A ROBINSON-SCHENSTED-KNUTH TYPE CORRESPONDENCE RELATED TO SCHUBERT POLYNOMIALS AND ITS APPLICATIONS
EXTENDED ABSTRACT

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ABSTRACT. We present an analog of the Robinson-Schensted-Knuth correspondence related to Schubert polynomials and some of its applications. This correspondence is based on a new insertion procedure for certain binary tableaux of staircase shape, which are the analogs of semistandard Young tableaux in the theory of Schubert polynomials. The main application is an extension of the theory of noncommutative Schur functions, due to S. Fomin and C. Greene, to Schubert polynomials; more precisely, we prove noncommutative versions of the Cauchy identity and the Pieri formula for Schubert polynomials. Finally, we use these results in the theory of Grothendieck polynomials, which are generalizations of Schubert polynomials related to the K-theory of flag varieties.

1. INTRODUCTION

The singular cohomology of flag varieties has attracted a great interest from people in various mathematical areas, ranging from algebraic geometry and topology, to representation theory, and more recently to algebraic combinatorics. The latter enters the picture with the construction by Lascoux and Schützenberger [11] of the *Schubert polynomials* as representatives for Schubert classes in type A. Since their work, many papers on the combinatorics of Schubert polynomials have appeared. Currently, one of the main open problems related to Schubert polynomials is finding a *Littlewood-Richardson rule* for them. This rule is concerned with a combinatorial description of the structure constants for the ring of polynomials with respect to its Schubert basis. In this work, we present a construction whose ultimate application we believe to be the mentioned Littlewood-Richardson rule. We base our assertion on an analogy with the classical case of symmetric functions. Indeed, our construction is an analog of the well-known *Robinson-Schensted-Knuth (R-S-K) correspondence.* It is based on an insertion procedure (this should be viewed as the analog of *Schensted's insertion*) for certain binary tableaux of staircase shape, which we call *rrc-graphs*, and which appear in the combinatorial definition of Schubert polynomials; in fact, these objects correspond to the semistandard Young tableaux in the theory of Schur functions. Our analog of the R-S-K correspondence is a bijection between binary tableaux of staircase shape and certain pairs of rrc-graphs. We show that this construction, as well as our insertion algorithm, have properties similar to those of the classical constructions. The main application of our insertion algorithm is an extension of the theory of *noncommutative Schur functions*, due to S. Fomin and C. Greene [3], to Schubert polynomials. We define noncommutative analogs of Schubert polynomials by considering a certain reading of rrc-graphs. Unlike noncommutative Schur functions, our noncommutative polynomials do not commute given certain relations between the variables. However, they satisfy a noncommutative version of the *Cauchy identity* for Schubert polynomials, given the same relations. The proof of this identity is based on a Pieri-type formula for our polynomials, and, ultimately, on our insertion algorithm. The mentioned noncommutative Pieri-type formula is considerably simpler than the Pieri formula for Schubert polynomials, because it only involves the weak Bruhat order. Finally, we discuss the special case when the variables satisfy the relations of the *degenerate Hecke algebra* $H_n(0)$, which is related to the theory of *Grothendieck polynomials.* Recall that these polynomials are representatives for the classes dual to the structure sheaves of Schubert varieties in the K-theory of the flag variety. Our Cauchy identity immediately implies a conjecture of Fomin and Kirillov concerning the expansion of a Grothendieck polynomial in the basis of Schubert polynomials; furthermore, it offers a combinatorial interpretation for the coefficients of this expansion.

Parts of this work will appear in [17].

2. The Main Construction

We start by recalling a combinatorial construction of Schubert polynomials. The objects underlying this construction, namely the *rc-graphs* used in [1], play a major role in this work.

The nilCoxeter algebra (of the symmetric group Σ_n, with n fixed throughout) is generated by elements $v_1, \dots, v_{n-1}$, subject to the relations

$$\begin{aligned} &v_i^2 = 0\,,\\ &v_i v_j = v_j v_i\,, \quad |i-j| \geq 2\,,\\ &v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}\,. \end{aligned} \tag{2.1}$$

This algebra has a basis consisting of elements which can be naturally identified with permutations in Σ_n (we will not attempt to distinguish them notationally, because we have made sure the context is clear each time); in particular, v_i is identified with the simple transposition $s_i = (i, i+1)$. According to Proposition 1.12 in [12] (see also [2, 6]), the Schubert polynomials $\mathfrak{S}_w(x) = \mathfrak{S}_w(x_1, \dots, x_{n-1})$, for w in Σ_n, can be defined by their generating function

$$\mathfrak{S}(x) := \prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i v_j)\,, \quad \text{that is,} \quad \mathfrak{S}(x) = \sum_{w \in \Sigma_n} \mathfrak{S}_w(x) w\,; \tag{2.2}$$

the variables x_i commute with v_j, and the noncommuting factors of the double product are evaluated in the specified order.

The definition of Schubert polynomials in (2.2) can be reformulated in terms of certain combinatorial objects called rc-graphs (essentially some binary tableaux of staircase shape), as in [1]; thus, it can be viewed as the analog of the combinatorial definition of Schur functions. We find more convenient for our purposes not to use the rc-graphs themselves, but a slight variation of them, which we call *reversed rc-graphs* (or, simply, *rrc-graphs*); these are binary tableaux of staircase shape obtained by reading the rows of rc-graphs from right to left. To be more precise, we define the collection $\mathcal{R}(w)$ of rrc-graphs associated with a permutation w in Σ_n to consist of the binary tableaux of staircase shape $R = (r_{ij})_{i+j\leq n}$ for which

$$v(R) := \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} v_{n-j}^{r_{ij}}$$

equals w in the nilCoxeter algebra. All binary tableaux considered from now on will have the same shape and size as the ones above.

Example 2.3. The following rrc-graph is associated with the permutation $(3,1,4,6,5,2)$:

$$\begin{array}{lllll} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & & \\ 0 & 0 & & & \\ 1 & & & & \end{array}$$

If we also define (for any binary tableau, in fact)

$$x(R) := \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} x_i^{r_{ij}},$$

then (2.2) can be reformulated as

$$\mathfrak{S}_w(x) = \sum_{R\in\mathcal{R}(w)} x(R)\,. \tag{2.4}$$

We are now ready to present our insertion algorithm for rrc-graphs. Let us first mention that a different insertion algorithm (for rc-graphs) was given in [1], and used in a combinatorial proof of Monk's formula. The main differences between the two algorithms are: a) it does not seem possible to extend the algorithm in [1] to a R-S-K type correspondence; b) the insertion of a decreasing sequence of elements using our procedure produces paths which are situated weakly above one another, unlike the algorithm in [1]. The latter fact prevented Bergeron and Billey from extending their proof of Monk's formula to the Pieri formula for Schubert polynomials.

Algorithm 2.5. (Insertion). Consider an integer i with $1 \leq i \leq n-1$, and an rrc-graph R in $\mathcal{R}(w)$ for some w in Σ_n. Assume it is possible to find a sequence of indices (the *insertion path*) $(k_0,l_0),(k_1,l_1),\ldots,(k_m,l_m)$, with $1 \leq k_p \leq n-1$, $1 \leq l_p \leq n-k_p$ for all $1 \leq p \leq m$, satisfying the following properties:

1. $k_0 = i$, $l_0 = 0$, and we set $r_{k_0l_0} := 0$;
2. if $r_{k_pl_p} = 0$ for some p with $0 \leq p \leq r-1$, then $k_{p+1} = k_p$ and $l_{p+1} = l_p + 1$;
3. if $r_{k_pl_p} = 1$ and $r_{k_{p-1}l_{p-1}} = 0$ for some p with $1 \leq p \leq r-1$, then $k_{p+1} = k_p - 1$ and $l_{p+1} = l_p$;
4. if $r_{k_pl_p} = 1$ and $r_{k_{p-1}l_{p-1}} = 1$ for some p with $1 \leq p \leq r-1$, then $k_{p+1} = k_p - 1$ and $l_{p+1} = l_p + 1$;
5. $k_m = 1$ and $r_{k_ml_m} = 1$.

If such a sequence exists, it is clearly unique. We define a new binary tableau $(i \to R) = (t_{ab})$ (the insertion of i into R) by simply setting $t_{k_pl_p} := r_{k_{p-1}l_{p-1}}$ if $1 \leq p \leq m$, and $t_{ab} := r_{ab}$ for all other pairs (a,b).

An example of insertion path is given in Fig. 3 (here $n = 11$ and $i = 4$). The insertion algorithm consists of shifting the entries along the insertion path; an extra 0 enters the tableau in position $(i, 1)$, and the entry equal to 1 at the end of the insertion path is removed (see Fig. 4).

							1		
						0	1		
				0	0	1			
			1						
		1							
	1								
0	1								

Fig. 3

							1		
						1	0		
				1	0	0			
			1						
		1							
	1								
0	0								

Fig. 4

We now present some properties of this insertion procedure.

Define the column index $j(i)$ as follows:

$$j(i) := \begin{cases} l_m & \text{if } k_{m-1} = 1 \\ l_m + 1 & \text{otherwise}. \end{cases} \tag{2.6}$$

We can show that we necessarily have $j(i) \leq n - 1$ and $r_{1,j(i)} = 1$.

Proposition 2.7. *The binary tableau $i \to R$ is an rrc-graph associated with the permutation $s_{n-j(i)}v(R)$, where $v(R)$ is viewed as an element of Σ_n; in particular, we have $l(s_{n-j(i)}v(R)) = l(v(R)) - 1$.*

We can reverse this algorithm.

Algorithm 2.8. (Reverse Insertion). Consider an integer j with $1 \leq j \leq n-1$, and an rrc-graph R in $\mathcal{R}(w)$ for some w in Σ_n. Find the unique sequence of indices (the reverse insertion path) $(k_0, l_0), (k_1, l_1), \dots, (k_m, l_m)$, with $1 \leq k_p \leq n-1$, $1 \leq l_p \leq n-k_p$, for all $1 \leq p \leq m$, satisfying the following properties:

1. $k_0 = 0$, $l_0 = j$, and we set $r_{k_0 l_0} := 1$;
2. if $r_{k_p l_p} = 0$ for some p with $0 \leq p \leq r-1$, then $k_{p+1} = k_p$ and $l_{p+1} = l_p - 1$;
3. if $r_{k_p l_p} = 1$ and $r_{k_p+1, l_p} = 0$ for some p with $1 \leq p \leq r-1$, then $k_{p+1} = k_p + 1$ and $l_{p+1} = l_p$;
4. if $r_{k_p l_p} = 1$ and $r_{k_p+1, l_p} = 1$ for some p with $1 \leq p \leq r-1$, then $k_{p+1} = k_p + 1$ and $l_{p+1} = l_p - 1$;
5. $l_m = 1$, and $r_{k_m l_m} = 1$ implies $r_{k_m+1, l_m} = 1$.

Note that it is not possible to have $k_p + l_p = n$ and $r_{k_p l_p} = 1$, so the entry r_{k_p+1, l_p} always exists in R if $r_{k_p l_p} = 1$. We define a new binary tableau $(R \leftarrow j) = (t_{ab})$ (the reverse insertion of j into R) by simply setting $t_{k_p l_p} := r_{k_{p-1} l_{p-1}}$ if $1 \leq p \leq m$, and $t_{ab} := r_{ab}$ for all other pairs (a, b). We let $i(j) := k_m$.

Proposition 2.9.

(a) If we can insert i into R, then $((i \to R) \leftarrow j(i)) = R$.

(b) If $v_j v(R) \neq 0$, then $(i(j) \to (R \leftarrow j)) = R$.

Based on the above results, we study the way in which successive insertions can be performed.

Proposition 2.10. *Let $1 \le i_1 < i_2 \le n-1$, and assume we can insert i_2 into R. Then it is also possible to insert i_1 into $i_2 \to R$, and the second insertion path stays weakly above the first one. Furthermore, if j_1 and j_2 denote the corresponding column indices defined in (2.6), we have $j_1 < j_2$.*

We now discuss the way to use the above insertion procedure to give an analog of the R-S-K correspondence. More precisely, we describe how to associate with an arbitrary binary tableau T, a pair of rrc-graphs $(R_1(T), R_2(T))$. We consider the lexicographic array $\binom{i_1,\dots,i_m}{j_1,\dots,j_m}$ of coordinates of 1's in T, and set

$$R_1(T) := (j_1 \to (j_2 \to \dots (j_m \to \mathbf{1}) \dots)),$$

where $\mathbf{1}$ denotes the tableau with all entries equal to 1. Indeed, Proposition 2.10 shows that all these insertions can be performed. Let $k_1, \dots, k_m$ denote the corresponding column indices defined in (2.6). We define $R_2(T)$ to be the binary tableau associated with the lexicographic array $\binom{i_1,\dots,i_m}{k_1,\dots,k_m}$.

Example 2.11. Consider the lexicographic array $\binom{1,1,2,3}{1,3,1,1}$, corresponding to a binary tableau which is *not* an rrc-graph. The algorithm described above produces the following results:

$$\left(\begin{matrix}1&1&1\\1&1&\\1&&\end{matrix}\;,\;\begin{matrix}0&0&0\\0&0&\\0&&\end{matrix}\right) \to \left(\begin{matrix}0&1&1\\1&1&\\1&&\end{matrix}\;,\;\begin{matrix}0&0&0\\0&0&\\1&&\end{matrix}\right) \to \left(\begin{matrix}0&0&1\\1&1&\\1&&\end{matrix}\;,\;\begin{matrix}0&0&0\\0&1&\\1&&\end{matrix}\right) \to$$
$$\left(\begin{matrix}0&1&0\\1&1&\\0&&\end{matrix}\;,\;\begin{matrix}0&0&1\\0&1&\\1&&\end{matrix}\right) \to \left(\begin{matrix}0&0&0\\1&1&\\0&&\end{matrix}\;,\;\begin{matrix}0&1&1\\0&1&\\1&&\end{matrix}\right).$$

We can prove the following result.

Theorem 2.12. *The correspondence defined above is a bijection between binary tableaux (of staircase shape with $n-1$ rows and $n-1$ columns) and pairs of rrc-graphs associated with permutations w and ww_0, for w in S_n. Furthermore, the number of 1's in the columns of T and the number of 0's in the rows of $R_1(T)$ coincide, while the number of 1's in the rows of T and $R_2(T)$ also coincide.*

This correspondence provides a bijective proof of the Cauchy identity for Schubert polynomials (2.13), exactly in the same way as the classical R-S-K correspondence proves the Cauchy identity for Schur functions.

$$\prod_{i+j\le n} (x_i + y_j) = \sum_{w\in\Sigma_n} \mathfrak{S}_w(x)\,\mathfrak{S}_{ww_0}(y)\,. \tag{2.13}$$

In fact, what we have done is to make explicit the combinatorial proof of the Cauchy identity in [5]; however, it seems very difficult to decode the above construction from the manipulations with wiring diagrams in that paper.

The following is the analog of the symmetry property of the classical R-S-K correspondence.

Proposition 2.14. *If a binary tableau T corresponds to the pair of rrc-graphs $(R_1(T), R_2(T))$, and $\overline{T}$ denotes the tableau obtained from T by changing 0's into 1's and viceversa, then the transpose of $\overline{T}$ corresponds to the pair $(R_2(T), R_1(T))$.*

3. Noncommutative Analogs of Schubert Polynomials

In their pioneering papers [19, 9], Lascoux and Schützenberger defined noncommutative Schur functions with variables in the free algebra by using a certain reading of the tableaux in the combinatorial definition of Schur functions. They went on to develop a theory of noncommutative Schur functions for the so-called *plactic algebra*. This theory was generalized by Fomin and Greene in [3]. They showed that, surprisingly, the noncommutative Schur functions commute if their variables $u_1, u_2, \ldots$ satisfy the "non-local Knuth relations"

$$\begin{aligned} u_i u_k u_j &= u_k u_i u_j\,, \quad i \le j < k\,, \quad |i-k| \ge 2\,,\\ u_j u_i u_k &= u_j u_k u_i\,, \quad i < j \le k\,, \quad |i-k| \ge 2\,, \end{aligned} \tag{3.1}$$

as well as the following "local commutation" relation:

$$(u_i + u_{i+1})u_{i+1}u_i = u_{i+1}u_i(u_i + u_{i+1})\,. \tag{3.2}$$

The above relations are satisfied in many well-known algebras, such as the plactic, nilplactic, nilCoxeter, and degenerate Hecke algebras. As a consequence of the above result, Fomin and Greene derived a Cauchy identity for the noncommutative Schur functions and a generalized Littlewood-Richardson rule for a large class of symmetric functions, including the stable Schubert and the stable Grothendieck functions.

We now extend the work of Fomin and Greene to Schubert polynomials. We start by defining noncommutative analogs of Schubert polynomials for every permutation w in the symmetric group Σ_n. Given a set of noncommuting variables $u_1, \ldots, u_{n-1}$ and some binary tableau T, we let

$$u(T) := \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} u_{n-i}^{1-t_{ij}}\,.$$

We now define the noncommutative analogs of Schubert polynomials $S_w(u)$ by

$$S_w(u) = S_w(u_1, \ldots, u_{n-1}) := \sum_{R \in \mathcal{R}(ww_0)} u(R)\,, \tag{3.3}$$

where w_0 is the longest permutation in Σ_n. Similarly, we can define $S_w(u_k, \ldots, u_{n-1})$ for every permutation w of the set $\{k, \ldots, n\}$; in this case, we consider binary tableaux $T = (t_{ij})$ with $1 \le i \le n-k$ and $1 \le j \le n+1-k-i$, and define

$$v(T) := \prod_{i=1}^{n-k} \prod_{j=1}^{n+1-k-i} v_{n-j}^{t_{ij}}\,, \quad u(T) := \prod_{j=1}^{n-k} \prod_{i=1}^{n+1-k-j} u_{n-i}^{1-t_{ij}}\,.$$

Note that

$$S_w(x) = \left(\prod_{i=1}^{n-1} x_i^i\right) \mathfrak{S}_{ww_0}(x_{n-1}^{-1}, \ldots, x_1^{-1})\,.$$

Clearly, the degree of $S_w(u)$ is $l(w)$. For instance, if w is the identity permutation, then $S_w(u) = 1$, and if $w = w_0$, then $S_w(u) = (u_{n-1} \dots u_1)(u_{n-1} \dots u_2) \dots u_{n-1}$. Let us consider another example.

Example 3.4. Consider the permutation $w = (2, 3, 4, 1)$ in Σ_4, for which $ww_0 = (1, 4, 3, 2) = s_2s_3s_2 = s_3s_2s_3$. The set $\mathcal{R}(ww_0)$ consists of the following rrc-graphs.

$$\begin{array}{lllll} 1\,1\,0 & 1\,1\,0 & 1\,0\,0 & 0\,1\,0 & 0\,0\,0 \\ 1\,0 & 0\,0 & 0\,1 & 1\,1 & 1\,1 \\ 0 & 1 & 1 & 0 & 1 \end{array}$$

Hence $S_w(u) = u_1u_2u_3 + u_2u_2u_3 + u_2u_3u_3 + u_3u_1u_3 + u_3u_3u_3$.

Note that the polynomials $S_w(u)$ are in some sense complementary to Schubert polynomials, because they are defined in terms of the entries equal to 0 of the rrc-graphs R, as opposed to the entries equal to 1, which are used to define Schubert polynomials. The main reason for considering these polynomials is that only in terms of them were we able to find a noncommutative version of the Cauchy identity for Schubert polynomials. Let us note that neither the polynomials $S_w(u)$, nor their variations obtained by changing the definition of $u(T)$ (for instance, by recording the 1's rather than the 0's) are directly related to the noncommutative Schubert polynomials defined by Lascoux and Schützenberger in [15] (see also [14, 18]).

The polynomials $S_w(u)$ are stable under the obvious embedding of the symmetric group $\Sigma_{\{k+1,\dots,n\}}$ into the symmetric group $\Sigma_{\{k,k+1,\dots,n\}}$. To state this property, we use the standard notation $1 \times w$ for the image of a permutation $w \in \Sigma_{\{2,\dots,n\}}$ in Σ_n.

Proposition 3.5. *For every permutation w in $\Sigma_{\{2,\dots,n\}}$, we have $S_w(u_2, \dots, u_{n-1}) = S_{1\times w}(u)$.*

Unlike the noncommutative Schur functions of Fomin and Greene, the polynomials $S_w(u)$ do not commute if the relations (3.1) and (3.2) are satisfied; they still do not commute if we replace the relations (3.1) with the stronger relation

$$u_iu_j = u_ju_i\,, \quad |i - j| \geq 2\,. \tag{3.6}$$

Nevertheless, we now show that our polynomials do satisfy a noncommutative version of the Cauchy identity for Schubert polynomials if relations (3.6) and (3.2) are satisfied (note that (3.1) and (3.2) do not suffice). So from now on, we assume that we are working in a noncommutative algebra containing elements $u_1, \dots, u_{n-1}$ which satisfy these relations. Clearly, we have to use different techniques from those in [3] to prove the Cauchy identity for $S_w(u)$. Indeed, the most difficult step for us is to prove a Pieri-type formula for our polynomials; this formula expresses the product of the noncommutative analog of an elementary symmetric polynomial with $S_w(u)$ as a sum of polynomials $S_{w'}(u)$, provided that $w(1) = 1$ (note that a priori it is not clear that such an expression should exist). Recall that the noncommutative analogs of the elementary symmetric polynomials $e_m(u)$ are defined by

$$e_m(u) := \sum_{n-1 \geq i_1 > i_2 > \dots > i_m \geq 1} u_{i_1}u_{i_2} \dots u_{i_m}\,.$$

Our Pieri formula is easiest to express using the nilCoxeter algebra, and the convention $S_0(u) := 0$.

Theorem 3.7. *Assuming that relations (3.6) and (3.2) are satisfied, the following identity holds for every integer m with $1 \leq m \leq n-1$, and every w in Σ_n with $w(1)=1$:*

$$e_m(u)\, S_w(u) = \sum_{n-1 \geq j_1 > j_2 > \ldots > j_m \geq 1} S_{v_{j_1} v_{j_2} \ldots v_{j_m} w}(u)\,.$$

The proof of this formula relies heavily on our insertion algorithm discussed in the previous section. Essentially, we insert elements corresponding to the indices of monomials in $e_m(u)$ into the rrc-graphs R corresponding to the monomials of $S_w(u)$, and investigate the effect on the reading word $u(R)$.

Based on Theorem 3.7, we can prove the noncommutative analog of the Cauchy identity (2.13) for Schubert polynomials.

Theorem 3.8. *If relations (3.6) and (3.2) are satisfied, we have*

$$\prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i u_j) = \sum_{w \in \Sigma_n} \mathfrak{S}_w(x)\, S_w(u)\,.$$

Proof. We use induction on n, which clearly starts at $n=1$. Assuming the identity holds for $n-1$, we have

$$\begin{aligned}
\prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i u_j) &= \left(\sum_{m=0}^{n-1} x_1^m e_m(u) \right) \left(\sum_{w \in \Sigma_{\{2,\ldots,n\}}} \mathfrak{S}_w(x_2, \ldots, x_{n-1})\, S_{1 \times w}(u) \right) \\
&= \sum_{w \in \Sigma_{\{2,\ldots,n\}}} \sum_{m=0}^{n-1} \sum_{n-1 \geq j_1 > \ldots > j_m \geq 1} x_1^m\, \mathfrak{S}_w(x_2, \ldots, x_{n-1})\, S_{v_{j_1} \ldots v_{j_m} (1 \times w)}(u) \\
&= \sum_{w \in \Sigma_n} \left(\sum_{m=0}^{n-1} \sum_{1 \times w' \in \Sigma_n^{(m)}(w)} x_1^m\, \mathfrak{S}_{w'}(x_2, \ldots, x_{n-1}) \right) S_w(u) \\
&= \sum_{w \in \Sigma_n} \mathfrak{S}_w(x)\, S_w(u)\,.
\end{aligned}$$

The first equality follows by induction and the stability property in Proposition 3.5; the second equality is an application of the Pieri-type formula in Theorem 3.7; finally, the fourth equality follows from a formula in [12] for expressing a Schubert polynomial as a univariate polynomial in the first variable with coefficients being Schubert polynomials in the rest of the variables. □

If u_i satisfy the relations (2.1) defining the nilCoxeter algebra, then the noncommutative Cauchy identity above implies $S_w(v) = w$, according to (2.2). This means that there is a unique rrc-graph R^* in $\mathcal{R}(ww_0)$ with $u(R^*) = w$, and for all the other rrc-graphs R we have $u(R) = 0$; the latter fact can actually be proved directly without difficulty. The following Proposition identifies the rrc-graph R^*.

Proposition 3.9. *Given w in Σ_n and assuming u_i satisfy (2.1), the unique rrc-graph R^* in $\mathcal{R}(w)$ with $u(R^*) = ww_0$ is the maximal one in lexicographic order (here $R^* = (r^*_{ij})$ is identified with the binary word $r^*_{11} \ldots r^*_{1,n-1} r^*_{21} \ldots r^*_{2,n-2} \cdots r^*_{1,n-1}$). The entries of this* *rrc-graph are*

$$r^*_{i,k-i+1} = \begin{cases} 1 & \text{if } i \le c_{n-k}(w^{-1}) \\ 0 & \text{otherwise}, \end{cases}$$

where $1 \le i \le k \le n-1$, *and* $c(w^{-1}) = (c_1(w^{-1}), \ldots, c_{n-1}(w^{-1}))$ *is the code of the inverse permutation to* w.

4. The Expansion of Grothendieck Polynomials in the Basis of Schubert Polynomials

We begin this section with a brief introduction to the cohomology and K-theory of flag varieties; for more information, we refer the reader to [7] and [10].

Let Fl_n be the variety of complete flags $0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ in $\mathbb{C}^n$; this is an irreducible algebraic variety of complex dimension $\binom{n}{2}$. Its integral cohomology ring $H^*(Fl_n)$ is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/I_n$, where I_n is the ideal generated by symmetric functions in $x_1, \ldots, x_n$ with constant term 0; here, the elements x_i are identified with the Chern classes of the dual line bundles $(V_i/V_{i-1})^*$. Recall that Fl_n is a disjoint union of cells indexed by permutations w in Σ_n, and that their closures are the so-called Schubert varieties X_w, of complex dimension $l(w)$. It is well-known that the cohomology class corresponding to X_w is represented by the Schubert polynomial $\mathfrak{S}_w(x)$.

The K-theory $K^0(Fl_n)$ of the flag variety is the Grothendieck ring of complex vector bundles over Fl_n under direct sum and tensor product. It is known that $K^0(Fl_n)$ is isomorphic to the same quotient of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ as the integral cohomology $H^*(Fl_n)$. This time we identify x_i with the K-theory Chern class $1 - 1/a_i$ of the line bundle $(V_i/V_{i-1})^*$, where a_i represents V_i/V_{i-1} in the Grothendieck ring. The classes dual to the structure sheaves of Schubert varieties form the natural basis of $K^0(Fl_n)$. The construction of these classes in the general case of flag varieties corresponding to Kac-Moody Lie algebras was given in [8]; this construction is based on certain divided difference operators, as shown below. For the flag variety Fl_n, the K-theory classes corresponding to Schubert varieties are represented by Grothendieck polynomials, which were introduced by Lascoux and Schützenberger in [13], and studied in more detail in [10]. In fact, in the latter paper, Lascoux defines the more general double Grothendieck polynomials, which are polynomials in two sets of variables related to the T-equivariant K-theory of Fl_n (see also [8]). Here we restrict ourselves to Grothendieck polynomials in only one set of variables.

Given a parameter β, we define polynomials $\mathfrak{G}^{(\beta)}_w(x) = \mathfrak{G}^{(\beta)}_w(x_1, \ldots, x_{n-1})$ by

$$\mathfrak{G}^{(\beta)}_{w_0}(x) := \prod_{i=1}^{n-1} x_i^{n-i},$$

$$\mathfrak{G}^{(\beta)}_w(x) = \pi^{(\beta)}_i \, \mathfrak{G}^{(\beta)}_{ws_i}(x), \quad \text{if } l(ws_i) = l(w) + 1.$$

Here $\pi^{(\beta)}_i$ is the operator on $\mathbb{Z}[x_1, \ldots, x_n]$ defined by

$$\pi^{(\beta)}_i \, f(x) = \frac{(1 + \beta x_{i+1})\, f(x) - (1 + \beta x_i)\, f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$

The Grothendieck polynomial indexed by the permutation w is $\mathfrak{G}_w^{(-1)}(x)$, which we denote simply by $\mathfrak{G}_w(x)$. Note that $\mathfrak{G}_w^{(0)}(x)$ is just the Schubert polynomial $\mathfrak{S}_w(x)$.

It is easy to check that the operators $\pi_i^{(\beta)}$ provide a faithful representation of the algebra $\mathcal{A}_n^{(\beta)}$ generated by $u_1, \dots, u_{n-1}$, subject to the following relations:

$$\begin{aligned} & u_i^2 = \beta u_i\,, \\ & u_i u_j = u_j u_i\,, \quad |i-j| \geq 2\,, \\ & u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}\,. \end{aligned} \tag{4.1}$$

The algebra $\mathcal{A}_n^{(\beta)}$ has a basis consisting of elements which can be identified with permutations in Σ_n in the same way as the basis elements for the nilCoxeter algebra were. Once again, we shall not attempt to distinguish notationally between elements and their products in Σ_n and $\mathcal{A}_n^{(\beta)}$. Note that $\mathcal{A}_n^{(0)}$ is the nilCoxeter algebra, and $\mathcal{A}_n^{(-1)}$ is the degenerate Hecke algebra $H_n(0)$.

Fomin and Kirillov gave a construction for the polynomials $\mathfrak{G}_w^{(\beta)}(x)$ similar to (2.2) in [4]. They proved that if u_i satisfy (4.1), and $\mathfrak{G}^{(\beta)}(x)$ is given by the same expression as $\mathfrak{S}(x)$, except that the v_i's are replaced by the u_i's, then we have

$$\mathfrak{G}^{(\beta)}(x) = \sum_{w \in \Sigma_n} \mathfrak{G}_w^{(\beta)}(x)\, w\,. \tag{4.2}$$

By taking a certain limit of the polynomials $\mathfrak{G}_w^{(\beta)}(x)$, we obtain power series in β, denoted $G_w^{(\beta)}(x_1, x_2, \dots)$, whose coefficients are symmetric functions in $x_1, x_2, \dots$. There is a generating function formula similar to (4.2) for these power series. We call $G_w^{(-1)}(x_1, x_2, \dots)$ a stable Grothendieck function.

It follows from (4.2) that the Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$ is a nonhomogeneous polynomial with monomials of degree greater or equal to $l(w)$; furthermore, the sign of the coefficient of any monomial of degree $l(w) + i$ is $(-1)^i$. On the other hand, the definition of Grothendieck polynomials implies that the lowest homogeneous component of $\mathfrak{G}_w(\mathbf{x})$ is the corresponding Schubert polynomial $\mathfrak{S}_w(x)$. Hence the transition matrix from Grothendieck to Schubert polynomials is triangular with 1's on the diagonal. The geometric idea underlying this observation is that the cohomology of Fl_n is the associated graded ring to $K^0(Fl_n)$ with respect to a certain filtration (see [8]).

Fomin and Greene used their theory of noncommutative Schur functions to show that the stable Grothendieck functions are nonnegative integer combinations of Schur functions, and gave a combinatorial interpretation for the coefficients of the expansion (see [3]). Here we extend their work, by providing explicit combinatorial information about the expansion of a Grothendieck polynomial in the basis of Schubert polynomials. We also confirm the conjecture of Fomin and Kirillov concerning the signs of the coefficients in this expansion.

Theorem 4.3. *The sign of the coefficient of the Schubert polynomial $\mathfrak{S}_{w'}(x)$ (where $l(w') \geq l(w)$) in the expansion of $\mathfrak{G}_w(\mathbf{x})$ is $(-1)^{l(w')-l(w)}$. Furthermore, the absolute value of this coefficient is equal to the number of rrc-graphs R with $v(R) = w'w_0$ and $u(R) = w$, where u_i satisfy (4.1) with $\beta = 1$.*

Note that $(-1)^{l(w')-l(w)}$ is precisely the value of the Möbius function of the Bruhat order on the symmetric group. Hence it is natural to expect the following result, conjectured by Lascoux.

Conjecture 4.4. *Any Schubert polynomial is a nonnegative integer combination of Grothendieck polynomials.*

In [16] we proved that Conjecture 4.4 is true in the Grassmannian case, and we presented a combinatorial interpretation for the coefficients in the corresponding expansion.

Concerning the terms which appear in the expansion of a Grothendieck polynomial in the basis of Schubert polynomials, we have the following result.

Proposition 4.5. *The Grothendieck polynomial $\mathfrak{G}_w(x)$ is a linear combination of Schubert polynomials $\mathfrak{S}_{w'}(x)$ with $w \leq w'$ in Bruhat order.*

We state one more conjecture, which was suggested by several computer experiments.

Conjecture 4.6. *We have that $\mathfrak{G}_w(x) = \mathfrak{S}_w(x)$ if and only if w is a dominant permutation (i.e., its code is a partition).*

As far as the geometric significance of the above results and conjectures is concerned, it is still mysterious to a considerable extent. The main reason for this is that the isomorphism between $K^0(Fl_n)$ and $H^*(Fl_n)$ defined above (using identification of Chern classes) is not entirely geometric. A geometrically defined isomorphism between $K^0(Fl_n) \otimes \mathbb{Q}$ and $H^*(Fl_n, \mathbb{Q})$ is the Chern character. However, the images of Schubert classes in K-theory under the Chern character are more complicated to describe then their images under the isomorphism used in this paper, although there is a connection between the two images.

Let us now consider an example to illustrate Theorem 4.3 and Proposition 4.5.

Example 4.7. We have the following expansion for the Grothendieck polynomial $\mathfrak{G}_{(1,4,3,2)}(x)$:

$$\mathfrak{G}_{(1,4,3,2)}(x) = \mathfrak{S}_{(1,4,3,2)}(x) - 2\mathfrak{S}_{(2,4,3,1)}(x) - \mathfrak{S}_{(3,4,1,2)}(x) + \mathfrak{S}_{(3,4,2,1)}(x)\,.$$

The two rrc-graphs counted by the coefficient of $\mathfrak{S}_{(2,4,3,1)}(x)$, and the rrc-graphs counted by the coefficients of $\mathfrak{S}_{(3,4,1,2)}(x)$ and $\mathfrak{S}_{(3,4,2,1)}(x)$ are listed below, in this order.

$$\begin{array}{lllllll} 0\,1\,0 & \quad & 0\,0\,0 & \quad & 0\,0\,1 & \quad & 0\,0\,0 \\ 0\,0 & & 0\,1 & & 0\,0 & & 0\,0 \\ 1 & & 1 & & 1 & & 1 \end{array}$$

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