightsquigarrow \mathcal{U} \setminus \mathcal{V}$. Then $\|\mathcal{A}\|$ is homotopy equivalent to*

$$\|\mathcal{A}_{\mathcal{V}}\| \vee \|\mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}\|.$$

In particular, if $\mathcal{V} = \{V\}$, then $\mathcal{A}$ is homotopy equivalent to

$$S^p \vee \|\mathcal{A}_{\mathcal{U} \setminus \{V\}}\|, \tag{2}$$

where $p = |V| - 1$; hence $\tilde{H}_p(\mathcal{A})$ is nontrivial.

Proof. Theorem 2.3 implies that $\mathcal{A}$ is homotopy equivalent to $\mathcal{A}_{\mathcal{U}}$; thus we may assume that $\mathcal{A} = \mathcal{A}_{\mathcal{U}} = \mathcal{A}_{\mathcal{V}} \cup \mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}$. Put $\mathcal{X} = \mathcal{A}_{\mathcal{V}} \cap \mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}$. By assumption, $\mathcal{X}$ contains no critical cells and is nonempty ($\phi, \{x\} \in \mathcal{X}$). Thus $\mathcal{X}$ is a sub-complex of $\mathcal{A}$ satisfying Theorem 2.3, which implies that $\|\mathcal{X}\|$ is contractible to a point. By the Contractible Subcomplex Lemma (see [Bj]), $\|\mathcal{A}\|$ is homotopy equivalent to the quotient $\|\mathcal{A}\| \,/\, \|\mathcal{X}\|$. By the same Lemma, $\|\mathcal{A}_{\mathcal{V}}\| \vee \|\mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}\|$ is homotopy equivalent to $(\|\mathcal{A}_{\mathcal{V}}\| \,/\, \|\mathcal{X}\|) \vee (\|\mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}\| \,/\, \|\mathcal{X}\|)$. Since clearly

$$\|\mathcal{A}\| \,/\, \|\mathcal{X}\| \simeq (\|\mathcal{A}_{\mathcal{V}}\| \,/\, \|\mathcal{X}\|) \vee (\|\mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}\| \,/\, \|\mathcal{X}\|),$$

the proof is finished. □

Corollary 2.7 *Let $\mathcal{V} \subseteq \mathcal{U} = \mathcal{U}(\mathcal{A}, \mathcal{P})$ be such that $\mathcal{U} \setminus \{V\} \not\rightarrow V$ and $V \not\rightarrow \mathcal{U} \setminus \{V\}$ for every $V \in \mathcal{V}$. Then $\|\mathcal{A}\|$ is homotopy equivalent to*

$$\left(\bigvee_{V \in \mathcal{V}} S^{|V|-1}\right) \vee \|\mathcal{A}_{\mathcal{U} \setminus \mathcal{V}}\| .$$

□

3 Graph-theoretical concepts

Let $G = (V, E)$ be a graph; V is the set of vertices and $E \subseteq \binom{V}{2}$ is the set of edges in G. The edge between a and b will be denoted ab or $\{a, b\}$. We will identify the graphs with their edge sets; $e \in G$ means that $e \in E$. Put $G \setminus e = (V, E \setminus \{e\})$ and $G + e = (V, E \cup \{e\})$. For $W \subset V$, let $G(W) = (W, E \cap \binom{W}{2})$.

A *monotone graph property* $\mathcal{A}$ is a nonempty family of graphs on a fixed vertex set such that the family is closed under deletion of edges and under permutations of the vertices. In particular, $\mathcal{A}$ is a simplicial complex on the set of edges.

For $0 < k < |V|$, say that G is *k-connected* if $G(V \setminus W)$ is connected for every $W \subset V$ such that $|W| < k$. $W \subset V$ *separates* G if $G(V \setminus W)$ is not connected. The property of being not k-connected is clearly a monotone graph property for each $k \geq 1$. For $0 < k < n$, let Δ_n^k be the complex of not k-connected graphs on the vertex set $\{1, \ldots, n\}$.

4 Not 3-connected graphs

We will consider the complex Δ_n^3 of not 3-connected graphs on the vertex set $V = \{1, \ldots, n\}$. Our purpose is to verify a conjecture in [BBLSW]:

Theorem 4.1 *$\|\Delta_n^3\|$ is homotopy equivalent to a wedge of $(n-3)\frac{(n-2)!}{2}$ spheres of dimension $2n-4$.*

We will find an acyclic matching on Δ_n^3 such that there are $(n-3)\frac{(n-2)!}{2}$ critical graphs with $2n-3$ edges. The graphs are easily described: For $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$ and $2 \leq k \leq n-2$, let $G_k(a_1, \ldots, a_n)$ be the graph with edge set

$$\{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_i a_{i+2} : 1 \leq i \leq k\} \cup \{a_k a_i : k+3 \leq i \leq n\}.$$

Put

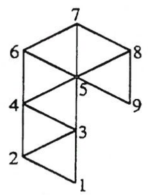

Figure 4.1: $G_5(1, 2, 3, 4, 5, 6, 7, 8, 9)$

$$\mathcal{U}_k = \{G_k(1, a_2, \ldots, a_{n-1}, n) : \{a_2, \ldots, a_{n-1}\} = \{2, \ldots, n-1\}, a_2 < a_3\}.$$

We will find an acyclic matching on Δ_n^3 such that the family of critical graphs is

$$\mathcal{U} = \bigcup_{k=2}^{n-2} \mathcal{U}_k.$$

Since $|\mathcal{U}| = (n-3)\frac{(n-2)!}{2}$ and since all graphs in $\mathcal{U}$ have the same number $2n-3$ of edges, Theorem 4.1 follows immediately from Corollary 2.7.

Proof of Theorem 4.1. We proceed in several steps.

Step 1. Begin by considering the edge $1n$. Take the matching $\mathcal{P}_{1n}(\Delta_n^3)$, and let $\mathcal{E}^n$ be the family of critical graphs with respect to this matching. One readily verifies that $\mathcal{E}^4$ consists of the graph $G_2(1,2,3,4)$ and nothing more. Hence from now on we may assume that $n \geq 5$. By the way, note that any acyclic matching on $\mathcal{E}^n$ will also be an acyclic matching on the family $\mathcal{E}(n)$ obtained from $\mathcal{E}^n$ by adding the edge $1n$ to every member of $\mathcal{E}^n$. In particular, our acyclic matching can be translated into an acyclic matching on the complex $\Delta^3(n)$ of 3-connected graphs (notations as in [Sh]).

Step 2. Let $G \in \mathcal{E}^n$. Let $X(G)$ be the set of pairs $\{x,y\}$ such that $G(V \setminus \{x,y\})$ is disconnected. Since $G + 1n$ is 3-connected, the set $X(G + 1n)$ is empty. This implies that any x, y such that $\{x,y\} \in X(G)$ separates 1 and n (any path from 1 to n must pass either x or y) and that for all $k \in V$ we have $\{1,k\}, \{k,n\} \notin X(G)$. For any $S = \{a,b\} \in X(G)$, let $M_1(S)$ be the set of vertices in the same component as 1 in $G(V \setminus S)$; $1 \in M_1(S)$. Put $M_2(S) = V \setminus (S \cup M_1(S))$. Since $G + 1n$ is 3-connected, $n \in M_2(S)$.

Let $S_G \in X(G)$ be such that the component in $G(V \setminus S_G)$ containing 1 is as small as possible. We have to show that S_G is uniquely determined, which is true if for any distinct $S_1 = \{a,b\}$ and $S_2 = \{c,d\} \in X(G)$, either $M_1(S_1) \subsetneqq M_1(S_2)$, $M_1(S_2) \subsetneqq M_1(S_1)$, or there is an S_3 such that $M_1(S_3) \subsetneqq M_1(S_i)$ for $i = 1,2$.

First suppose that (say) $a = c$ and $b \neq d$. Since $G + 1n$ is 3-connected, G is 2-connected. Hence if $d \in M_2(S_1)$, then $M_1(S_2) \supseteq M_1(S_1) \cup \{b\}$, while if $d \in M_1(S_1)$, then $M_2(S_2) \supseteq M_2(S_1) \cup \{b\}$, which implies that $M_1(S_2) \subseteq M_1(S_1) \setminus \{d\}$.

Now suppose that a, b, c, d are all different. If $S_2 \subseteq M_2(S_1)$, then clearly $M_1(S_2) \supseteq M_1(S_1) \cup S_1$, while if $S_2 \subset M_1(S_1)$, then $M_1(S_2) \subseteq M_1(S_1) \setminus S_2$. It remains to consider the case (say) $a \in M_1(S_2)$, $b \in M_2(S_2)$, $c \in M_1(S_1)$, and $d \in M_2(S_1)$. Since there are no edges between $M_1(S_1) \cap M_1(S_2)$ and $M_2(S_1) \cup M_2(S_2)$, it is clear that $\{a,c\} \in X(G)$. Note that $M_1(\{a,c\}) = M_1(S_1) \cap M_1(S_2)$ is properly included in $M_1(S_i)$ for $i = 1,2$ ($a, c \notin M_1(\{a,c\})$). It follows that S_G is uniquely determined.

For any $M \subset V$ and $x, y \notin M$, put

$$\mathcal{E}^n(M,x,y) = \{G \in \mathcal{E}^n : S_G = \{x,y\}, M = M_1(S_G)\}.$$

If $G \subseteq H$, then it is clear that $M_1(S_G) \subseteq M_1(S_H)$. Moreover, if $G \subset H$ and $S_G \neq S_H$, then $M_1(S_G) \subsetneqq M_1(S_H)$ (consider the discussion above). In particular,

$$\{\mathcal{E}^n(M,x,y) : M \subset V, x < y\}$$

satisfies the conditions in Concatenation Lemma 2.2.

Step 3. The following lemma will imply Theorem 4.1:

Lemma 4.2 *Let $x, y \in \{2, \ldots, n-1\}$, $x < y$. There is an acyclic matching on $\mathcal{E}^n(\{1\}, x, y)$ such that the unmatched graphs are*

$$G_k(1, x, y, a_4, \ldots, a_{n-1}, n)$$

($2 \leq k \leq n-2, \{a_4, \ldots, a_{n-1}\} = \{2, \ldots, n-1\} \setminus \{x, y\}$). For $M \neq \{1\}$, $\mathcal{P}_{xy}(\mathcal{E}^n(M, x, y))$ is a complete acyclic matching on $\mathcal{E}^n(M, x, y)$.

Proof of Lemma 4.2. The lemma is certainly true for $n = 4$. We will use induction over n to prove the lemma. Let $\mathcal{E}_0^n(M, x, y)$ be the family of critical graphs in $\mathcal{E}^n(M, x, y)$ with respect to the matching $\mathcal{P}_{xy}(\mathcal{E}^n(M, x, y))$. We want to show that $\mathcal{E}_0^n(M, x, y)$ is empty unless $M = \{1\}$. Note that $\mathcal{E}_0^n(M, x, y)$ is the family of all $G \in \mathcal{E}^n(M, x, y)$ containing the edge xy and having the property that $G' := G \setminus xy + 1n$ is not 3-connected. Furthermore, for any $\{c, d\} \in X(G')$, $\{c, d\} \cap \{x, y\} = \phi$.

Put $M_1 = M_1(\{x, y\}) \ni 1$ and $M_2 = M_2(\{x, y\}) \ni n$. Moreover, for an arbitrarily chosen $\{c, d\} \in X(G')$, let N_1 and N_2 be the components in $G'(V \setminus \{c, d\})$. Since G' is not 3-connected, 1 and n must be in the same component; assume that $1, n \in N_1 \cup \{c, d\}$. Furthermore, assume that $d \in M_1$ and $c \in M_2$. Let a, b be such that $\{a, b\} = \{x, y\}$, $b \in N_1$, and $a \in N_2$. The situation for G is as in Figure 4.2. Namely, there is no edge between $M_i \cap N_j$ and $M_{3-i} \cup N_{3-j}$.

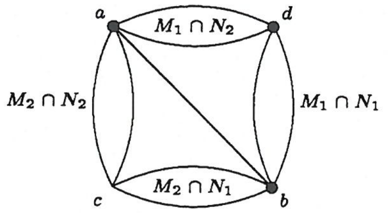

Figure 4.2: A graph in $\mathcal{E}_0(M, x, y)$; $ab = xy$ and $1, n \in N_1 \cup \{c, d\}$.

Examining Figure 4.2, one may deduce that $M_1 \cap N_2 = M_2 \cap N_2 = \phi$, because otherwise $\{a, c\}$ or $\{a, d\}$ separates $G + 1n$. Note that if $1 \in M_1 \cap N_1$, then $M_1(\{b, d\}) = M_1 \cap N_2 \subsetneq M_1$, which is a contradiction to the fact that $S_G = \{a, b\}$; hence $d = 1$. Moreover, since $\{1, b\} \notin X(G)$, we must have $M_1 \cap N_1 = \phi$. In particular, $\mathcal{E}_0^n(M, x, y)$ is nonempty if and only if $M = \{1\}$. If $c = n$, then we have $M_2 \cap N_1 = \phi$, which implies that $n = 4$. This is a contradiction; hence $c \neq n$. The situation is illustrated in Figure 4.3.

For $v \in V$, put $N_G(v) = \{w \in V \setminus \{v\} : vw \in E\}$ and $\deg v = |N_G(v)|$. When $M = \{1\}$, there are two cases; either $\deg y = 3$ (which is true if $y = a$ or if $\deg b = 3$) or $\deg y > 3$. For $z \neq 1, x, y, n$, put

$$\mathcal{F}_1^n(x, y, z) = \{G : N_G(y) = \{1, x, z\}\} \cap \mathcal{E}_0^n(\{1\}, x, y)$$

and

$$\mathcal{F}_2^n(x, y, z) = \{G : \deg y > 3, N_G(x) = \{1, y, z\}\} \cap \mathcal{E}_0^n(\{1\}, x, y).$$

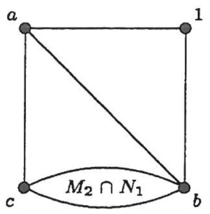

Figure 4.3: A graph in $\mathcal{E}_0^n(\{1\}, x, y)$; $ab = xy$ and $n \in M_2 \cap N_1$.

The partition

$$\{\mathcal{F}_1^n(x,y,z), \mathcal{F}_2^n(x,y,z) : z \neq 1, x, y, n\}$$

of $\mathcal{E}_0^n(\{1\}, x, y)$ satisfies the conditions in Concatenation Lemma 2.2. We will show that there is an acyclic matching on $\mathcal{F}_1^n(x,y,z)$ with critical graphs

$$G_2(1, x, y, z, a_5, \ldots, a_{n-1}, n)$$

and an acyclic matching on $\mathcal{F}_2^n(x,y,z)$ with critical graphs

$$G_k(1, x, y, z, a_5, \ldots, a_{n-1}, n),\ 3 \leq k \leq n-2$$

($\{a_5, \ldots, a_{n-1}\} = \{2, \ldots, n-1\} \setminus \{x, y, z\}$).

Case 1 Consider a graph $G \in \mathcal{F}_1^n(x,y,z)$. One readily verifies that there is a unique maximal path

$$P_G = (x_1, x_2, \ldots, x_t)$$

with $x_1 = 1$, $x_2 = y$, and $x_3 = z$ such that $N_G(x_k) = \{x_{k-1}, x_{k+1}, x\}$ for all $k \in \{2, \ldots, t-1\}$. Note that if $k < t$, then $x_k \neq n$ (because otherwise $G + 1n \in \Delta_n^3$; remove x and n). If $x_t = n$, then (by the same reason) all vertices in $V \setminus \{x\}$ are contained in the path; thus $t = n$ and $G = G_2(x_1, \ldots, x_n)$. For the other graphs in $\mathcal{F}_1^n(x,y,z)$, put

$$\mathcal{F}_1^n(x,y,z,x_4,\ldots,x_t) = \mathcal{F}_1^n(x,y,z) \cap \{G : P_G = (1,y,z,x_4,\ldots,x_t)\}.$$

By the maximality property of P_G, x_t is adjacent to x implies that x_t is adjacent either to exactly 2 vertices or to more than 3 vertices. However, by the 3-connectivity of $G + 1n$, the first case implies that $x_t = n$. Thus x_t is adjacent to more than 3 vertices. In particular, the families $\mathcal{F}_1^n(x,y,z,x_4,\ldots,x_t)$ satisfy the conditions in Concatenation Lemma 2.2, since t cannot increase if we add an edge.

Suppose that $K = G + 1n \setminus xx_t$ is not 3-connected and that $p, q \in V$ have the property that $K' = K(V \setminus \{p,q\})$ is disconnected. Obviously x_t and x belong to different components in K'. This means that, say, $p = x_{t-1}$. Since $\deg x_t \geq 3$ in K, the component in K' containing x_t must contain something more than x_t, and it does not contain n. Therefore, $K(V \setminus \{x_t, q\})$ is disconnected, which is a contradiction to the fact that $G + 1n$ is 3-connected. If x_t is not adjacent to x, then certainly $G + xx_t \in \mathcal{F}_1^n(x,y,z,x_4,\ldots,x_t)$. Thus $\mathcal{P}_{xx_t}(\mathcal{F}_1^n(x,y,z,x_4,\ldots,x_t))$ is a complete matching on $\mathcal{F}_1^n(x,y,z,x_4,\ldots,x_t)$.

Case 2 Finally consider $\mathcal{F}_2^n(x,y,z)$. Let $\hat{\mathcal{E}}^{n-1}(x,y,z)$ be the family of graphs H on the vertex set $V \setminus \{1\}$ such that $H \in \Delta^3_{n-1}$, $H + xn \notin \Delta^3_{n-1}$, and $N_G(x) = \{y,z\}$. We want to prove that $G \mapsto G(V \setminus \{1\})$ is a bijection from $\mathcal{F}_2^n(x,y,z)$ to $\hat{\mathcal{E}}^{n-1}(x,y,z)$. First we show how this will imply Lemma 4.2. By induction, Lemma 4.2 holds for $\hat{\mathcal{E}}^{n-1}(x,y,z)$, since $\hat{\mathcal{E}}^{n-1}(x,y,z)$ is equal to $\mathcal{E}^{n-1}(\{1\},y,z)$ with x replaced by 1. Hence there is an acyclic matching on $\hat{\mathcal{E}}^{n-1}(x,y,z)$ such that the unmatched graphs are of the form

$$G_k(x,y,z,a_5,\dots,a_{n-1},n),$$

where $2 \le k \le n-3$ and $\{a_5,\dots,a_{n-1}\} = \{2,\dots,n-1\} \setminus \{x,y,z\}$. Note that if we add 1, $1x$, and $1y$ to $G_k(x,y,z,a_5,\dots,a_{n-1},n)$, then we obtain the graph $G_{k+1}(1,x,y,z,a_5,\dots,a_{n-1},n)$. Thus choosing the acyclic matching on $\mathcal{F}^n(x,y,z)$ corresponding in the obvious way to the chosen acyclic matching on $\hat{\mathcal{E}}^{n-1}(x,y,z)$, we obtain Lemma 4.2 and hence Theorem 4.1.

To obtain the bijection, let $H \in \hat{\mathcal{E}}^{n-1}(x,y,z)$, and let $\{p,q\} \in X(H)$; note that $\{p,q\} \cap \{x,n\} = \phi$. Since $H + xn$ is 3-connected, $H(V \setminus \{1,p,q\})$ consists of two connected components, one containing x and one containing n. Therefore, $(G+1n)(V \setminus \{p,q\})$ is connected, where G is the graph obtained by adding the vertex 1 and the edges $1x, 1y$ to H. Since $H = G(V \setminus \{1\})$ is 2-connected, it follows that $X(G+1n) = \phi$, that is, $G \in \mathcal{F}_2^n(x,y,z)$.

Furthermore, if $G \in \mathcal{F}_2^n(x,y,z)$, then $H = G(V \setminus \{1\}) \in \hat{\mathcal{E}}^{n-1}(x,y,z)$. Namely, suppose that $H + xn$ is not 3-connected. Then there is a $\{p,q\} \in X(H)$ such that $H(V \setminus \{1,p,q\})$ contains a connected component that does not contain x, y, or n. However, then the very same component will occur in $G(V \setminus \{p,q\})$, which implies that $\{p,q\}$ separates $G+1n$, a contradiction. Thus $G \mapsto G(V \setminus \{1\})$ is a bijection from $\mathcal{F}_2^n(x,y,z)$ to $\hat{\mathcal{E}}^{n-1}(x,y,z)$. □

5 Concluding remarks

Since Δ^3_n is not contractible, the complex is evasive (see [KSS]). Given a decision tree for a simplicial complex Δ, say that a set S is evasive if $S \in \Delta$ and $S \cup \{x\} \notin \Delta$, where x is the last element checked in the decision tree for S. Recent results by Robin Forman imply that any decision tree must contain at least $m(\Delta)$ evasive sets, where $m(\Delta)+1$ is the minimal number of cells needed to form a CW complex that is homotopy equivalent to Δ. Namely, a decision tree induces an acyclic matching on Δ with unmatched sets precisely the evasive sets; see [J] for details. Say that a decision tree is *optimal* for Δ if the lower bound $m(\Delta)$ is attained by the tree. There is an optimal decision tree for the complex Δ^1_n of disconnected graphs, and we claim that there exists an optimal decision tree also for the complex Δ^2_n of not 2-connected graphs; the decision trees are described in [J]. However, the problem of finding an optimal decision tree for Δ^3_n seems to be unsolved.

Finally, we mention that we have been able to construct a basis for the nontrivial homology group $\tilde{H}_{2n-3}(\Delta^3(n))$ of the CW complex $\Delta^3(n)$ of 3-connected graphs on n vertices. As the basis is not very easy to describe, we have decided not to include the result in this extended abstract. As far as we know, the related problem of determining the action of the symmetric group on $\tilde{H}_{2n-3}(\Delta^3(n))$ is still unsolved.

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