THE RING STRUCTURE ON THE COHOMOLOGY OF COORDINATE SUBSPACE ARRANGEMENTS

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ABSTRACT. Every simplicial complex $\Delta \subset 2^{[n]}$ on the vertex set $[n] = \{1, \ldots, n\}$ defines a real resp. complex arrangement of coordinate subspaces in $\mathbb{R}^n$ resp. $\mathbb{C}^n$ via the correspondence $\Delta \ni \sigma \mapsto \operatorname{span}\{e_i : i \in \sigma\}$. The linear structure of the cohomology of the complement of such an arrangement is explicitly given in terms of the combinatorics of Δ and its links by the Goresky–MacPherson formula. Here we derive, by combinatorial means, the ring structure on the integral cohomology in terms of data of Δ. We provide a non-trivial example of different cohomology rings in the real and complex case. Furthermore, we give an example of a coordinate arrangement that yields non-trivial multiplication of torsion elements.

1. INTRODUCTION & RESULTS

This article is concerned with coordinate subspace arrangements, a family of (linear) subspace arrangements in real and complex space associated with simplicial complexes. For a detailed survey of subspace arrangements we refer to [Bj]; all we need here is given in Section 2. Associated with any subspace arrangement are its link and its complement. The homology of the link, the cohomology of the complement, and in particular its ring structure, have motivated a lot of research [Ar], [BZ], [Br], [CP], [FZ], [GM], [OS], [OT], [Zi].
The Goresky–MacPherson formula for the homology of the link is the starting point of our investigation. By analyzing Alexander duality combinatorially in the case of coordinate subspace arrangements, we give a complete combinatorial description of the ring structure of the integral cohomology. In this analysis the duality of the cross polytope and the cube plays a crucial role.
This work was motivated by a result of S. Yuzvinsky [Yu] on the rational cohomology ring structure of complex arrangements. We can give a partial answer to Conjecture 6.6 on the integral cohomology ring structure of complex arrangements of his article. Our modeling of the cohomology of the complement was inspired by the article [BC] of E. Babson and C. Chan.

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We provide an example of a simplicial complex not containing faces of cardinality $n-1$, so that the complement of the associated real coordinate subspace arrangement is connected, that yields different ring structures for the cohomology of the complement of the associated real and complex arrangement. This answers a question by Gasharov, Peeva and Welker [GPW].
Finally, we give an example of a coordinate subspace arrangement that yields non trivial multiplication of torsion elements.

Results. Our main result – the description of the ring structure on the cohomology of the complement C_Δ of a coordinate subspace arrangement – is based on the Goresky–MacPherson formula for the link (cf. [GM]). After applying Alexander duality it is given in our situation by

$$\tilde{H}^i(C_\Delta;\mathbb{Z}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}_{n-i-|\sigma|-2}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}).$$

To describe the multiplication in $\tilde{H}^*(C_\Delta;\mathbb{Z})$ it suffices to describe how to multiply classes $[u]$ and $[v]$ that correspond to $[c] \in H_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})$ and $[c'] \in H_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z})$ under the Goresky–MacPherson isomorphism. Note that there is a double grading of cohomology classes by assigning the grade (r,σ) to $[u]$.
Our main result is the following.

Theorem 1.1. *Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let C_Δ denote the complement of the associated real coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ is given by the homomorphisms*

$$\tilde{H}_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})\otimes\tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z}) \longrightarrow \tilde{H}_{r+r'+2}(\mathrm{link}_\Delta\,\sigma\cap\sigma';\mathbb{Z})$$
$$[c]\otimes[c'] \longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \text{if } \sigma\cup\sigma' = [n],\\ 0 & \text{otherwise.}\end{cases}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon\in\{\pm1\}$ is a sign depending on n,σ,σ',r,r' computed in Section 3. If C_Δ is not connected there is additional non-trivial multiplication of cohomology classes in dimension zero.

This implies in particular that the multiplication respects the double grading of cohomology classes. The condition $\sigma\cup\sigma'=[n]$ is the "standard codimension condition" (cf., e.g., [Yu], [HRW, Proposition 6]). As a consequence we obtain the following Corollary, which answers Conjecture 6.6 in [Yu] in the case of coordinate subspace arrangements.

Corollary 1.2. *Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let $C_\Delta^{\mathbb{C}}$ denote the complement of the associated complex coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$ is given by the homomorphisms*

$$\tilde{H}_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})\otimes\tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z}) \longrightarrow \tilde{H}_{r+r'+2}(\mathrm{link}_\Delta\,\sigma\cap\sigma';\mathbb{Z})$$
$$[c]\otimes[c'] \longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \text{if } \sigma\cup\sigma' = [n],\\ 0 & \text{otherwise.}\end{cases}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon\in\{\pm1\}$ a sign depending on n,r,r' computed in Section 3.

The fact that the sign ε depends on σ and σ' in the real case, but not in the complex case, is the reason why in general there is no (dimension-shifting) isomorphism of graded rings between the cohomology rings of the real and complex arrangement associated with Δ (compare Corollary 2.2 and Section 4).

Example 1.3. There is a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices such that the following holds.

- ▷ The complement of the associated real arrangement is connected.
- ▷ The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ differs from $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$.

Example 1.4. There is a simplicial complex $\Delta \subset 2^{[10]}$ on ten vertices such that the cohomology ring of the complement of the associated real (or complex) arrangement yields non-trivial multiplication of torsion elements.

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2. Objects, Tools and Facts

In this section we recall basic facts on coordinate subspace arrangements, provide combinatorial models for their links and complements, and describe Lefschetz duality in the framework of cubical cohomology for the complement of a coordinate subspace arrangement.

2.1. **Coordinate Subspace Arrangements.** Simplicial complexes give rise to real and complex subspace arrangements. For that, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, resp. $\{e_1^{\mathbb{C}}, \ldots, e_n^{\mathbb{C}}\}$ the standard basis of $\mathbb{C}^n$. Let $\Delta \subset 2^{[n]}$ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. We define that always $\emptyset \in \Delta$ is a face. To avoid trivial cases we assume throughout the article that $\Delta \neq 2^{[n]}$ and $n \geq 2$. The *(real) coordinate subspace arrangement* in $\mathbb{R}^n$ associated with Δ is

$$\mathcal{A}_\Delta = \{\operatorname{span}_{\mathbb{R}}\{e_{i_0}, \ldots, e_{i_k}\} : \{i_0, \ldots, i_k\} \in \Delta\},$$

the *(complex) coordinate subspace arrangement* in $\mathbb{C}^n$ associated with Δ is

$$\mathcal{A}_\Delta^{\mathbb{C}} = \{\operatorname{span}_{\mathbb{C}}\{e_{i_0}^{\mathbb{C}}, \ldots, e_{i_k}^{\mathbb{C}}\} : \{i_0, \ldots, i_k\} \in \Delta\}.$$

For every subspace arrangement we have the notion of the link and the complement, which in our case we denote by L_Δ and C_Δ, resp. $L_\Delta^{\mathbb{C}}$ and $C_\Delta^{\mathbb{C}}$.

$$L_\Delta = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}_\Delta \qquad C_\Delta = \mathbb{R}^n \setminus \bigcup \mathcal{A}_\Delta$$
$$L_\Delta^{\mathbb{C}} = \mathbb{S}^{2n-1} \cap \bigcup \mathcal{A}_\Delta^{\mathbb{C}} \qquad C_\Delta^{\mathbb{C}} = \mathbb{C}^n \setminus \bigcup \mathcal{A}_\Delta^{\mathbb{C}}$$

2.2. **Models for the Real Case.** We introduce combinatorial models Λ_Δ and Γ_Δ for L_Δ and C_Δ. Consider the n-dimensional cross polytope $Q^n = \operatorname{conv}\{\pm e_i : i = 1, \dots, n\}$. Its proper faces form a simplicial complex, which we denote by ∂Q^n. Let Λ_Δ be the subcomplex of ∂Q^n of all simplices that are contained in $\bigcup \mathcal{A}_\Delta$.

$$\Lambda_\Delta = \{\{\varepsilon_0 e_{i_0}, \dots, \varepsilon_k e_{i_k}\} : \{i_0, \dots, i_k\} \in \Delta, (\varepsilon_0, \dots, \varepsilon_k) \in \{\pm 1\}^{k+1}\}$$

Let Γ_Δ be the "mirror complex" of $\mathcal{A}_\Delta$ (cf. [BBC]), i.e., the faces of the n-cube $C^n = [-1,1]^n$ disjoint to $\bigcup \mathcal{A}_\Delta$ considered as a polytopal subcomplex of the cube.

$$\Gamma_\Delta = \{c : c \text{ a proper face of } C^n,\ [n] \setminus \{\text{varying coordinates of } c\} \notin \Delta\}$$

The underlying spaces $|\Lambda_\Delta|$ and $|\Gamma_\Delta|$ are homeomorphic, resp. homotopy equivalent, to the link L_Δ and the complement C_Δ, see e.g. [Mu, p. 414].

2.3. **From Complex to Real Arrangements.** As far as the topology is concerned any complex coordinate arrangement can be modeled as a real subspace arrangement. Let $\Delta \subset 2^{[n]}$ be a simplicial complex on the vertex set $\{1, \dots, n\}$. Let $\pi : [2n] \longrightarrow [n]$ the map defined by $2i-1, 2i \mapsto i$ for $i \in [n]$. Define the *"complexification" of* Δ by

$$\Delta^{\mathbb{C}} = \{\sigma \subset [2n] : \pi(\sigma) \in \Delta\}.$$

For an example of a "complexification" and the following Lemma see Figure 1.

Lemma 2.1.

- ▷ *Under the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ the spaces $\bigcup \mathcal{A}_\Delta^{\mathbb{C}}$ and $\bigcup \mathcal{A}_{\Delta^{\mathbb{C}}}$ correspond to each other.*
- ▷ *For $\sigma \in \Delta^{\mathbb{C}}$ the following homotopy equivalence holds*

$$\operatorname{link}_{\Delta^{\mathbb{C}}} \sigma \simeq \begin{cases} * & \text{if } \pi^{-1}(\pi(\sigma)) \neq \sigma, \\ \operatorname{link}_\Delta \pi(\sigma) & \text{if } \pi^{-1}(\pi(\sigma)) = \sigma. \end{cases}$$

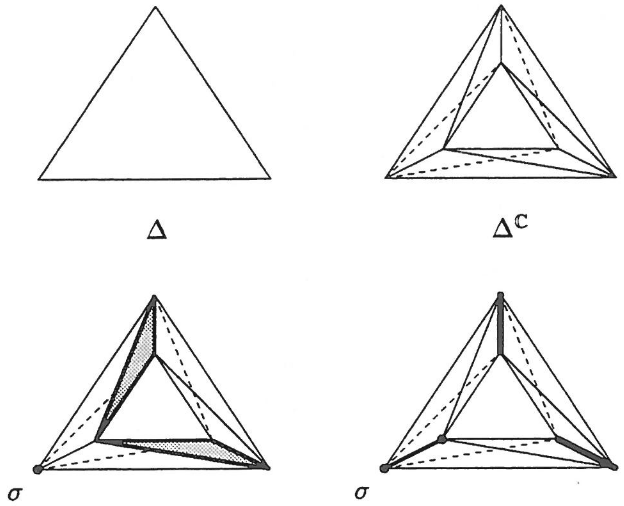
□

2.4. **The Goresky–MacPherson Theorem.** Let $\mathcal{A}$ be a (linear) subspace arrangement in $\mathbb{R}^n$ with link $L = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}$ and complement $C = \mathbb{R}^n \setminus \bigcup \mathcal{A}$. Denote by P the intersection poset of $\mathcal{A}$ ordered by reversed inclusion, and by $d{:}P \longrightarrow \mathbb{N}$ the dimension function. For $v \in P$ let $P_{<v}$ be the subposet of all elements in P that are smaller than v. For any finite poset Q denote by $\Delta(Q)$ the order complex of Q.

Theorem (Goresky-MacPherson [GM, Part III]). *The homology of the link $L_\mathcal{A}$, and the cohomology of the complement $C_\mathcal{A}$, of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$ can be computed from the data (P, d) and n:*

$$\tilde{H}_i(L_\mathcal{A}; \mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{i-d(v)}(\Delta(P_{<v}); \mathbb{Z}),$$

$$\tilde{H}^i(C_\mathcal{A}; \mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{n-i-d(v)-2}(\Delta(P_{<v}); \mathbb{Z}).$$



$\sigma \neq \pi^{-1}(\pi(\sigma))$ $\qquad$ $\sigma = \pi^{-1}(\pi(\sigma))$

$\operatorname{link}_{\Delta^{\mathbb{C}}} \sigma$ is a cone $\qquad$ $\operatorname{link}_{\Delta^{\mathbb{C}}} \sigma = (\operatorname{link}_{\Delta} \pi(\sigma))^{\mathbb{C}}$

FIGURE 1. Example for the "complexification" of a complex Δ

This theorem, originally proven by means of stratified Morse theory in [GM], was given an elementary proof by Ziegler and Živaljević in [ZŽ].

2.5. **The Goresky–MacPherson Theorem for coordinate subspace arrangements.** In the situation of a real coordinate subspace arrangement $\mathcal{A}_\Delta$ the order complexes $\Delta(P_{<v})$ can be described more explicitly. The poset P is given by the face poset of the simplicial complex Δ ordered by inverse inclusion. The poset $P_{<\sigma}$ then is isomorphic to the opposite face lattice of $\operatorname{link}_\Delta \sigma = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}$. Thus we obtain the following formulation of the Goresky–MacPherson theorem.

Theorem. *Let $\Delta \subset 2^{[n]}$ be a simplicial complex with vertex set $\{1, \ldots, n\}$. Then*

$$\tilde{H}_i(L_\Delta; \mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{i-|\sigma|}(\operatorname{link}_\Delta \sigma; \mathbb{Z}),$$
$$\tilde{H}^i(C_\Delta; \mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z}).$$

Here $|\sigma|$ denotes the cardinality of σ, i.e., $|\sigma| = \dim \sigma + 1$.

In view of section 2.2 this yields the following result for the associated complex coordinate subspace arrangement.

Corollary 2.2. *For simplicial complexes* $\Delta \subset 2^{[n]}$ *we have*

$$\tilde{H}_i(L_\Delta^{\mathbb{C}};\mathbb{Z}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}_{i-2|\sigma|}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})$$
$$\tilde{H}^i(C_\Delta^{\mathbb{C}};\mathbb{Z}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}_{2n-i-2|\sigma|-2}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}),$$

and hence there is a dimension-shifting group isomorphism between the (co)homologies of the real and complex coordinate subspace arrangements. Every homology class

$$[c] \in \tilde{H}_{n-i-|\sigma|-2}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}) = \tilde{H}_{2n-(n+|\sigma|+i)-2|\sigma|-2}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})$$

corresponds to

$$[u] \in \tilde{H}^i(C_\Delta;\mathbb{Z})$$

and to

$$[u^{\mathbb{C}}] \in \tilde{H}^{n+|\sigma|+i}(C_\Delta^{\mathbb{C}};\mathbb{Z}).$$

The correspondence $[u] \longmapsto [u^{\mathbb{C}}]$ *sets up the isomorphism.*

2.6. **A Homology Model and a Map into the Link.** We establish a simplicial version of the Ziegler–Živaljević [ZŽ] proof for the Goresky–MacPherson theorem. Let $\Delta \subset 2^{[n]}$ be a simplicial complex. We construct a simplicial complex $\mathfrak{L}_\Delta$ together with a simplicial map $\Phi : \mathfrak{L}_\Delta \longrightarrow \Lambda_\Delta$ to the link that induces an isomorphism in homology. Let $\mathfrak{L}_\Delta$ be the following one-point union of spaces.

$$\mathfrak{L}_\Delta = (\dot{\bigcup_{\sigma\in\Delta}} \partial Q^{|\sigma|} * \mathrm{link}_\Delta\,\sigma)/\sim \; = (\Delta \,\dot{\cup} \dot{\bigcup_{\sigma\in\Delta\setminus\{\emptyset\}}} \partial Q^{|\sigma|} * \mathrm{link}_\Delta\,\sigma)/\sim$$

The one-point union is given by the following identifications $\sim$. For each $\sigma = \{i_0 < \ldots < i_k\} \in \Delta$, $\sigma \neq \emptyset$, identify $e_1 \in \partial Q^{|\sigma|} * \mathrm{link}_\Delta\,\sigma$ with the vertex $i_0 \in \Delta = \partial Q^{|\emptyset|} * \mathrm{link}_\Delta\,\emptyset$. Compare Figure 2.

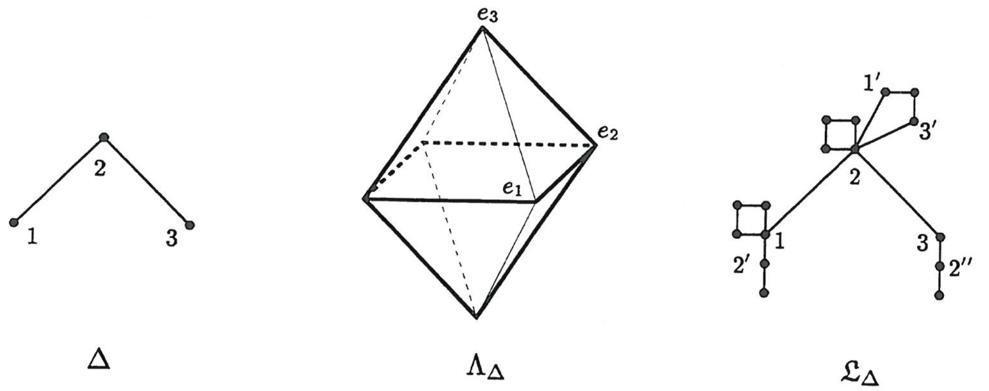


FIGURE 2. An easy example for the model space $\mathfrak{L}_\Delta$

We get the map Φ by defining it on the pieces $\partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma$. Let

$$\phi_\sigma : \partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma \longrightarrow \Lambda_\Delta$$

be defined by the simplicial homeomorphism

$$\partial Q^{|\sigma|} \longrightarrow \operatorname{span}_{\mathbb{R}}\{e_{i_0}, \dots, e_{i_k}\} \cap \partial Q^n,$$

$\sigma = \{i_0 < \dots < i_k\}$, such that $\phi_\sigma(e_{j+1}) = e_{i_j}$, in particular $\phi_\sigma(e_1) = e_{i_0}$. On $\operatorname{link}_\Delta \sigma$ the map ϕ_σ is defined by

$$\{j_0, \dots, j_l\} \longmapsto \{e_{j_0}, \dots, e_{j_l}\} \in \Lambda_\Delta$$

for $\{j_0, \dots, j_l\} \in \operatorname{link}_\Delta \sigma$. By construction all these maps fit together and yield a simplicial map Φ.

Proposition 2.3. *The map Φ induces an isomorphism in homology. (Indeed, it is a homotopy equivalence.)*

Sketch of proof. The proof works as in [ZŽ] by induction on the cardinality of Δ. In the induction step one removes a maximal simplex of Δ and uses the Mayer-Vietoris sequence along with the induction hypotheses (resp. the Glueing Lemma, to obtain the homotopy equivalence). □

2.7. **Cubical Cohomology.** The homotopy model Γ_Δ of the complement C_Δ is a subcomplex of the boundary of the cube. We compute its cohomology by using "cubical cohomology." We give a short overview of the most important notations and the formula for the cup product (see also [Ma]).

Let Γ be a subcomplex of the n-cube C^n, and let $T \in \Gamma$ be a t-dimensional cube. We use two descriptions of T:

Denote the projection to the i-th coordinate by π_i. On the one hand, we can identify T with a vector in $\{+,-,*\}^n$, where the i-th coordinate is $+$, $-$ or $*$ iff $\pi_i(T) = \{+1\}$, $\{-1\}$, resp. $[-1,+1]$. On the other hand, there are three sets $T_+, T_-, T_* \subseteq \{1, \dots, n\}$ that uniquely define the cube,

$$T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, T_*),$$

where $|T_*| = t$ and the following holds for the coordinate projections.

$$\begin{aligned} \pi_i(T) &= \{+1\} && \text{for } i \in T_+, \\ \pi_j(T) &= \{-1\} && \text{for } j \in T_-, \\ \pi_k(T) &= [-1,+1] && \text{for } k \in T_*. \end{aligned}$$

Let $C_t(\Gamma)$ be the free abelian group generated by the t-cubes in Γ. In order to get a boundary map we begin by defining face operators. Let $T \in \Gamma$ be a t-dimensional cube $T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, \{k_1 < \dots < k_t\})$. For $A = \{a_1, \dots, a_p\} \subseteq \{1, \dots, t\}$ and $\varepsilon = \pm 1$ define the $(t-p)$-cube

$$D_A^\varepsilon T = \begin{cases} (T_+ \cup \{k_{a_1}, \dots, k_{a_p}\}, T_-, \{k_1, \dots, k_t\} \setminus \{k_{a_1}, \dots, k_{a_p}\}) & \text{if } \varepsilon = +1, \\ (T_+, T_- \cup \{k_{a_1}, \dots, k_{a_p}\}, \{k_1, \dots, k_t\} \setminus \{k_{a_1}, \dots, k_{a_p}\}) & \text{if } \varepsilon = -1. \end{cases}$$

$D^{\varepsilon}_{A}T$ is the face of T obtained by fixing the varying coordinates $\{k_{a_1}, \dots, k_{a_p}\}$ to ε. A boundary operator is now defined by

$$\partial_t : C_t(\Gamma) \longrightarrow C_{t-1}(\Gamma),$$
$$T \longmapsto \sum_{a=1}^{t} (-1)^a (D^{+1}_{\{a\}}T - D^{-1}_{\{a\}}T).$$

The homology of the resulting *cubical chain complex* $(C_*(\Gamma), \partial_*)$ is canonically isomorphic to singular homology. The cup product formula in this situation is given on the chain level by the following. Let $u \in \mathrm{Hom}(C_p(\Gamma), \mathbb{Z})$ and $v \in \mathrm{Hom}(C_q(\Gamma), \mathbb{Z})$, then for a $(p+q)$-cube T we obtain

$$(u \cup v)(T) = \sum \rho_{H,K} \cdot u(D^{+1}_{H}T) v(D^{-1}_{K}T),$$

where the sum is taken over all q-subsets H of $\{1, \dots, p+q\}$, K is the complement of H, and $\rho_{H,K}$ is the sign of the permutation HK of $\{1, \dots, p+q\}$, i.e., the signature of the shuffle (H, K).

2.8. **Lefschetz Duality for the Cross Polytope.** As a crucial part of Alexander duality, we describe Lefschetz duality explicitly for simplicial homology of the cross polytope and cubical cohomology of the cube (cf. [Mu]).

Theorem (Lefschetz Duality). *Let (X, A) be a compact, orientable, triangulated relative homology n-manifold. Then there is an isomorphism*

$$H_k(X, A) \cong H^{n-k}(|X| \setminus |A|).$$

Outline of the proof. Let X^- be the simplicial complex consisting of all simplices of the barycentric subdivision $\mathrm{sd}\, X$ that are disjoint from $|A|$. Then

- ▷ $|X^-|$ is a deformation retract of $|X| \setminus |A|$.
- ▷ $|X^-|$ equals the union of all blocks $D(\sigma)$ dual to simplices $\sigma \in X$ that are not in A.

Now there is a chain isomorphism

$$C^k(X, A) \xrightarrow{\cong} D_{n-k}(X^-),$$

where $D_*(X^-)$ denotes the dual chain complex of X^-. Dualization yields

$$C_k(X, A) \cong \mathrm{Hom}(C^k(X, A), \mathbb{Z}) \xleftarrow{\cong} \mathrm{Hom}(D_{n-k}(X^-), \mathbb{Z}).$$

The inverse map $C_k(X, A) \longrightarrow \mathrm{Hom}(D_{n-k}(X^-), \mathbb{Z})$ is given by $\sigma \mapsto D(\sigma)^*$, where σ is a k-simplex of X not in A. This induces the desired isomorphism. □

Lefschetz duality is dealing with the complex X^-, whose underlying space is the union of the dual blocks $D(\sigma)$, $\sigma \in X \setminus A$. In case X is the boundary of the cross polytope Q^n, the dual blocks $|D(\sigma)|$, $\sigma \in X$, correspond to the faces of the boundary of the n-dimensional cube C^n. See Figure 3.

Let now $A = \Lambda_\Delta$ be the subcomplex of $X = \partial Q^n$ given by the arrangement

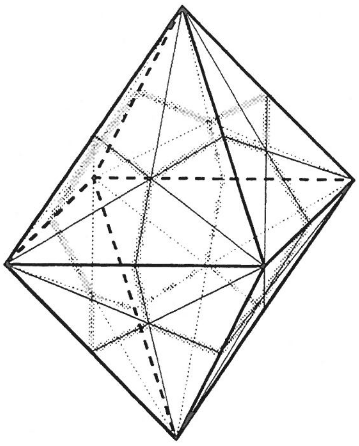


FIGURE 3. The 3-dimensional cross polytope with the 1-skeleton of the 3-dimensional cube in the barycentric subdivision

associated with a simplicial complex Δ (Section 2.2). Then there is a chain isomorphism from the dual block complex of $(\partial Q^n)^-$ to the cubical chain complex of Γ_Δ

$$D_j((\partial Q^n)^-) \longrightarrow C_j(\Gamma_\Delta),$$

which yields a chain isomorphism

$$\begin{aligned}\Psi : C_k(\partial Q^n, \Lambda_\Delta) \longrightarrow \operatorname{Hom}(D_{n-1-k}((\partial Q^n)^-), \mathbb{Z}) &\longrightarrow \operatorname{Hom}(C_{n-1-k}(\Gamma_\Delta), \mathbb{Z}) \\ \sigma \longmapsto \Psi(\sigma),&\end{aligned}$$

where

$$\Psi(\sigma) = (-1)^{i_0+\dots+i_k}(-1)^{|T_-(\sigma)|}(T_+(\sigma), T_-(\sigma), T_*(\sigma))^*,$$

for $\sigma = \langle \varepsilon_0 e_{i_0}, \dots, \varepsilon_k e_{i_k} \rangle \in X \setminus A$, $i_0 < \dots < i_k$, with

$$\begin{aligned} T_+(\sigma) &= \{i_j \in [n] : \varepsilon_j = +1\}, \\ T_-(\sigma) &= \{i_j \in [n] : \varepsilon_j = -1\}, \\ T_*(\sigma) &= [n] \setminus (T_+(\sigma) \cup T_-(\sigma)). \end{aligned}$$

The signs in $\Psi(\sigma)$ result from the condition that Ψ must commute with the respective boundary maps.

3. ABOUT THE PROOF OF THEOREM 1.1

The proof of the main theorem is based on an explicit description of the Alexander duality, using the description of Lefshetz duality above. By hand, the computation of the cup product follows. One has to distinguish certain cases depending on the union of the simplices σ and σ' appearing in the main theorem.

4. Example of a Simplicial Complex yielding different Ring Structures

Let $[u], [v], [w]$ be cohomology classes of the complement of a real coordinate subspace arrangement corresponding to homology classes of links of Δ, such that $[u] \cup [v] = [w]$. Then our results imply that for the corresponding cohomology classes of the complement of the associated complex arrangement we have (see Corollary 2.2)

$$[u^{\mathbb{C}}] \cup [v^{\mathbb{C}}] = \pm [w^{\mathbb{C}}].$$

Hence it arises the question if we can choose signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$ consistently such that it becomes a (dimension-shifting) ring isomorphism. An example of different ring structures containing hyperplanes was given in [GPW]: the existence of hyperplanes lead to additional multiplication in the real case. Our example shows that this is not the only case where non-isomorphic rings occur.

Remark 4.1. There is a (dimension shifting) ring isomorphism of $\tilde{H}^*(C_\Delta; \mathbb{Z}_2)$ and $\tilde{H}^*(C_\Delta^{\mathbb{C}}; \mathbb{Z}_2)$.

4.1. The Example: Different Sign Patterns.

We construct a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices given by four facets $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$, and investigate the multiplication of cohomology classes stemming from the links of these facets in the case of the associated real and complex arrangement. For the real and complex case the resulting sign pattern implies that there is no ring isomorphism between $\tilde{H}^*(C_\Delta)$ and $\tilde{H}^*(C_\Delta^{\mathbb{C}})$. The facets are given by the following scheme which also helps for computing the signs appearing in the multiplication. A black box in position (ρ, j) indicates that $j \in \rho$.

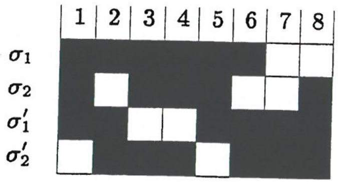


Figure 4. The facets of Δ

The sign patterns arising in the real and in the complex case are given by the following table.

		Real Sign	Complex Sign
σ_1	σ_1'	-1	-1
σ_1	σ_2'	-1	-1
σ_2	σ_1'	$+1$	-1
σ_2	σ_2'	-1	-1

Clearly, there is no consistent way of assigning signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$.

5. Example of non trivial multiplication of torsion elements

We construct a simplicial complex $\Delta \subset 2^{[10]}$. Let $\sigma := \{1,2,3,4,5,6\}$ and $P \subset 2^{\{1,2,3,4,5,6\}}$ be a six-vertex triangulation of the projective plane. Let $\sigma' = \{7,8,9,10\}$, and let S be a simplicial 1-sphere on four vertices as a subcomplex of $2^{\{7,8,9,10\}}$. Now define $\Delta = P * 2^{\sigma'} \cup 2^{\sigma} * S$. Then the homotopy type of Δ is $\Sigma(P * 2^{\sigma'} \cap 2^{\sigma} * S) = \Sigma(P * S)$. Hence Δ has the homotopy type of a threefold suspended projective plane. Now $\text{link}_\Delta(\sigma * \emptyset) = \emptyset * S$ and $\text{link}_\Delta(\emptyset * \sigma') = P * \emptyset$. Let $[c] \in \tilde{H}_1(\text{link}_\Delta(\sigma * \emptyset)) \cong \mathbb{Z}$ and $[c'] \in \tilde{H}_1(\text{link}_\Delta(\emptyset * \sigma')) \cong \mathbb{Z}_2$ be generating homology classes. They correspond to elements $[u] \in \tilde{H}^{10-1-6-2}(\Gamma_\Delta)$ and $[v] \in \tilde{H}^{10-1-4-2}(\Gamma_\Delta)$. Their cup product corresponds to a generating class

$$[\langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'] \in \tilde{H}_{10-4-0-2}(\text{link}_\Delta \emptyset) \cong \mathbb{Z}_2$$

for $i_\sigma \in \{7,8,9,10\}$ and $i_{\sigma'} \in \{1,2,3,4,5,6\}$.
Note that this example works for the real as well as for the complex case.

6. Questions and Remarks

- ▷ A very natural question is as to what extent our methods can be used to treat more general subspace arrangements.
- ▷ It is easy to see that if $\Delta \subset 2^{[n]}$ is a simplicial complex such that
  - ▷ $\dim \Delta \leq n-3$, i.e., the associated real arrangement does not contain hyperplanes, and
  - ▷ Δ is Cohen-Macaulay over $\mathbb{Z}$,

  then the ring structure of $\tilde{H}^*(C_\Delta; \mathbb{Z})$ is trivial. Using the specific description of the multiplication it would be nice to derive a better characterization of simplicial complexes yielding trivial multiplication. Confer also [HRW].

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