The Number of Subgroups in the Fundamental Groups of Orientable Circle Bundles over Surfaces

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Abstract

The aim of this work is to count subgroups of a given index in the fundamental group of an orientable S^1-bundle over a compact surface. The number of subgroups of index n turns out to be independent of the orientability of the base surface, closed or bordered, and is expressed as a linear combination of the numbers of surface subgroups.

Résumé

Le but de cet article est de calculer le nombre de sous-groupes d'indice donné dans le groupe fondamental d'un fibré en circles sur une surface compacte. Nous prouvons que le nombre de sous-groupes d'indice n est indépendant de l'orientabilité de la base lorsque celle-ci est une surface fermée ou à bord. Le nombre des sous-groupes est exprimé comme une combinaison linéaire des nombres de sous-groupes du groupe fondamental de la base.

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1 Background

1.1. Recall some known facts from algebraic topology. (cf., e.g., [Ma91, Ch.IV]). There are three types of connected compact surfaces. Let $\mathcal{S}_g$ denote a closed orientable surface of genus $g \geq 0$, $\mathcal{K}_p$ a closed non-orientable surface of genus $p \geq 1$ and $\mathcal{D}_r$ a *bordered* surface of characteristic $\chi = 1 - r$, $r \geq 0$. In the first two cases, the Euler characteristic is, resp., $\chi = 2 - 2g$ and $\chi = 2 - p$. $\mathcal{D}_r$ presents, in fact, a family of not homeomorphic surfaces, orientable or not. The simplest one is a 2-disc with r holes.

Let $\mathcal{M}, \mathcal{M}'$ and $\mathcal{N}$ be connected manifolds. Two (unbranched) coverings $\rho : \mathcal{M} \to \mathcal{N}$ and $\rho' : \mathcal{M}' \to \mathcal{N}$ are said to be *equivalent* if there exists a homeomorphism $\eta : \mathcal{M} \to \mathcal{M}'$ such that $\rho = \rho' \circ \eta$.

The equivalence classes of n-sheeted covering of $\mathcal{N}$ are in one-to-one correspondence with the conjugacy classes of subgroups of index n in the fundamental group $\pi_1(\mathcal{N})$. This makes natural the investigation of subgroups of fundamental groups and, in particular, the enumeration of them.

Denote $\Gamma_g = \pi_1(\mathcal{S}_g)$ and $\Phi_p = \pi_1(\mathcal{K}_p)$. Furthermore, $\pi_1(\mathcal{D}_r) = F_r$, the free group of rank r.

The number of subgroups of a given index in the free group F_r was determined by M. Hall in 1949 (see formula (2.14) below). An explicit formula for the number of conjugacy classes of subgroups of a given index in F_r was obtained by one of the authors [Li71] (see also [Li98]). Both problems for the group Γ_g were completely solved by the second author in [Me79] and [Me83], respectively. For Φ_p they were solved in [MP86].

In the present work we extend the developed technique to some 3-manifolds.

Our aim is to find the number of subgroups of a finite index in the fundamental group of the orientable S^1-bundles over a compact surface, or, in other words, of the three-dimensional Seifert fibre spaces *without* exceptional fibres [Sc83]. We restrict ourselves to the *orientable* Seifert spaces. There are four types of such spaces [FM97, Th.10.1]: in the ordinary notation introduced by H. Seifert, these are the types (O,o) and (O,n) over a bordered or closed base surface.

The structure of an orientable S^1-bundle over a surface with non-empty boundary is completely determined by the base surface. In the closed cases, such a bundle is determined, up to fibre preserving homeomorphism, by its base surface and one more parameter e called its *Euler number* ([FM97, Th.10.3], cf. also [Sc83, §3]). In our cases, e may be an arbitrary integer.

1.2. As is typical for enumerative and other combinatorial questions about subgroups, we rely upon the well-known interconnection between subgroups of a group and its transitive permutational representations (cf., e.g., [Lu95]). However, one feature of the present approach looks, possibly, somewhat unusual: we *do not make use* of the ordinary recurrence relation between the number of transitive representations and that of arbitrary ones (or, equivalently, the logarithmic formula in terms of the corresponding generating functions). Instead, the transitivity constraint combined with commutativity conditions make it possible to manipulate within centralizers of regular permutations. Owing to this, we find the number of subgoups by applying results and adapting our approach developed in the above-mentioned papers; a slight generalization ("anti-commuting") is required, moreover, for non-orientable base surfaces. The key idea in the closed cases is to present all additional restrictions in the form of systems of linear congruences modulo ℓ, the order of the regular permutation. For the problems under consideration, these systems, non-homogeneous in general, prove to be uniform, i.e. they possess an *equal* number of solutions.

The number of subgroups of index n for the four types of S^1-bundles is expressed uniformly as a linear combination of the numbers of surface subgroups of index $m = n/\ell$, $\ell | n$. The corresponding coefficients prove to meet the following *pattern*:

$$\ell^{-\chi m+1} \cdot \begin{cases} 1 & \text{for orientable bundles over bordered surfaces} \\ \ell \text{ or } 0 & \text{for orientable bundles over closed surfaces.} \end{cases} \tag{1.1}$$

Non-vanishing the last factor in the second case depends on n, ℓ and the Euler number e of the bundle. In both cases, the results prove to be *independent* of the orientability of the base surface.

2 Definitions and preliminary results

Below, $M_G(n)$ denotes the number of subgroups of index n in a group G.

A. Permutations

$\mathbf{S}_n := \mathbf{S}(V)$ denotes the symmetric group of permutations acting on a set $V := V_n$, $|V_n| = n$ (usually $V_n = \{1, 2, \ldots, n\}$). We distinguish one element $v_0 \in V$ (e.g., $v_0 := 1$) and call it the *root*. $\mathbb{1} = \mathbb{1}_n$ stands for the identity permutation. We denote by v^h the result of applying a permutation h to an element v.

For enumerative aims, it is convenient to express some known, or easily provable, properties of permutational representations of finitely presented groups in terms of permutation tuples. In fact, both languages will be used interchangeably.

2.1. Definitions. 1. An r-tuple of permutations $\boldsymbol{a} = (a_1, a_2, \ldots, a_r)$ where $a_i \in \mathbf{S}_n$, $i = 1, 2, \ldots, r$, is called a *transitive permutation r-tuple* of degree n (or simply a transitive tuple) if the permutation group $A = \langle a_1, a_2, \ldots, a_r \rangle$ generated by them is transitive.

2. A permutation is called *regular* if it consists of independent cycles of an equal length.

3. If two r-tuples of permutations $\boldsymbol{a} = (a_1, a_2, \ldots, a_r)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_r)$ are conjugate element-wise by a common permutation $h \in \mathbf{S}_n$, that is,

$$h a_i h^{-1} = b_i, \quad i = 1, 2, \ldots, r, \tag{2.1}$$

then they are called *similar* (by h). We write $\boldsymbol{b} = h\boldsymbol{a}h^{-1}$. If, moreover, one can find

$h \in \mathbf{S}_n$ that meets equality (2.1) and leaves the root v_0 fixed, then the tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ are called *root-similar.*

4. If $\boldsymbol{b} = \boldsymbol{a}$ in (2.1), i.e. a_i commute with h:

$$ha_ih^{-1} = a_i, \quad i = 1, 2, \ldots, r, \tag{2.2}$$

then the tuple $\boldsymbol{a}$ is said to commute with h. If, instead,

$$ha_ih = a_i, \quad i = 1, 2, \ldots, r, \tag{2.2$^-$}$$

or, equivalently, $a_iha_i^{-1} = h^{-1}$, then a_i and $\boldsymbol{a}$ as a whole are said to *anti-commute* with h. More generally, "semi-similarity" between $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined with respect to h by the rule

$$ha_ih^{-\varepsilon_i} = b_i, \quad i = 1, 2, \ldots, r, \tag{2.1$^\pm$}$$

given a vector $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ where all $\varepsilon_i = \pm 1$. The tuple $\boldsymbol{a}$ is said to *semi-commute* with h if

$$ha_ih^{-\varepsilon_i} = a_i, \quad i = 1, 2, \ldots, r. \tag{2.2$^\pm$}$$

The next simple statement is crucial in what follows.

2.2. Lemma. *A permutation h with which a transitive permutation tuple semi-commutes is regular.*

Proof. Let a tuple $\boldsymbol{a}$ semi-commute with h. By (2.2$^-$), if a permutation anti-commutes with h, it anti-commutes with all powers of h. A permutation is regular if and only if all its non-identity powers are point-free. Therefore it suffices to prove that $h = \mathbb{1}$ whenever $v^h = v$ for some $v \in V$.

Suppose that permutations a and b anti-commute with h. Since a anti-commutes with h^{-1} as well, we have $ah^{-1}a^{-1} = h$ and $bhb^{-1} = h^{-1}$. Substituting h^{-1} from the second equality into the first one, we obtain $abhb^{-1}a^{-1} = h$, i.e. ab commutes with h. Likewise, if c commutes with h, then ac and ca anti-commute with h. Therefore the group $A = \langle a_1, a_2, \ldots, a_r \rangle$ consists of permutations which commute or anti-commute with h. By transitivity, for an arbitrary $w \in V$, there exists $a \in A$ such that $w^a = v$. Then $aha^{-1} = h^\varepsilon$ where $\varepsilon = \pm 1$. Hence, $w^{h^\varepsilon} = w^{aha^{-1}} = v^{ha^{-1}} = v^{a^{-1}} = w$. That is, $w^h = w$. $\square$

2.3. Lemma. *For any group G, $M_G(n) = |\mathfrak{T}_G(n)|/(n-1)!$ where $\mathfrak{T}_G(n)$ denotes the set of transitive permutational representations of G of degree n.*

The well-known interconnection between subgroups of a group and its transitive permutational representations [Ha59, Ch.5] can be formulated for a finitely generated group as follows (by abuse of notation we denote both the generators and their images in $\mathbf{S}_n$ by the same variables a_i):

2.4. Proposition. *Given a finitely generated group*

$$G = \langle a_1, a_2, \ldots, a_r : \ R_1 = 1, R_2 = 1, \ldots \rangle,$$

there is a one-to-one correspondence between the subgroups of index $n \geq 1$ in G and the root-similarity classes of transitive permutation r-tuples $(a_1, a_2, \ldots, a_r)$ of degree n that satisfy the defining relations $\{R_j = \mathbb{1}\}$, $j = 1, 2 \ldots$

Specifically, a transitive permutational representation $\varphi : \ G \to \mathbf{S}_n$ corresponds to the subgroup $H_\varphi \subseteq G$ which is the preimage of the root stabilizer, i.e. H_φ consists of all words whose images fix the root.

2.5. Centralizer of a regular permutation. Let h be a regular permutation of degree n and order ℓ; for definiteness, put

$$h := h_\ell = (1, 2, \ldots, \ell)(\ell + 1, \ldots, 2\ell) \ldots ((m-1)\ell + 1, \ldots, n) \tag{2.3}$$

where $m = n/\ell$. Then the centralizer $C(h)$ of h is the *wreath product* [Ha59, 5.9]

$$C(h) \cong \mathbb{Z}_\ell \wr \mathbf{S}_m$$

where $\mathbb{Z}_\ell$ is the cyclic group meant as the *additive* group of residues modulo ℓ.

2.6. Lemma. 1. *Any permutation a that commutes with a regular permutation h of order ℓ and degree $n = \ell m$ can be uniquely written in the form*

$$a = (c_1, c_2, \ldots, c_m; \widehat{a}) \tag{2.4}$$

where $c_i = c_i(a) \in \mathbb{Z}_\ell$, $i = 1, \ldots, m$, and $\widehat{a} \in \mathbf{S}_m$. And conversely, any such permutation a commutes with h.

2. *If $b = (d_1, d_2, \ldots, d_m; \widehat{b})$ is another element of $\mathbb{Z}_\ell \wr \mathbf{S}_m$, then*

$$ab = (c_1 + d_{1^{\hat{a}}}, c_2 + d_{2^{\hat{a}}}, \ldots, c_m + d_{m^{\hat{a}}}; \widehat{ab}). \tag{2.5}$$

Furthermore,

$$a^{-1} = (-c_{1^{\hat{a}^{-1}}}, -c_{2^{\hat{a}^{-1}}}, \ldots, -c_{m^{\hat{a}^{-1}}}; \widehat{a}^{-1}). \tag{2.6}$$

In particular, $\widehat{a^{-1}} = \widehat{a}^{-1}$.

3. *If $a \in C(h)$ is a regular permutation, then $\widehat{a}$ is also regular. In particular, $h = (1, \ldots, 1; \mathbb{1}_m)$ (so that, $\widehat{h} = \mathbb{1}_m$) and $\mathbb{1}_n = (0, \ldots, 0; \mathbb{1}_m)$.*

4. *If permutations $a_1, a_2, \ldots, a_r$ commute with h and form a transitive tuple, then the tuple $(\widehat{a_1}, \widehat{a_2}, \ldots, \widehat{a_r})$ is also transitive.*

From formulae (2.5) and (2.6) we obtain immediately (since $(j^{\hat{a}\hat{b}})^{\hat{a}^{-1}} = j^{\hat{a}\hat{b}\hat{a}^{-1}}$)

$$aba^{-1} = (c_1 + d_{1^{\hat{a}}} - c_{1^{\hat{a}\hat{b}\hat{a}^{-1}}}, c_2 + d_{2^{\hat{a}}} - c_{2^{\hat{a}\hat{b}\hat{a}^{-1}}}, \ldots, c_m + d_{m^{\hat{a}}} - c_{m^{\hat{a}\hat{b}\hat{a}^{-1}}}; \widehat{aba^{-1}}) \tag{2.7}$$

and for the commutator $[a, b] = aba^{-1}b^{-1}$,

$$[a, b] = (c_1 + d_{1^{\hat{a}}} - c_{1^{\hat{a}\hat{b}\hat{a}^{-1}}} - d_{1^{[\hat{a},\hat{b}]}}, \ldots, c_m + d_{m^{\hat{a}}} - c_{m^{\hat{a}\hat{b}\hat{a}^{-1}}} - d_{m^{[\hat{a},\hat{b}]}}; [\widehat{a}, \widehat{b}]). \tag{2.8}$$

Important for the sequel is the following.

2.7. Lemma [Me83]. *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_r)$ *be a transitive permutation tuple of degree* m. *Then the linear system of* m *equations in* mr *unknowns*

$$x_j^1 - x_{ja_1}^1 + x_j^2 - x_{ja_2}^2 + \ldots + x_j^r - x_{ja_r}^r = 0, \quad j = 1, \ldots, m,$$

has rank $m-1$.

2.8. Anti-commuting. Let $C^-(h)$ denote the set of permutations that anti-commute with a regular permutation h. As is clear from the proof of Lemma 2.2, $C(h) \cup C^-(h)$ is the group of permutations that commute or anti-commute with h. Therefore $C^-(h) = C(h)q$ where q is an arbitrary permutation anti-commuting with h. It is possible to select, as such, an involution preserving all cycles of h. Namely, if

$$h = (v_{1,1}, v_{1,2}, \ldots, v_{1,\ell}) \ldots (v_{m,1}, v_{m,2}, \ldots, v_{m,\ell}),$$

then we may put $q := q_h \in C^-(h)$ where q_h is defined by

$$v_{i,j}{}^{q_h} = v_{i,\ell-j+1} \quad \text{for all } i, j. \tag{2.9}$$

2.9. Lemma. 1. *Any permutation* a *that anti-commutes with a regular permutation* h *of order* ℓ *and degree* $n = \ell m$ *can be uniquely written in the form*

$$a = (c_1, c_2, \ldots, c_m; \widehat{a})q_h \tag{2.10}$$

where $c_i = c_i(a) \in \mathbb{Z}_\ell$, $i = 1, \ldots, m$, *and* $\widehat{a} \in \mathbf{S}_m$. *And conversely, any such permutation* a *anti-commutes with* h.

2. *Also,*

$$(c_1, c_2, \ldots, c_m; \widehat{a})q_h = q_h(-c_1, -c_2, \ldots, -c_m; \widehat{a}). \tag{2.11}$$

B. Fundamental groups

2.10. Closed surfaces. The fundamental groups $\Gamma_g = \pi_1(\mathcal{S}_g)$ and $\Phi_p = \pi_1(\mathcal{K}_p)$ (see **1.1**) possess the following well-known presentations:

$$\Gamma_g = \langle a_i, b_i : \prod_{i=1}^{g} [a_i, b_i] = 1, \ i = 1, \ldots, g \rangle \tag{2.12}$$

(this is trivial for $g = 0$) and

$$\Phi_p = \langle a_i : \prod_{i=1}^{p} a_i^2 = 1, \ i = 1, \ldots, p \rangle, \quad p \geq 1. \tag{2.13}$$

2.11. Subgroups of free groups. According to Proposition 2.4, the number $M_{F_r}(n)$ of subgroups of index n in the free group F_r is equal to the number of root-similarity classes of all transitive r-tuples. Therefore by the well-known general method of counting connected combinatorial objects, one can obtain the following famous recurrence formula of M. Hall [Ha59, Th.7.2.9] (see also [St98, Ex.5.13(a)]):

$$M_{F_r}(n) = n(n!)^{r-1} - \sum_{t=1}^{n-1}((n-t)!)^{r-1} M_{F_r}(t), \qquad M_{F_r}(1) = 1. \tag{2.14}$$

2.12. Subgroups of surface groups. According to [Me83] (cf. also [Me88]),

$$M_{\Gamma_g}(n) = R_\nu(n) \tag{2.15}$$

where $\nu = 2g-2$,

$$R_\nu(n) = n \sum_{t=1}^{n} \frac{(-1)^{t+1}}{t} \sum_{\substack{i_1+i_2+\cdots+i_t=n \\ i_1,i_2,\ldots,i_t \geq 1}} \beta_{i_1}\beta_{i_2}\ldots\beta_{i_t}, \tag{2.16}$$

$\beta_k = \sum_{\lambda \in D_k} (k!/f^{(\lambda)})^\nu$, D_k is the set of all irreducible representations of the symmetric group $\mathbf{S}_k$ and $f^{(\lambda)}$ is the degree of the representation λ.

Also [MP86], for $\nu = p-2$,

$$M_{\Phi_p}(n) = R_\nu(n). \tag{2.17}$$

2.13. Circle bundles. Let $\mathcal{B}$ be an orientable Seifert 3-manifold over a compact surface $\mathcal{F}$. Its fundamental group can be presented as follows [FM97, Prop.10.4].

- If $\mathcal{F}$ is an orientable surface of genus g with $k \geq 1$ boundary components, then

$$\pi_1(\mathcal{B}) = \langle a_i, b_i, d_j, h : \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{k} d_j = 1,\ a_i h a_i^{-1} = h,\ b_i h b_i^{-1} = h,\ d_j h d_j^{-1} = h \rangle. \tag{2.18}$$

Here the characteristic $\chi = 2 - 2g - k$.

- If $\mathcal{F}$ is a non-orientable surface of genus p with $k \geq 1$ boundary components, then

$$\pi_1(\mathcal{B}) = \langle a_i, d_j, h : \prod_{i=1}^{p} a_i^2 \prod_{j=1}^{k} d_j = 1,\ a_i h a_i^{-1} = h^{-1},\ d_j h d_j^{-1} = h^{-1} \rangle. \tag{2.19}$$

Here the characteristic $\chi = 2 - p - k$.

- If $\mathcal{F}$ is the orientable closed surface $\mathcal{S}_g$ of genus $g \geq 0$ and $e \in \mathbb{Z}$ is the Euler number of the bundle $\mathcal{B}$, then

$$\pi_1(\mathcal{B}) = \langle a_i, b_i, h : \prod_{i=1}^{g}[a_i, b_i] = h^e,\ a_i h a_i^{-1} = h,\ b_i h b_i^{-1} = h \rangle. \tag{2.20}$$

- If $\mathcal{F}$ is the non-orientable closed surface $\mathcal{K}_p$ of genus $p \geq 1$ and $e \in \mathbb{Z}$ is the Euler number of the bundle $\mathcal{B}$, then

$$\pi_1(\mathcal{B}) = \langle a_i, h : \prod_{i=1}^{p} a_i^2 = h^e,\ a_i h a_i^{-1} = h^{-1} \rangle. \tag{2.21}$$

In these presentations, the index i ranges over the interval $[1, g]$ in the orientable cases or $[1, p]$ in the non-orientable cases, and j ranges over $[1, k]$ in the bordered cases.

3 Bundles over bordered surfaces

3.1. Theorem. *Let $\Delta = \Delta_r$ denote the fundamental group of an orientable S^1-bundle $\mathcal{B}$ over a bordered surface $\mathcal{D}_r$ of Euler characteristic $\chi = 1 - r$. Then the number $M_\Delta(n)$ of subgroups of index n in Δ is given by the formula*

$$M_\Delta(n) = \sum_{m|n,\, \ell m = n} M_{F_r}(m)\ell^{(r-1)m+1}. \tag{3.1}$$

Proof (sketch). Suppose that the base surface Δ_r of the bundle $\mathcal{B}$ is *orientable* and its boundary consists of $k \geq 1$ components. Now, Δ_r is the group presented by (2.18) where the genus g meets the equality $r = 2g + k - 1$. After renaming the generators appropriately we obtain the following presentation:

$$\Delta_r = \langle a_i, h : a_i h a_i^{-1} = h;\ i = 1, \dots, r\rangle \cong F_r \times \mathbb{Z}. \tag{3.2}$$

Let $\mathcal{T}^+_{r+1}(n)$ denote the set of all transitive $(r+1)$-tuples $(a_1, a_2, \dots, a_r, h)$ of degree n such that all a_i commute with h. Then by Lemma 2.3, $M_\Delta(n) = |\mathcal{T}^+_{r+1}(n)|/(n-1)!$.

Consider a tuple $(a_1, a_2, \dots, a_r, h) \in \mathcal{T}^+_{r+1}(n)$. Since h commutes with itself as well, by Lemma 2.2 it is a regular permutation. Let ℓ denote its order and $m = n/\ell$. By Lemma 2.6 we may write $a_i = (c_1^i, c_2^i, \dots, c_m^i; \widehat{a_i})$, $\quad i = 1, 2, \dots, r$. Here c_j^i may be *arbitrary* elements of $\mathbb{Z}_\ell$ while $(\widehat{a_1}, \widehat{a_2} \dots, \widehat{a_r})$ is an arbitrary transitive tuple of degree m. Due to this fact, we come easily to (3.1).

The arguments for *non-orientable* base surfaces are similar. The only difference is that the appropriate tuples $(a_1^-, \dots, a_r^-)$ *anti-commute* with h (in fact, such tuples bijectively correspond to the tuples $(a_1, \dots, a_r)$ of the orientable case). □

4 Bundles over closed surfaces

4.1. Theorem. *Let $\Theta = \Theta_\nu^e$ denote the fundamental group of an orientable S^1-bundle $\mathcal{B}$ with Euler number e over a closed surface $\mathcal{F}$ of Euler characteristic $\chi = -\nu$. Then the number $M_\Theta(n)$ of subgroups of index n in Θ is given by the formula*

$$M_\Theta(n) = \sum_{\substack{m|n,\, \ell m = n \\ \ell^2 | n\cdot(e,n)}} R_\nu(m)\ell^{\nu m + 2} \tag{4.1}$$

where $R_\nu(n)$ is the number of subgroups of index n in the fundamental group $\pi_1(\mathcal{F})$ determined by formula (2.16) and (e, n) denotes the g.c.d. of the numbers e and n.

Proof. (A) Let the base surface be *orientable* of genus g: $\mathcal{F} = \mathcal{S}_g$ where $\nu = 2g - 2$. Now the group Θ is determined by presentation (2.20).

Expression (4.1) is trivial for $g = 0$, so that suppose $g > 0$. Let $\mathcal{T}^{(+,e)}_{2g+1}(n)$ denote the set of all transitive $(2g+1)$-tuples $(a_1, b_1, \dots, a_g, b_g, h)$ of degree n such that all a_i, b_i commute with h and the equality

$$[a_1, b_1] \dots [a_g, b_g] = h^e \tag{4.2}$$

is valid. By Lemma 2.3,

$$M_{\Theta_v^e}(n) = |\mathcal{T}_{2g+1}^{(+,e)}(n)|/(n-1)!. \tag{4.3}$$

Consider a permutation tuple $(a_1, b_1, \ldots, a_g, b_g, h) \in \mathcal{T}_{2g+1}^{(+,e)}(n)$. Since h commutes with itself as well, by Lemma 2.2 it is a regular permutation. Let ℓ be its order and $m = n/\ell$. By Lemma 2.6 we write

$$a_i = (c_1^i, c_2^i, \ldots, c_m^i; \widehat{a_i}), \quad b_i = (d_1^i, d_2^i, \ldots, d_m^i; \widehat{b_i}), \quad i = 1, 2, \ldots, g,$$

and

$$h^e = (\overline{e}, \ldots, \overline{e}; \mathbb{1}_m)$$

where $c_j^i, d_j^i \in \mathbb{Z}_\ell$, $\widehat{a_i}, \widehat{b_i} \in \mathbf{S}_m$ and $\overline{e}$ is the residue of e modulo ℓ.

For reducing long expressions we will write (c_j) instead of $(c_1, c_2, \ldots, c_m)$ and so on with j standing for the 'generic' subscript.

Applying multiplication formulae (2.5) – (2.8) repeatedly we obtain

$$\begin{aligned}[a_1, b_1] \ldots [a_g, b_g] \;=\; & (c_{j\alpha_1}^1 + d_{j\beta_1}^1 - c_{j\gamma_1}^1 - d_{j\delta_1}^1 + \cdots + c_{j\alpha_g}^g + d_{j\beta_g}^g - c_{j\gamma_g}^g - d_{j\delta_g}^g; \\ & [\widehat{a_1}, \widehat{b_1}] \ldots [\widehat{a_g}, \widehat{b_g}])\end{aligned} \tag{4.4}$$

where $\alpha_i = \delta_{i-1}$, $\beta_i = \delta_{i-1}\widehat{a_i}$, $\gamma_i = \delta_{i-1}\widehat{a_i}\widehat{b_i}\widehat{a_i}^{-1}$ and $\delta_i = \delta_{i-1}[\widehat{a_i}, \widehat{b_i}]$ for $i = 1, 2, \ldots, g$, with $\delta_0 = \mathbb{1}_m$.

Substituting permutations a_i and b_i into equation (4.2) we obtain by formula (4.4), the following system of m linear equations in the group $\mathbb{Z}_\ell$:

$$c_{j\alpha_1}^1 + d_{j\beta_1}^1 - c_{j\gamma_1}^1 - d_{j\delta_1}^1 + \cdots + c_{j\alpha_g}^g + d_{j\beta_g}^g - c_{j\gamma_g}^g - d_{j\delta_g}^g \equiv e \pmod{\ell}, \tag{4.5}$$

$j = 1, \ldots, m$, and one equation in the group $\mathbf{S}_m$

$$[\widehat{a_1}, \widehat{b_1}] \ldots [\widehat{a_g}, \widehat{b_g}] = \mathbb{1}_m. \tag{4.6}$$

Note that, given a regular permutation h, the tuple $(a_1, b_1, \ldots, a_g, b_g, h)$ belongs to the set $\mathcal{T}_{2g+1}^{(+,e)}(n)$ if and only if the elements $a_i = (c_1^i, \ldots, c_m^i; \widehat{a_i})$ and $b_i = (d_1^i, \ldots, d_m^i; \widehat{b_i})$ satisfy jointly the system of equations (4.5) and (4.6), and moreover, the $2g$-tuple $(\widehat{a_1}, \widehat{b_1}, \ldots, \widehat{a_g}, \widehat{b_g})$ is transitive. Indeed, the presence of $h = (1, \ldots, 1; \mathbb{1}_m)$ ensures the transitivity (i.e. belonging to the same orbit) inside each ℓ-block and does not affect interconnections between the ℓ-blocks.

Now, the number of transitive $2g$-tuples $(\widehat{a_1}, \widehat{b_1} \ldots, \widehat{a_g}, \widehat{b_g})$ satisfying commutator equation (4.6) is equal to $(m-1)! M_{\Gamma_g}(m)$.

Equations (4.5) form almost the same system that was used in [Me83] for the count of n-coverings of $\mathcal{S}_g$, except that now it is in general non-homogeneous. From Lemma 2.7 it follows easily (see [Me83]) that for any transitive tuple $(\widehat{a_1}, \widehat{b_1} \ldots, \widehat{a_g}, \widehat{b_g})$, the rank of (4.5) is equal to

$$m - 1. \tag{4.7}$$

This means that just one nontrivial linear combination of the left-hand parts, namely, their sum is identically equal to zero. This ensures the following necessary and sufficient condition that the system of equations (4.5) be solvable:

$$me \equiv 0 \pmod{\ell}. \tag{4.8}$$

It is clear that, given a transitive tuple $(\widehat{a_1}, \widehat{b_1}, \ldots, \widehat{a_g}, \widehat{b_g})$, if e, ℓ and m meet congruence (4.8), the number of solutions of the system of equations (4.5) modulo ℓ is equal to $\ell^{2gm-(m-1)}$. Thus, h contributes $(m-1)!M_{\Gamma_g}(m)\ell^{2gm-(m-1)}$ tuples into $\mathcal{T}^{(+,e)}_{2g+1}(n)$ if (4.8) holds and it contributes nothing otherwise. Summing over all $n!/(\ell^m m!)$ regular permutations and all admissible m we obtain

$$|\mathcal{T}^{(+,e)}_{2g+1}(n)| = \sum_{\substack{m|n,\ \ell m=n \\ me\equiv 0 (\mathrm{mod}\ \ell)}} (m-1)!M_{\Gamma_g}(m)\ell^{2gm-(m-1)}\frac{n!}{\ell^m m!}. \tag{4.9}$$

The conditions $\ell \,|\, n$ & $\ell \,|\, me$ where $m = n/\ell$ are equivalent to $\ell^2 \,|\, n^2$ & $\ell^2 \,|\, ne$, which in turn are equivalent to $\ell^2 \,|\, n \cdot (e, n)$. Hence in view of (4.3) and (2.15), formula (4.9) turns into (4.1).

(B) Now let the base surface of the bundle $\mathcal{B}$ be *non-orientable* of genus p: $\mathcal{F} = \mathcal{K}_p$ where $\nu = p - 2$. The group Θ is determined by presentation (2.21). In accordance with it, let $\mathcal{T}^{(-,e)}_{p+1}(n)$ be the set of all transitive $(p+1)$-tuples $(a_1, \ldots, a_p, h)$ of degree n such that all a_i anti-commute with h and the equality

$$a_1^2 a_2^2 \ldots a_p^2 = h^e \tag{4.10}$$

is valid. By Lemma 2.3,

$$M_{\Theta^e_\nu}(n) = |\mathcal{T}^{(-,e)}_{p+1}(n)|/(n-1)!. \tag{4.11}$$

Consider a permutation tuple $(a_1, \ldots, a_p, h) \in \mathcal{T}^{(-,e)}_{p+1}(n)$. Since all a_i anti-commute with h and h commutes with itself, it is a regular permutation (Lemma 2.2). Let ℓ be its order and $m = n/\ell$. By Lemma 2.9,

$$a_i = (c^i_1, c^i_2, \ldots, c^i_m; \widehat{a_i})q_h, \quad i = 1, 2, \ldots, p, \tag{4.12}$$

and

$$h^e = (\overline{e}, \ldots, \overline{e}; \mathbb{1}_m)$$

where $c^i_j \in \mathbb{Z}_\ell$, $\widehat{a_i} \in \mathbf{S}_m$ and $\overline{e}$ is the residue of e modulo ℓ.
By formulae (2.11) and (2.5), we have

$$\begin{aligned} a_i^2 &= (c^i_1, c^i_2, \ldots, c^i_m; \widehat{a_i})q_h \cdot q_h(-c^i_1, -c^i_2, \ldots, -c^i_m; \widehat{a_i}) \\ &= (c^i_1, c^i_2, \ldots, c^i_m; \widehat{a_i}) \cdot (-c^i_1, -c^i_2, \ldots, -c^i_m; \widehat{a_i}) \\ &= (c^i_1 - c^i_{1^{a_i}}, c^i_2 - c^i_{2^{a_i}}, \ldots, c^i_m - c^i_{m^{a_i}}; \widehat{a_i}^2), \end{aligned}$$

whence

$$a_1^2 \ldots a_p^2 = (c^1_j - c^1_{j^{\hat{a}_1}} + c^2_{j^{\hat{a}_1^2}} - c^2_{j^{\hat{a}_1^2\hat{a}_2}} + \cdots + c^p_{j^{\hat{a}_1^2\ldots\hat{a}_{p-1}^2}} - c^p_{j^{\hat{a}_1^2\ldots\hat{a}_{p-1}^2\hat{a}_p}}; \widehat{a_1}^2 \ldots \widehat{a_p}^2). \tag{4.13}$$

Substituting (4.12) into equation (4.10) we obtain by formula (4.13), the following system of m equations in the group $\mathbb{Z}_\ell$:

$$c^1_j - c^1_{j^{\hat{a}_1}} + \cdots + c^p_{j^{\hat{a}_1^2 \ldots \hat{a}_{p-1}^2}} - c^p_{j^{\hat{a}_1^2 \ldots \hat{a}_{p-1}^2 \hat{a}_p}} \equiv e \pmod{\ell}, \quad j = 1, \ldots, m, \tag{4.14}$$

and one equation in the group $\mathbf{S}_m$,

$$\widehat{a_1}^2 \ldots \widehat{a_p}^2 = \mathbb{1}_m. \tag{4.15}$$

Note again that, given a regular permutation h, the tuple $(a_1, \ldots, a_p, h)$ belongs to the set $\mathcal{T}_{p+1}^{(-,e)}(n)$ if and only if the elements $a_i = (c_1^i, \ldots, c_m^i; \widehat{a_i})$ satisfy jointly the system of equations (4.14) and (4.15), and moreover, the p-tuple $(\widehat{a_1}, \ldots, \widehat{a_p})$ is transitive. Just as in the above cases, the presence of $h = (1, \ldots, 1; \mathbb{1}_m)$ ensures the transitivity inside each ℓ-block and does not affect interconnections between the ℓ-blocks.

Now, the number of transitive p-tuples $(\widehat{a_1}, \ldots, \widehat{a_p})$, satisfying equation (4.15) is equal to $(m-1)! M_{\Phi_p}(m)$.

By Lemma 2.7, the rank of linear system (4.14) is equal to $m - 1$. This ensures the following necessary and sufficient condition that the system of equations (4.14) be solvable: $me \equiv 0 \pmod{\ell}$. But this is just condition (4.8) as above, and the remainder of the proof is the same as in the part (A) due to formula (2.17). □

4.2. Corollary. 1. *$M_{\Theta_\nu^e}(n)$, as a function of e, reaches the maximal value if and only if $e \equiv 0 \pmod{n}$, and it reaches the minimal value if $e = 1$.*

2. *For $\nu = 0$ and any e, $M_{\Theta_0^e}(n)$ is a multiplicative arithmetic function of n.*

5 Concluding remarks

1. Our method can also be applied to two types (out of the four), (N,o) and (N,n,I), of *non-orientable* Seifert fibre spaces without exceptional fibres. But formulae and their proofs are somewhat more combersome, although they are also independent of the orientability of the base surface and meet a pattern similar to (1.1). Furthermore, it is also possible to combine this approach with the technique developed by us previously and based on the enumerative Burnside lemma in order to count the *coverings* of the corresponding 3-manifolds.

2. We do not know whether there exists a direct, "geometrical" interpretation of the obtained simple reductive formulae and of their independence of the surface orientability. In this respect it is interesting to note that the function $M_{\Phi_p \times \mathbb{Z}}(n)$, which corresponds to the *non-orientable* fibre bundle $\mathcal{K}_p \times S^1$, *differs* from $M_{\Gamma_g \times \mathbb{Z}}(n) = M_{\Theta_\nu^0}(n)$ in spite of the equality $M_{\Phi_p}(n) = M_{\Gamma_g}(n)$, $p = 2g$ (see **2.12**). On the other hand, the coefficients (1.1) can be interpreted as the "lifting factors" for the corresponding surface subgroups.

3. According to [Li98], the assertion of Lemma 2.2 can be naturally reformulated as the manifestation of the claim that the component-wise action of permutations on transitive tuples by "semi-conjugacy" $(2.1^{\pm})$ is *orthogonal* to their ordinary action, which partially explains the simplicity of formulae.

4. $M_{\Theta_\nu^e}(n) \sim R_\nu(n) \sim 2n(n!)^\nu$ as $\nu \to \infty$. Evidently, this is also valid as $n \to \infty$ for any e and $\nu > 0$. However, at present we do not possess a proof.

5. Let $N_G(n)$ denote the number of *conjugacy classes* of subgroups of index n in a group G. As was pointed out by R. Stanley [St98, Ex.5.13(c)], expression (3.1) for the particular case (3.2) of the direct product $F_r \times \mathbb{Z}$ is, in fact, equivalent to the formula for $N_{F_r}(n)$ given in [Li71] (cf. [Li98]). This follows easily from the general reductive formula [St98, (5.125)]

$$M_{G\times\mathbb{Z}}(n) = \sum_{m|n} mN_G(m)$$

valid for an arbitrary finitely generated group G.

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