ON A BIJECTION BETWEEN LITTLEWOOD–RICHARDSON TABLEAUX AND RIGGED CONFIGURATIONS

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Abstract

We define a bijection between Littlewood–Richardson tableaux and rigged configurations and show that it preserves the appropriate statistics. This proves in particular a quasi-particle expression for the generalized Kostka polynomials $K_{\lambda R}(q)$ labeled by a partition λ and a sequence of rectangles R. The generalized Kostka polynomials are q-analogues of multiplicities of the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_n$ with highest weight λ in the tensor product $W(R_1) \otimes \cdots \otimes W(R_L)$.

Résumé

Nous définissons une bijection entre les tableaux de Littlewood–Richardson et les configurations gréées et montrons qu'elle préserve les statistiques appropriées. Cela démontre en particulier l'expression quasi-particulaire des polynômes généralisés de Kostka $K_{\lambda R}(q)$, labellés selon une partition λ et une séquence de rectangles R. Les polynômes de Kostka sont des q-analogues de multiplicité de $W(\lambda)$ dans le produit tensoriel $W(R_1) \otimes \cdots \otimes W(R_L)$. Où $W(\lambda)$ est la représentation irréductible de dimension finie de $\mathfrak{gl}_n$ avec le plus grand poids λ.

1. Introduction

This extended abstract is based on the preprint [6].

Recently generalizations of the Kostka polynomials were introduced and many of their properties studied [5, 10, 11, 12, 13, 14]. These generalized Kostka polynomials $K_{\lambda R}(q)$ are labeled by a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and a sequence of rectangles $R = (R_1, \dots, R_L)$, that is, each $R_i = (\eta_i^{\mu_i})$ is a partition of rectangular shape. They are q-analogues of the multiplicities of the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_n$ with highest weight λ in the tensor product $W(R_1) \otimes \cdots \otimes W(R_L)$

$$K_{\lambda R}(1) = [W(\lambda) : W(R_1) \otimes \cdots \otimes W(R_L)]. \tag{1}$$

When all R_i are single rows (in which case $R_i = (\eta_i)$), the generalized Kostka polynomial reduces to the Kostka polynomial $K_{\lambda\eta}(q)$. The multiplicity $K_{\lambda R}(1)$ is equal to the cardinality of the set of Littlewood–Richardson tableaux [1].

In refs. [5, 10] a representation of the generalized Kostka polynomials in terms of rigged configurations was conjectured. This extends the results of Kirillov and Reshetikhin [4] who, in their study of the XXX model using Bethe Ansatz techniques, obtained an expression of the Kostka polynomials as the generating function of rigged configurations.

Date: February 1999.

The second author is supported by the "Stichting Fundamenteel Onderzoek der Materie".
The third author is partially supported by NSF grant DMS-9800941.

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In fact, the representation of the (generalized) Kostka polynomials in terms of rigged configurations is exactly in quasi-particle form. In recent years, much research has been devoted to the study of quasi-particle representations of characters of conformal field theories and configuration sums of exactly solvable lattice models. These quasi-particle representations are physically interesting [2, 3] because they reflect the particle structure of the underlying model as opposed to the "bosonic" representations coming from the Feigin and Fuchs construction. For example, the quasi-particle representations give rise to the exclusion statistics of the quasi-particles and also reflect the different integrable perturbations of the underlying conformal field theory. These notions are highly important to understand real physical systems in two dimensions which can be described by conformal field theories such as, for example, the fractional quantum Hall effect.

The quasi-particle representation of the generalized Kostka polynomials was recently proven in ref. [6]. This talk will report on this proof, which is based on a statistic preserving bijection between Littlewood–Richardson (LR) tableaux and rigged configurations. The next section contains a brief review of the definition of the generalized Kostka polynomials as the generating function of LR tableaux. In section 3 rigged configurations are defined and the quasi-particle representation of the generalized Kostka polynomials is stated. The bijection between LR tableaux and rigged configurations is subject of section 4. A proof that this bijection indeed preserves the statistics, thereby proving the quasi-particle representation of the generalized Kostka polynomials, is sketched in section 5.

2. Generalized Kostka polynomials and LR tableaux

The multiplicity $K_{\lambda R}(1)$ as defined in (1) is equal to the cardinality of the set of Littlewood–Richardson tableaux. There are several ways to define LR tableaux. Here we define the set $\mathrm{CLR}(\lambda; R)$ where "C" indicates a column labeling. Later we will also need the set of row LR tableaux denoted by $\mathrm{RLR}(\lambda; R)$. For a given sequence of rectangles $R = (R_1, \dots, R_L)$ define the standard tableaux Z_i ($1 \le i \le L$) of shape $R_i = (\eta_i^{\mu_i})$ by inserting the numbers

$$(c-1)\mu_i + \sum_{j=1}^{i-1} |R_i| < k \le c\mu_i + \sum_{j=1}^{i-1} |R_i|$$

into the c^{th} column of R_i. For example, for $R = ((2,2),(3,3))$ we have

$$Z_1 = \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix} \quad \text{and} \quad Z_2 = \begin{matrix} 5 & 7 & 9 \\ 6 & 8 & 10 \end{matrix}.$$

This means that Z_i is a standard tableau over the alphabet $B_i = \{|R_1| + \cdots + |R_{i-1}| + 1 < \cdots < |R_1| + \cdots + |R_i|\}$. For a tableau T denote by $T|_B$ the skew tableau given by the restriction of T to the alphabet B. The row-reading word of a skew tableau T is given by $\mathrm{word}(T) = \cdots w_2 w_1$ where w_i is the word of the i^{th} row of T. Denote by $P(w)$ the Schensted P-tableau of the word w and define $P(T) := P(\mathrm{word}(T))$. Finally denote the set of all standard tableaux of shape λ by $\mathrm{ST}(\lambda)$. Then the set $\mathrm{CLR}(\lambda; R)$ is defined as

$$\mathrm{CLR}(\lambda; R) = \{T \in \mathrm{ST}(\lambda) | P(T|_{B_i}) = Z_i\}.$$

It was shown in [10, Section 6] and [11] that the set $\mathrm{CLR}(R) = \cup_\lambda \mathrm{CLR}(\lambda; R)$ has the structure of a graded poset with covering relation given by the R-cocyclage and grading function given by the generalized charge, denoted c_R. The generalized Kostka polynomial is the generating function of LR tableaux with the charge statistics [10, 11]

$$K_{\lambda R}(q) = \sum_{T \in \mathrm{CLR}(\lambda; R)} q^{c_R(T)}. \tag{2}$$

This extends the charge representation of the Kostka polynomials $K_{\lambda\eta}(q)$ of Lascoux and Schützenberger [8, 9].

3. Quasi-particle representation of the generalized Kostka polynomials

Recall that $R = (R_1, \ldots, R_L)$ such that R_j has μ_j rows and η_j columns for $1 \le j \le L$. For a partition λ denote by λ^t its transpose and set $R^t = (R_1^t, \ldots, R_L^t)$. A $(\lambda^t; R^t)$-configuration is a sequence of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots)$ with the size constraints

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j^t - \sum_{a=1}^{L} \mu_a \max(\eta_a - k, 0). \tag{3}$$

For a partition ρ, define $m_n(\rho)$ as the number of parts equal to n and $Q_n(\rho) = \rho_1^t + \rho_2^t + \cdots + \rho_n^t$, the size of the first n columns of ρ. Let $\xi^{(k)}(R)$ be the partition whose parts are the heights of the rectangles in R of width k. The vacancy numbers for the $(\lambda^t; R^t)$-configuration ν are the numbers (indexed by $k \ge 1$ and $n \ge 0$) defined by

$$P_n^{(k)}(\nu) = Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) + Q_n(\xi^{(k)}(R)) \tag{4}$$

where $\nu^{(0)}$ is the empty partition by convention. In particular $P_0^{(k)}(\nu) = 0$ for all $k \ge 1$. The $(\lambda^t; R^t)$-configuration ν is admissible if $P_n^{(k)}(\nu) \ge 0$ for all $k, n \ge 1$, and the set of admissible $(\lambda^t; R^t)$-configurations is denoted by $\mathrm{C}(\lambda^t; R^t)$. Define the cocharge of a $(\lambda^t; R^t)$-configuration ν by

$$\mathrm{cc}(\nu) = \sum_{k,n \ge 1} \alpha_n^{(k)} (\alpha_n^{(k)} - \alpha_n^{(k+1)})$$

where $\alpha_n^{(k)}$ is the size of the n-th column in $\nu^{(k)}$. Finally, the q-binomial is given by

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{(q)_{m+n}}{(q)_m (q)_n}$$

for $m, n \in \mathbb{Z}_{\ge 0}$ and zero otherwise where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$. With this notation we can state the following quasi-particle expression of the generalized Kostka polynomials.

Theorem 1. *For λ a partition and R a sequence of rectangles*

$$K_{\lambda R}(q) = \sum_{\nu \in \mathrm{C}(\lambda^t; R^t)} q^{\mathrm{cc}(\nu)} \prod_{k,n \ge 1} \begin{bmatrix} P_n^{(k)}(\nu) + m_n(\nu^{(k)}) \\ m_n(\nu^{(k)}) \end{bmatrix}. \tag{5}$$

Expression (5) can be reformulated as the generating function over rigged configurations. To this end we need to define certain labelings of the rows of the partitions in a configuration. For this purpose one should view a partition as a multiset of positive integers. A rigged partition is by definition a finite multiset of pairs (n,x) where n is a positive integer and x is a nonnegative integer. The pairs (n,x) are referred to as strings; n is referred to as the length or size of the string and x as the label or quantum number of the string. A rigged partition is said to be a rigging of the partition ρ if the multiset consisting of the sizes of the strings, is the partition ρ. So a rigging of ρ is a labeling of the parts of ρ by nonnegative integers, where one identifies labelings that differ only by permuting labels among equal-sized parts of ρ.

A rigging J of the $(\lambda^t; R^t)$-configuration ν is a sequence of riggings of the partitions $\nu^{(k)}$ such that every label x of a part of $\nu^{(k)}$ of size n, satisfies the inequalities

$$0 \le x \le P_n^{(k)}(\nu). \tag{6}$$

The pair (ν, J) is called a rigged configuration. The set of riggings of admissible $(\lambda^t; R^t)$-configurations is denoted by $\mathrm{RC}(\lambda^t; R^t)$. Let $(\nu, J)^{(k)}$ be the k-th rigged partition of (ν, J). A string $(n,x) \in (\nu, J)^{(k)}$ is said to be singular if $x = P_n^{(k)}(\nu)$, that is, its label takes on the maximum value.

The set of rigged configurations is endowed with a natural statistic cc [5, (3.2)] defined by

$$\mathrm{cc}(\nu, J) = \mathrm{cc}(\nu) + \sum_{k,n \ge 1} |J_n^{(k)}|$$

for $(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)$. Here $|\rho|$ is the size of the partition ρ and $J_n^{(k)}$ denotes the partition inside the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size n. Since the q-binomial $\begin{bmatrix} P+m \\ m \end{bmatrix}$ is the generating function of partitions with at most m parts each not exceeding P, Theorem 1 is equivalent to the following theorem.

Theorem 1'. *For λ a partition and R a sequence of rectangles*

$$K_{\lambda R}(q) = \sum_{(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)} q^{\mathrm{cc}(\nu, J)}. \tag{7}$$

To establish this theorem we describe a bijection $\phi_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ in the following section and sketch a proof in section 5 that this bijection preserves the statistics, that is $c_R(T) = \mathrm{cc}(\phi_R(T))$.

Observe that the definition of the set $\mathrm{RC}(\lambda^t; R^t)$ is completely insensitive to the order of the rectangles in the sequence R. However the notation involving the sequence R is useful when discussing the bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ between LR tableaux and rigged configurations, since the ordering on R is essential in the definition of $\mathrm{CLR}(\lambda; R)$. Exchanging two rectangles in R induces a bijection on the LR tableaux that is explicitly described in [11]. It coincides with the automorphism of conjugation of [9] in the case when each R_j is a single row (using a suitable labeling of LR tableaux).

4. The bijection between LR tableaux and rigged configurations

In this section we define a bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ between LR tableaux and rigged configurations. The bijection which preserves the statistics is

$$\phi_R = \mathrm{comp} \circ \overline{\phi}_R$$

where comp : $\mathrm{RC}(\lambda; R) \to \mathrm{RC}(\lambda; R)$ complements the rigging labels. That is, for $(\nu, J) \in \mathrm{RC}(\lambda; R)$ a string $(n, x) \in (\nu, J)^{(k)}$ is mapped to $(n, P_n^{(k)}(\nu) - x)$.

There are several ways to define a bijection between LR tableaux and rigged configurations depending on the labeling of the LR tableaux (rowwise or columnwise labeling) and the way to assign the rigging labels (quantum or coquantum labeling). The bijection $\overline{\phi}_R$ uses the columnwise and quantum labeling and is defined recursively based on the following two operations on sequences of rectangles $R = (R_1, \dots, R_L)$:

I. Let $R^\wedge$ be the sequence of rectangles obtained from R by splitting off the last column of R_L; formally, $R_j^\wedge = R_j$ for $1 \le j \le L-1$, $R_L^\wedge = ((\eta_L - 1)^{\mu_L})$ and $R_{L+1}^\wedge = (1^{\mu_L})$.

II. If the last rectangle of R is a single column, let $\overline{R}$ be given by removing one cell from the column R_L; $\overline{R}_j = R_j$ for $1 \le j \le L-1$ and $\overline{R}_L = (1^{\mu_L - 1})$.

Remark 1. Given any sequence of rectangles, there is a unique sequence of transformations of the form $R \to R^\wedge$ or $R \to \overline{R}$ resulting in the empty sequence, where $R \to R^\wedge$ is only used when the last rectangle of R has more than one column.

For both transformations on sequences of rectangles, there are natural (injective) maps on the corresponding sets of LR tableaux and rigged configurations. The analogue of transformation I on LR tableaux is the inclusion

$$\imath^\wedge : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda; R^\wedge).$$

When the last rectangle of R is a single column define

$$\mathrm{CLR}(\lambda^-; \overline{R}) = \bigcup_{\rho \lessdot \lambda} \mathrm{CLR}(\rho; \overline{R})$$

where $\rho \lessdot \lambda$ means that $\rho \subset \lambda$ and λ/ρ is a single cell. Define the injective map

$$\begin{aligned} \mathrm{CLR}(\lambda; R) &\to \mathrm{CLR}(\lambda^-; \overline{R}) \\ T &\mapsto T^- \end{aligned}$$

where T^- is the LR tableau obtained by removing the maximum letter from T.

The analogue of transform I for rigged configurations is given by the map

$$\jmath^\wedge : \mathrm{RC}(\lambda^t; R^t) \to \mathrm{RC}(\lambda^t; (R^\wedge)^t)$$

by declaring that $\jmath^\wedge(\nu, J)$ is obtained from $(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)$ by adding a singular string of length μ_L to each of the first $\eta_L - 1$ rigged partitions. Note that $\jmath^\wedge$ is the identity map if R_L is a single column. It is shown in [6] that $\jmath^\wedge$ is a well-defined injection that preserves the vacancy numbers of the underlying configurations.

Suppose the last rectangle of R is a single column. Define the set

$$\mathrm{RC}(\lambda^{-t}; \overline{R}^t) = \bigcup_{\rho \lessdot \lambda} \mathrm{RC}(\rho^t; \overline{R}^t).$$

The key algorithm on rigged configurations is given by the map

$$\overline{\delta}:\mathrm{RC}(\lambda^t;R^t)\to\mathrm{RC}(\lambda^{-t};\overline{R}^t),$$

defined as follows. Let $(\nu,J)\in\mathrm{RC}(\lambda^t;R^t)$. Define $\overline{\ell}^{(0)}=\mu_L$. By induction select the singular string in $(\nu,J)^{(k)}$ whose length $\overline{\ell}^{(k)}$ is minimal such that $\overline{\ell}^{(k-1)}\le\overline{\ell}^{(k)}$. Let $\overline{\mathrm{rk}}(\nu,J)$ denote the smallest k for which no such string exists, and set $\overline{\ell}^{(k)}=\infty$ for $k\ge\overline{\mathrm{rk}}(\nu,J)$. Then $\overline{\delta}(\nu,J)$ is obtained from (ν,J) by shortening each of the selected singular strings by one, changing their labels so that they remain singular, and leaving the other strings unchanged. It is shown in [6] that the map $\overline{\delta}$ is a well-defined injection such that $\overline{\delta}(\nu,J)\in\mathrm{RC}(\rho^t;\overline{R}^t)$ where ρ is obtained from λ by removing the corner cell in the column of index $\overline{\mathrm{rk}}(\nu,J)$.

The bijection $\overline{\phi}_R:\mathrm{CLR}(\lambda;R)\to\mathrm{RC}(\lambda^t;R^t)$ is defined inductively based on Remark 1.

Definition-Proposition 2. For each sequence R there exists a unique bijection $\overline{\phi}_R:\mathrm{CLR}(\lambda;R)\to\mathrm{RC}(\lambda^t;R^t)$, such that:

1. If the last rectangle of R is a single column, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CLR}(\lambda;R) \arrow[r, "-"] \arrow[d, "\overline{\phi}_R"'] & \mathrm{CLR}(\lambda^-;\overline{R}) \arrow[d, "\overline{\phi}_{\overline{R}}"] \\
\mathrm{RC}(\lambda^t;R^t) \arrow[r, "\overline{\delta}"'] & \mathrm{RC}(\lambda^{-t};\overline{R}^t).
\end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{CLR}(\lambda;R) \arrow[r, "i^\wedge"] \arrow[d, "\overline{\phi}_R"'] & \mathrm{CLR}(\lambda;R^\wedge) \arrow[d, "\overline{\phi}_{R^\wedge}"] \\
\mathrm{RC}(\lambda^t;R^t) \arrow[r, "j^\wedge"'] & \mathrm{RC}(\lambda^t;(R^\wedge)^t).
\end{tikzcd}$$

The proof of this Definition-Proposition is given in [6].

5. Sketch of the proof of Theorem 1'

For the proof of Theorem 1' it remains to show that the bijection ϕ_R preserves the statistics.

Lemma 3. *Let $T\in\mathrm{CLR}(\lambda;R)$. Then $c_R(T)=\mathrm{cc}(\phi_R(T))$.*

The proof of this lemma is given in full length in [6]. Here we only sketch the main ideas.

There are further important maps on the sets of LR tableaux and rigged configurations. The maps which play a central rôle in the proof are the transposition maps on LR tableaux and rigged configurations and a statistic preserving embedding on LR tableaux. Let us briefly review their definitions and some of their properties.

Denote by $\mathrm{tr}:\mathrm{ST}(\lambda)\to\mathrm{ST}(\lambda^t)$ the ordinary transposition of standard tableaux. Analogous to the definition of $\mathrm{CLR}(\lambda;R)$ let us define the set

$$\mathrm{RLR}(\lambda;R)=\{T\in\mathrm{ST}(\lambda)|P(T|_{B_i})=Z_i^t\}$$

where $Z_i^!$ is the standard tableau of shape $R_i = (\eta_i^{\mu_i})$ obtained by inserting the numbers

$$(r-1)\eta_i + \sum_{j=1}^{i-1} |R_i| < k \le r\eta_i + \sum_{j=1}^{i-1} |R_i|$$

into the r^{th} row of R_i. There is a bijection $\gamma_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RLR}(\lambda; R)$ given by relabeling as follows. Suppose the letter j occurs in Z_i in cell s. Then to obtain $\gamma_R(T)$ from $T \in \mathrm{CLR}(\lambda; R)$ replace the letter j in T by the letter occurring in cell s of $Z_i^!$ for all letters j. The transpose map tr restricts to a bijection $\mathrm{tr} : \mathrm{CLR}(\lambda; R) \to \mathrm{RLR}(\lambda^t; R^t)$. Then the LR-transpose

$$\mathrm{tr}_{\mathrm{LR}} : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda^t; R^t)$$

is defined as $\mathrm{tr}_{\mathrm{LR}} := \mathrm{tr} \circ \gamma_R$.

An analogous RC-transpose bijection exists for the set of rigged configurations denoted by $\mathrm{tr}_{\mathrm{RC}} : \mathrm{RC}(\lambda^t; R^t) \to \mathrm{RC}(\lambda; R)$, which was described in [5, Section 9]. Let $(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)$ and let ν have the associated matrix m_{ij} as in [5, (9.2)]

$$m_{ij} = \alpha_j^{(i-1)} - \alpha_j^{(i)}$$

for $i, j \ge 1$, where $\alpha_j^{(i)}$ is the size of the j-th column of the partition $\nu^{(i)}$, recalling that $\nu^{(0)}$ is defined to be the empty partition. The configuration ν^t in $(\nu^t, J^t) = \mathrm{tr}_{\mathrm{RC}}(\nu, J)$ is defined by its associated matrix m^t given by

$$m_{ij}^t = -m_{ji} - \#\{a | (i,j) \in R_a\} + \begin{cases} 1 & \text{if } (i,j) \in \lambda \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \ge 1$. Here $(i,j) \in \lambda$ means that the cell (i,j) is in the Ferrers diagram of the partition λ with i specifying the row and j the column. Recall that the rigging J is determined by partitions $J_n^{(k)}$ inside the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size n. The partition $J_k^{t(n)}$ corresponding to $(\nu^t, J^t) = \mathrm{tr}_{\mathrm{RC}}(\nu, J)$ is defined as the transpose of the complementary partition to $J_n^{(k)}$ in the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$.

It is shown in [6] that the diagram

$$\begin{tikzcd} \mathrm{CLR}(\lambda; R) \arrow[r, "\mathrm{tr}_{\mathrm{LR}}"] \arrow[d, "\phi_R"'] & \mathrm{CLR}(\lambda^t; R^t) \arrow[d, "\phi_{R^t}"] \\ \mathrm{RC}(\lambda^t; R^t) \arrow[r, "\mathrm{tr}_{\mathrm{RC}}"'] & \mathrm{RC}(\lambda; R) \end{tikzcd} \tag{8}$$

commutes.

Let rows(R) be obtained from the sequence of rectangles R by slicing all the rectangles of R into single rows. In refs. [11, 10] an embedding

$$\theta_R : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda; \mathrm{rows}(R))$$

was defined and it was shown that θ_R preserves the charge c_R of LR tableaux. This embedding stems from an analogous embedding on column-strict tableaux given by Lascoux and Schützenberger [7, 9]. For rigged configurations, it follows immediately from the definitions that there is an inclusion $\mathrm{RC}(\lambda^t;R^t) \subseteq \mathrm{RC}(\lambda^t;\mathrm{rows}(R)^t)$. It is shown in [6] that the diagram

$$\begin{array}{ccc} \mathrm{CLR}(\lambda;R) & \xrightarrow{\theta_R} & \mathrm{CLR}(\lambda;\mathrm{rows}(R)) \\ \phi_R \big\downarrow & & \big\downarrow \phi_{\mathrm{rows}(R)} \\ \mathrm{RC}(\lambda^t;R^t) & \xrightarrow[\text{inclusion}]{} & \mathrm{RC}(\lambda^t;\mathrm{rows}(R)^t) \end{array} \tag{9}$$

commutes.

Now the proof of Lemma 3 follows directly from (8) and (9). Since the embedding θ_R preserves the statistics one can reduce the proof of Lemma 3 to the case that all rectangles in R are single rows using (9). By (8) it may be assumed that R consists of single columns only. Finally applying (9) again, it is sufficient to establish Lemma 3 for R a sequence of single boxes only. In this case the lemma is verified explicitly in [6].

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