EXTENDED PATTERN AVOIDANCE

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Abstract

A 0-1 matrix is said to be extendably τ-avoiding if it can be the upper left corner of a τ-avoiding permutation matrix. This concept arose in [EL], where the surprising result that the number of extendably 321-avoiding rectangles are enumerated by the ballot numbers was proved. Here we study the other five patterns of length three. The main result is that the six patterns of length three divides into only two cases, no easy symmetry can explain this. An other result is that the Simion-Schmidt-West-bijection for permutations avoiding patterns 12τ and 21τ works also for extended pattern avoidance.

The results and proofs use many properties of the Catalan numbers and refinements of the Catalan numbers.

Keywords: avoiding pattern, Catalan number, ballot number

1. Introduction

1.1. **Notation.** Given a permutation $\pi \in S_n$, let it be represented by a permutation matrix, with 1's in positions $(i, \pi(i))$. Fix any t of these 1's and delete all rows and columns that do not contain any of them. The result is a permutation matrix for a permutation $\tau \in S_t$. It is said that π contains the **pattern** τ. A permutation that does not contain the pattern τ is said to be τ**-avoiding**.

We will for convenience switch from 1's and 0's to dots and empty positions.

An $r \times k$-rectangle with d dots is said to be **extendably** τ**-avoiding** if it can be extended with dots to the right and below the rectangle to form a τ-avoiding permutation matrix. Figure 1 gives an example of an 10×9 rectangle with 6 dots that is extendably 213-avoiding.

Definition For a pattern $\tau \in S_t$ and integers $0 \leq d \leq r, k$, let $S_\tau(r, k, d)$ denote the number of extendably τ-avoiding $r \times k$ matrices with d dots. □

Note that there can be many different ways to extend the rectangle which does not influence $S_\tau(r, k, d)$. See for example Figure 1, where in rows 6 and 7 we could replace the grey dots with the dashed ones.

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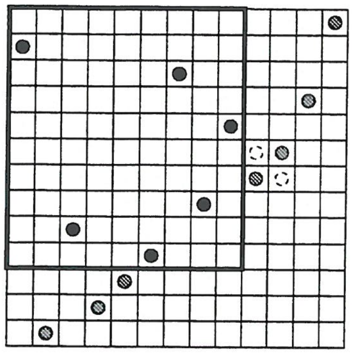

FIGURE 1. An extendably 213-avoiding 10×9 rectangle with 6 dots.

The Catalan numbers and different refinements will be essential in both the results and the proofs. We have collected a number of facts about them in an Appendix.

1.2. **Results.** The concept of extendably avoiding a pattern was first studied in [EL] in connection with the essential set of a permutation. It was proved that the number $S_{321}(r,k,d)$ was equal to the ballot numbers, a refinement of the Catalan numbers. There is no evident reason why the ballot numbers occur in this context. The present paper calculates the $S_\tau(r,k,d)$ for the other five patterns τ of length three. There is a priori no reason to expect that any of these six formulae should be equal and therefore it was a surprise when it turned out that there are not six different formulae for the six different patterns of length three but only two formulae. The only symmetry that will simplify our proofs is $S_{312}(r,k,d) = S_{231}(k,r,d)$.

Theorem 1.1 (Main Theorem). *The number of extendably τ-avoiding $r \times k$ rectangular matrices with d dots is:*

1. *for* $\tau = 321, 312, 231$ *and* 132
$$S_\tau(r,k,d) = \binom{r+k}{d} - \binom{r+k}{d-1},$$
2. *for* $\tau = 123$ *and* 213
$$S_\tau(r,k,d) = \sum_{s=1}^{d}\sum_{t=1}^{d} C_d(s,t)\binom{r+s-d}{s}\binom{k+t-d}{t},$$

where $C_d(s,t) = \binom{2d-s-t-2}{d-s-1} - \binom{2d-s-t-2}{d-1}$*, is the double ballot refinement of the Catalan numbers, see Appendix.*

Note that $\binom{r+k}{d} - \binom{r+k}{d-1} = C_{r+k-d+1}(r+k-2d+1)$ is a ballot number, a fact we will use in the proofs later. One might think it is not surprising that the ballot numbers show up since they are a natural refinement of the Catalan numbers $C_n = \sum_{t=1}^{n} C_n(t)$. However, we see no way in this context to sum up the ballot numbers occurring here to obtain a Catalan number. Indeed, the extendably τ-avoiding matrices are not a subset of τ-avoiding permutations. Another non-expected fact is that in case 1 the value only depends on $r+k$ and d. The proof is given in Section 2.

We can immediately deduce the following corollary.

Corollary 1.2. *The number of r-letter words on the alphabet $1,\dots,n$ that can be extended to a τ-avoiding permutation is*

$$\binom{r+n}{r} - \binom{r+n}{r-1}, \quad \textit{for } \tau = 321, 312, 231 \textit{ and } 132$$

$$\sum_{t=1}^{r} C_r(t)\binom{n+t-r}{t}, \quad \textit{for } \tau = 123 \textit{ and } 213$$

where $C_r(t)$ is the ballot number.

PROOF Follows from the Main Theorem with $r=d$ and $n=k$. □

In Section 3 we prove a general theorem for longer patterns using the Simoion-Schmidt-West bijection [SS, W1].

Theorem 1.3. *For any $\tau \in S_{k-2}$ we have*

$$S_{12\tau}(r,k,d) = S_{21\tau}(r,k,d).$$

2. Proof of the Main theorem

Case 321: The case $\tau = 321$ was proved in [EL].

Case 132: First we map to a 132-avoiding permutation matrix of size $r+k-d+2$.

Given an $r \times k$ rectangle R with d dots that extendably avoids 132, we add a zeroth row with a dot in square $(0, k+1)$, we add a zeroth column with a dot in square $(r+1, 0)$ and then we extend the rectangle with dots to the right and below such that we obtain a 132-avoiding permutation matrix of size $r+k-d+2$ with the dots in first column and first row as described, see Figure 2. Because of the dots $(0,k+1)$ and $(r+1,0)$ there is only one way to do the extension and still be 132-avoiding.

There are $C_{r+k-d+2}(k-d+1, r-d+1)$ such matrices, see equation (1) in the Appendix. However we only obtain those which have d dots in the area corresponding to R, which is the same as having zero dots in area B of Figure 2. It is easy to see that if there is a dot in B then there is a dot in $(r+k-d+1, r+k-d+1)$. We therefore have to subtract the number of 132-avoiding permutations $\pi \in S_{r+k-d+2}$ with $\pi(1) = k+2, \pi(r+2) = 1$ and $\pi(r+k-d+2) = r+k-d+2$, which is the same as the number of 132-avoiding permutations $\pi \in S_{r+k-d+1}$ with $\pi(1) = k+2, \pi(r+2) = 1$, that is $C_{r+k-d+1}(k-d, r-d)$. Hence we get,

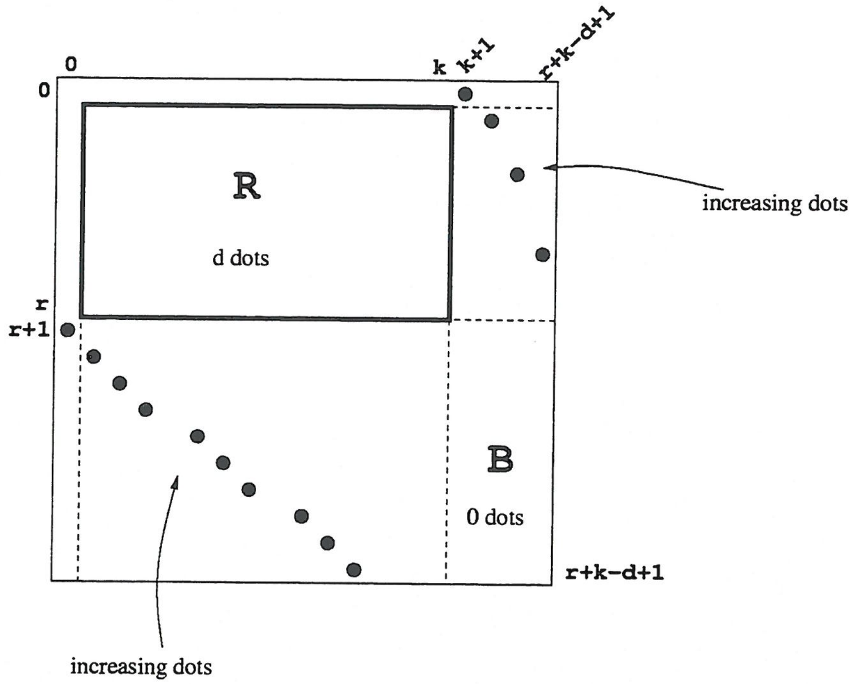

FIGURE 2. The Case 132.

$$S_{132}(r,k,d) = C_{r+k-d+2}(k-d+1, r-d+1) - C_{r+k-d+1}(k-d, r-d) =$$
$$\left[\binom{r+k}{k} - \binom{r+k}{d-1}\right] - \left[\binom{r+k}{k} - \binom{r+k}{d}\right] = \binom{r+k}{d} - \binom{r+k}{d-1}.$$

Case 312 and 231:

We study the 312 case. First we want to establish the following recursion.

Lemma 2.1. *For any r, k, d with $k > d \geq 1$ we have*

$$S_{312}(r,k,d) = S_{312}(r,k-1,d) + \sum_{i=1}^{d} C_{i-1} S_{312}(r-i, k-i, d-i).$$

PROOF Let R denote an extendably 312-avoiding $r \times k$ rectangle with d dots. If there is no dot in the first column of R then we can just remove it, this case gives the first term. Assume the dot in the first column is in row i. Since $d < k$, there is an empty column c which in the extended matrix gives a dot below row i in column c. This means that rows $1, \ldots, i-1$ must all have dots in R otherwise we could not extend to a 312-avoiding matrix. By the same reasoning the dots in these rows must be in columns $2, \ldots, i$ and form any 312-avoiding permutation matrix, there are C_{i-1} such. The other $d-i$ dots must be in the lower $r-i \times k-i$ rectangle which must be extendably 312-avoiding, there are $S_{312}(r-i, k-i, d-i)$ such. See Figure 3. $\square$

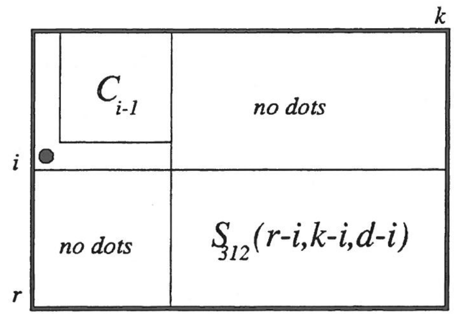

FIGURE 3. The Case 312.

We want to show that $S_{312}(r, k, d) = C_{r+k-d+1}(r + k - 2d + 1)$ and plugging this into Lemma 2.1 we get recursion (2) in the Appendix. We are done by induction over k, if we prove the theorem for $k = d$.

Lemma 2.2. *For any $r, d \geq 1$ we have*

$$S_{312}(r, d, d) = \sum_{i=1}^{d} C_{i-1} S_{312}(r-i, d-i, d-i) + \sum_{i=d+1}^{r} S_{312}(i-1, d-1, d-1).$$

PROOF Let R denote an extendably 312-avoiding $r \times d$ rectangle with d dots. Assume the dot in the first column of R is in row i. If $1 \leq i \leq d$ we argue as in the proof of Lemma 2.1 and get the first sum. If $d < i \leq r$ then there can not be any dots in rows $i+1, \ldots, r$ in R and there are $S_{312}(i-1, d-1, d-1)$ possibilities to fill in rows $1, \ldots, i-1$ and columns $2, \ldots, d$ with $d-1$ dots. This gives the second sum. $\square$

By induction over d, we have $\sum_{i=d+1}^{r} S_{312}(i-1, d-1, d-1) = \sum_{i=d+1}^{r} C_i(i-d+1)$, which by equation (3) in the Appendix is equal to $C_r(r-d)$. We are once again in the situation of recursion (2) in the Appendix and we are done by induction.

Reflection of the permutation matrices in the main diagonal gives $S_{312}(r, k, d) = S_{231}(k, r, d)$ and since the formula for $S_{312}(r, k, d)$ is symmetric in r and k we are done also with the case 231.

Case 123 and 213: We will do the $\tau = 213$ case. $\tau = 123$ will then follow from Theorem 1.3.

Let R denote an extendably 213-avoiding $r \times k$ rectangle with d dots. Assume that there is a dot (x, k) in column k and that there are $s \geq 1$ dots in R that are in rows $x+1, \ldots, r$. Since R is extendably 213-avoiding, these s dots have to be in rows $r-s+1, \ldots, r$. Similarly assume that there is a dot (r, y) in row r and that there are $t \geq 1$ dots in R that are in columns $y+1, \ldots, k$. Again, these t dots have to be in columns $k-t+1, \ldots, k$, see Figure 1 and 4. Note that because R

is 213-avoiding all the dots in rows $1, \dots, x-1$ have to be in increasing order, and similarly for the dots in columns $1, \dots, y-1$.

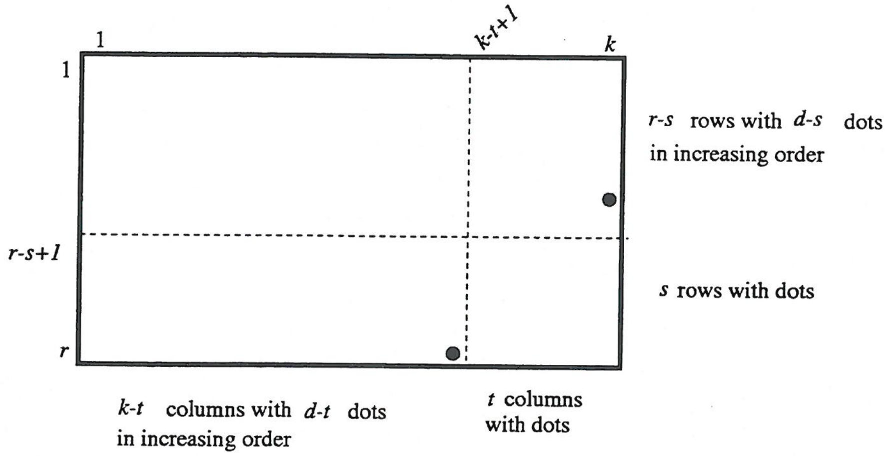

FIGURE 4. The Case 213.

If we were to remove all empty rows and columns of R we would get a 213-avoiding $d \times d$ permutation matrix. Vice versa we could start with a 213-avoiding permutation matrix with $\pi(d) = d - t$ and $\pi(d-s) = d$ and construct an R by inserting $r - d$ empty rows among the first $r - s$ rows and $k - d$ empty columns among the first $k - t$ columns of R. Since we know that the dots in these rows and columns are increasing, this will preserve the property of being extendably 213-avoiding. In this way we construct $C_d(d-s, d-t)\binom{r-s}{r-d}\binom{k-t}{k-d}$. Substitute variables and we have the the double sum

$$\sum_{s=1}^{d-1}\sum_{t=1}^{d-1} C_d(s,t)\binom{r+s-d}{s}\binom{k+t-d}{t}.$$

Which R have we missed? All those that have no dot in the last column or no dot in the last row or a dot in (r,k). That is all cases when all the dots have to be in increasing order. This gives a total of $\binom{r}{d}\binom{k}{d}$.

The Main Theorem is proved. □

3. THE BIJECTION

In this section we will define a bijection that will prove Theorem 1.3. We are using the bijection between permutations avoiding 12τ and permutations avoiding 21τ in [W1], which was inspired by the bijection in [SS]. We are brief in our description and the intrested reader is referred to [W1, BW] for more details.

First some definitions. Assume we are given the pattern $\tau \in S_t$ and an extendably τ-avoiding $r \times k$-rectangle R with d dots. If $\tau(t-1) < \tau(t)$ then R can be extended to a τ-avoiding permutation π with $\pi(r+1) > \pi(r+2) > \cdots > \pi(r+k-d)$

whereas if $\tau(t-1) > \tau(t)$ then R can be extended to a τ-avoiding permutation π with $\pi(r+1) < \pi(r+2) < \cdots < \pi(r+k-d)$. Similarly we can always extend R in columns $k+1, \ldots, r+k-d$ with either increasing or decreasing dots. We will call this extension of R the **standard extension.**

With a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$ we associate a **Ferrers board** which has top row of length λ_1, second row of length λ_2 etc.

We now need to extend the concept of containing a pattern to Ferrers boards. We say that λ contains the pattern $\tau \in S_t$ if there are rows $1 \leq r_1 < r_2 < \ldots < r_t \leq s$ and columns $1 \leq c_1 < \ldots < c_t \leq \lambda_1$ such that the restriction of λ to these rows and columns form the permutation matrix of τ and that every square (r_j, c_i) falls within the board. Let $S_\tau(\lambda, s)$ be the number of τ-avoiding ways to fill in s dots on λ.

Also define a partial order on partitions by $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s) \leq_{term} \mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s)$ if $\lambda_i \leq \mu_i$ for all $1 \leq i \leq s$.

With these definitions we have the following lemma.

Lemma 3.1. *Given $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 1)$ with $\lambda_1 = s$ then*

$$S_{12}(\lambda, s) = S_{21}(\lambda, s) = \begin{cases} 1, & \text{if } \lambda \geq_{term} (s, s-1, \ldots, 3, 2, 1) \\ 0, & \text{otherwise} \end{cases}$$

PROOF See [BW]. □

BIJECTION [essentially due to West] Let $\pi \in S_{r+k-d}$ be the standard extension of a 12τ-avoiding $r \times k$ permutation matrix R with d dots. A square (i, j) in the permutation matrix π will be called **dominant** if the pattern τ can be found among the dots in rows $i+1, \ldots, r+k-d$ and columns $j+1, \ldots, r+k-d$. Note that the set of dominant squares form a Ferrers board λ. Let $D(\pi)$ be the dots in π that are in dominant squares. $D(\pi)$ includes only dots in R, since we have choosen the standard extension. Restrict λ to the rows and columns that contain a dot in $D(\pi)$ and get a new board λ'. If λ' is empty then we do nothing. If it is nonempty we know by Lemma 3.1 that the dots have to be in the one and only 12-avoiding way to fill the board which we map to the only 21-avoiding way to fill in the board. Do the corresponding change of dots in R and the bijection is done. It is clearly well-defined since the entire change of dots takes place within R. □

4. Appendix: Catalan numbers and Ballot refinements

A huge amount of mathematical objects, see [St], is enumerated by the Catalan numbers $1, 1, 2, 5, 14, 42, 132, \ldots$ $C_n = \binom{2n}{n} - \binom{2n}{n-1}$. One important instance is pattern avoiding permutations. It is a well known theorem that this is the Catalan numbers for every pattern of length three, see e.g. [K] or [SS].

There are several different interesting refinements of the Catalan numbers. One is the ballot numbers $C_n(t)$ which we may define as

$$C_n(t) = |\{\pi \in S_n : \pi \text{ is 213-avoiding and } \pi(t) = n\}|.$$

Reflecting the permutation matrix in the main diagonal we see that we could replace $\pi(t) = n$ with $\pi(n) = t$. The same refinement can be found for all six patterns of length three. Some readers might recognize $C_n(t)$ as the number of Dyck paths (i.e. paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that do not go below the x-axis) with last peak of height t.

Lemma 4.1. *The ballot number* $C_n(t) = \binom{2n-t-1}{n-t} - \binom{2n-t-1}{n-t-1}$.

PROOF Induction over n. Given a 213-avoiding permutation $\pi \in S_n$ we want to expand this to a 213-avoiding permutation $\pi' \in S_{n+1}$ by defining

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i < x \\ n+1 & \text{if } i = x \\ \pi(i-1) & \text{if } i > x, \end{cases}$$

for some x. It is clear that this is possible if and only if $\pi(t) = n, t \geq x - 1$. Hence

$$C_{n+1}(x) = \sum_{t=x-1}^{n} C_n(t) = \sum_{t=x-1}^{n} \binom{2n-t-1}{n-t} - \binom{2n-t-1}{n-t-1} = \binom{2n-x+1}{n-x+1} - \binom{2n-x+1}{n-x}.$$

□

That the same refinement exists, for the appropriate statistics, is clear by symmetry for 132,312 and 231. It is also true for 123 and 321, but a slight adjustment of the proof is necessary, see for example [W2].

We are also concerned with a double refinement of the Catalans.

$$C_n(s,t) = |\{\pi \in S_n : \pi \text{ is 213-avoiding and } \pi(s) = n, \pi(n) = t\}|.$$

I have not seen these numbers discussed in print, but I am sure that they and the lemma below have been rediscovered plenty of times. For example, they also count the number of Dyck paths with first peak of height t and last peak of height s.

Lemma 4.2. *For* $1 \leq s,t < n$ *we have*

$C_n(s,t) = \binom{2n-s-t-2}{n-t-1} - \binom{2n-s-t-2}{n-s-t-1}$.

$C_n(s,n) = C_n(n,t) = 0$ *and*

$C_n(n,n) = 1$.

PROOF Similar to the proof of Lemma 4.1. □

Again it is by symmetry clear that the corresponding refinement exists for 132,312 and 231. In this paper we need

$$C_n(s,t) = |\{\pi \in S_n : \pi \text{ is 132-avoiding and } \pi(n+1-s) = 1, \pi(1) = n+1-t\}|. \tag{1}$$

It is not immediate that the same is true for 123 and 321, but in fact we have the following that is even stronger.

Lemma 4.3.

$$C_n(s,t) = |\{\pi \in S_n : \pi \textit{ is 123-avoiding and } \pi(s) = n, \pi(n) = t\}| = |\{\pi \in S_n : \pi \textit{ is 123-avoiding and } \pi(s) = 1, \pi(1) = t\}|.$$

PROOF Omitted. □

$s\backslash t$	1	2	3
1	1	1	
2	1	1	
3			1

$s\backslash t$	1	2	3	4
1	2	2	1	
2	2	2	1	
3	1	1	1	
4				1

$s\backslash t$	1	2	3	4	5
1	5	5	3	1	
2	5	5	3	1	
3	3	3	2	1	
4	1	1	1	1	
5					1

TABLE 1. Tables of $C_3(s,t)$, $C_4(s,t)$ and $C_5(s,t)$.

We also need the following two recursions.

The ballot number $C_n(t)$ satisfies

$$C_n(t) = C_{n-1}(t-1) + \sum_{i=1}^{n-t} C_{i-1}C_{n-i}(t). \tag{2}$$

The perhaps easiest way to see this is to think of the Dyck paths where $(2i, 0)$ is the first place the path hits the x-axis.

For every $n \geq 2, s \geq 0$ we have

$$\sum_{i=2}^{n} C_{i+s}(i) = C_{n+s}(n-1). \tag{3}$$

This recursion is easily proved using Lemma 4.1. $\sum_{i=2}^{n} C_{i+s}(i) = \sum_{i=2}^{n} \binom{2s+i-1}{s} - \sum_{i=2}^{n} \binom{2s+i-1}{s-1} = \binom{2s+n}{s+1} - \binom{2s+1}{s+1} - \left[\binom{2s+n}{s} - \binom{2s+1}{s}\right] = \binom{2s+n}{s+1} - \binom{2s+n}{s}$.

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