A quantum Sylvester theorem and representations of Yangians

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Abstract

We give a quantum analogue of Sylvester's theorem where the minors of a numerical matrix are replaced with the quantum minors of the matrix formed by the generators of the Yangian for the Lie algebra $\mathfrak{gl}(n)$. We then use it to explicitly construct the highest vectors and to find the highest weights of the so-called elementary representations of the Yangian.

Résumé

Nous donnons un analogue quantique du théorème de Sylvester où on remplace les mineurs d'une matrice numérique par les mineurs quantiques de la matrice des générateurs du yangien à l'algèbre de Lie $\mathfrak{gl}(n)$. On l'applique pour la construction explicite des vecteurs de plus haut poid et pour trouver les plus hauts poids des représentations dites élémentaires du yangien.

0 Introduction

The classical Sylvester theorem provides certain relations between the minors of a numerical matrix. A generalization of this theorem for matrices over an arbitrary noncommutative ring was obtained by Gelfand and Retakh [5]. This result was used by Krob and Leclerc [6] to find a quantum analogue of Sylvester's theorem for the quantized algebra of functions on $GL(n)$.

In this paper we use a different approach based on R-matrix calculations to prove a quantum Sylvester theorem for the $\mathfrak{gl}(n)$-Yangian $\mathrm{Y}(n)$ (Theorem 1.2). The first part of the theorem provides a natural algebra homomorphism $\pi : \mathrm{Y}(n) \to \mathrm{Y}(n+m)$, while the second part gives a quantum analog of the Sylvester identity where the minors of a numerical matrix are replaced with the quantum minors of the matrix formed by the standard generators of $\mathrm{Y}(n)$. The image of the composition $\varepsilon \circ \pi$ of the homomorphism π with the natural epimorphism $\varepsilon : \mathrm{Y}(n+m) \to \mathrm{U}(\mathfrak{gl}(n+m))$ turns out to be contained in the centralizer $\mathrm{A} = \mathrm{U}(\mathfrak{gl}(n+m))^{\mathfrak{gl}(m)}$ thus providing us with a homomorphism $\mathrm{Y}(n) \to \mathrm{A}$.

Let us now consider a finite-dimensional irreducible representation $L(\lambda)$ of the Lie algebra $\mathfrak{gl}(n+m)$ with the highest weight λ and denote by $L(\lambda)^+_\mu$ the subspace in $L(\lambda)$ of $\mathfrak{gl}(m)$-highest vectors of weight μ. It is well-known (see e.g. [3, Section 9.1]) that $L(\lambda)^+_\mu$ is an irreducible representation of the algebra A and so, $L(\lambda)^+_\mu$ becomes a $\mathrm{Y}(n)$-module which can be shown to be irreducible.

A different homomorphism $\mathrm{Y}(n) \to \mathrm{A}$ was constructed earlier by Olshanski [12, 13], and the corresponding representation of $\mathrm{Y}(n)$ in $L(\lambda)^+_\mu$ was studied by Nazarov and Tarasov [11]. It turns out that these two $\mathrm{Y}(n)$-module structures on $L(\lambda)^+_\mu$ coincide, up to an automorphism of $\mathrm{Y}(n)$ (see Corollary 2.3). These modules play an important role in the classification of the representations of $\mathrm{Y}(n)$ with a semisimple action of the Gelfand–Tsetlin subalgebra; see [2, 11]. In particular, it was proved in [11, Theorem 4.1] that, up to an automorphism of $\mathrm{Y}(n)$, any such module is isomorphic to a tensor product of representations of the form $L(\lambda)^+_\mu$.

We explicitly construct the highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ and calculate its highest weight. We also identify $L(\lambda)^+_\mu$ as a module over the Yangian for the Lie algebra $\mathfrak{sl}(n)$ by calculating its Drinfeld polynomials; cf. [11].

1 Quantum Sylvester's theorem

A detailed description of the algebraic structure of the Yangian for the Lie algebra $\mathfrak{gl}(n)$ is given in the expository paper [10]. In this section we reproduce some of those results and use them to prove a quantum analogue of Sylvester's theorem.

The *Yangian* $\mathrm{Y}(n) = \mathrm{Y}(\mathfrak{gl}(n))$ is the complex associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \le i, j \le n$, and the defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)), \tag{1.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots \in \mathrm{Y}(n)[[u^{-1}]]$$

and u is a formal variable. Introduce the matrix

$$T(u) := \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \in \mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\,\mathbb{C}^n,$$

where the E_{ij} are the standard matrix units. Then the relations (1.1) are equivalent to the single relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{1.2}$$

Here $T_1(u)$ and $T_2(u)$ are regarded as elements of $\mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n$, the subindex of $T(u)$ indicates to which copy of $\mathrm{End}\,\mathbb{C}^n$ this matrix corresponds, and

$$R(u) = 1 - Pu^{-1}, \qquad P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}.$$

The *quantum determinant* $\mathrm{qdet}\,T(u)$ of the matrix $T(u)$ is a formal series in u^{-1} with coefficients from $\mathrm{Y}(n)$ defined by

$$\mathrm{qdet}\,T(u) = \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}(p)\, t_{p(1)1}(u) \cdots t_{p(n)n}(u-n+1). \tag{1.3}$$

The coefficients of the quantum determinant $\mathrm{qdet}\,T(u)$ are algebraically independent generators of the center of the algebra $\mathrm{Y}(n)$.

Introduce the series $t^{a_1\cdots a_s}_{b_1\cdots b_s}(u) \in \mathrm{Y}(n)[[u^{-1}]]$ where $a_i, b_i \in \{1,\dots,n\}$ by the following equivalent formulas

$$\begin{aligned} t^{a_1\cdots a_s}_{b_1\cdots b_s}(u) &= \sum_{\sigma \in \mathfrak{S}_s} \mathrm{sgn}(\sigma)\, t_{a_{\sigma(1)}b_1}(u) \cdots t_{a_{\sigma(s)}b_s}(u-s+1) \\ &= \sum_{\sigma \in \mathfrak{S}_s} \mathrm{sgn}(\sigma)\, t_{a_1 b_{\sigma(1)}}(u-s+1) \cdots t_{a_s b_{\sigma(s)}}(u). \end{aligned}$$

In particular, $t^{a}_{b}(u) = t_{ab}(u)$. The series $t^{a_1\cdots a_s}_{b_1\cdots b_s}(u)$ can be shown to be antisymmetric with respect to permutations of the upper indices and of the lower indices; see [10]. Note that $t^{1\cdots n}_{1\cdots n}(u) = \mathrm{qdet}\,T(u)$ by (1.3).

Proposition 1.1 *We have the relations*

$$[t^{a_1\cdots a_k}_{b_1\cdots b_k}(u), t^{c_1\cdots c_l}_{d_1\cdots d_l}(v)] = \sum_{p=1}^{\min\{k,l\}} \frac{(-1)^{p-1}\,p!}{(u-v-k+1)\cdots(u-v-k+p)}$$

$$\sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_p}} \Big(t^{a_1\cdots c_{j_1}\cdots c_{j_p}\cdots a_k}_{b_1\cdots\cdots b_k}(u)\, t^{c_1\cdots a_{i_1}\cdots a_{i_p}\cdots c_l}_{d_1\cdots\cdots d_l}(v) - t^{c_1\cdots\cdots c_l}_{d_1\cdots b_{i_1}\cdots b_{i_p}\cdots d_l}(v)\, t^{a_1\cdots\cdots a_k}_{b_1\cdots d_{j_1}\cdots d_{j_p}\cdots b_k}(u) \Big).$$

Here the p-tuples of upper indices $(a_{i_1},\ldots,a_{i_p})$ *and* $(c_{j_1},\ldots,c_{j_p})$ *are respectively interchanged in the first summand on the right hand side while the p-tuples of lower indices* $(b_{i_1},\ldots,b_{i_p})$ *and* $(d_{j_1},\ldots,d_{j_p})$ *are interchanged in the second summand.*

We may regard the series $t^{a_1\cdots a_s}_{b_1\cdots b_s}(u)$ as matrix elements of certain operators in the space $\mathbb{C}^n\otimes\cdots\otimes\mathbb{C}^n$ with coefficients in $\mathrm{Y}(n)[[u^{-1}]]$; see [10]. To prove the proposition we use a generalization of (1.2) for such operators [10].

The Poincaré–Birkhoff–Witt theorem for the Yangians (see e.g. [10]) implies that the Yangian $\mathrm{Y}(n)$ can be identified with the subalgebra in $\mathrm{Y}(n+m)$ generated by the coefficients of the series $t_{ij}(u)$ with $1\le i,j\le n$. For any indices $1\le i,j\le n$ introduce the following series with coefficients in $\mathrm{Y}(n+m)$

$$\widetilde{t}_{ij}(u)=t^{1\cdots m,m+i}_{1\cdots m,m+j}(u)$$

and combine them into the matrix $\widetilde{T}(u)=(\widetilde{t}_{ij}(u))$. For subsets $\mathcal{P}$ and $\mathcal{Q}$ of the set $\{1,\ldots,n+m\}$ and an $(n+m)\times(n+m)$-matrix X we shall denote by $X_{\mathcal{P}\mathcal{Q}}$ the submatrix of X whose rows and columns are enumerated by $\mathcal{P}$ and $\mathcal{Q}$ respectively. Set $\mathcal{A}=\{1,\ldots,m\}$.

Theorem 1.2 *The mapping*

$$t_{ij}(u)\mapsto\widetilde{t}_{ij}(u),\qquad 1\le i,j\le n$$

defines an algebra homomorphism $\mathrm{Y}(n)\to\mathrm{Y}(n+m)$. *Moreover, one has the identity*

$$\operatorname{qdet}\widetilde{T}(u)=\operatorname{qdet}T(u)\operatorname{qdet}T(u-1)_{\mathcal{A}\mathcal{A}}\cdots\operatorname{qdet}T(u-n+1)_{\mathcal{A}\mathcal{A}}.\qquad(1.4)$$

The first part of the theorem follows from Proposition 1.1. The relation (1.4) is a noncommutative generalization of Sylvester's identity and is proved by using R-matrix representations of the quantum determinants; see [10].

2 Elementary representations of the Yangian

Here we use Theorem 1.2 to identify the elementary representations of $\mathrm{Y}(n)$ by constructing their highest vectors.

2.1 Yangian action on the multiplicity space

Let $\lambda=(\lambda_1,\ldots,\lambda_{n+m})$ be an $(n+m)$-tuple of complex numbers satisfying the condition $\lambda_i-\lambda_{i+1}\in\mathbb{Z}_+$ for $i=1,\ldots,n+m-1$. Denote by $L(\lambda)$ the irreducible finite-dimensional representation of the Lie algebra $\mathfrak{gl}(n+m)$ with the highest weight λ. It contains a unique nonzero vector ξ (the highest vector) such that

$$\begin{aligned}E_{ii}\xi&=\lambda_i\xi\qquad\text{for}\quad i=1,\ldots,n+m,\\ E_{ij}\xi&=0\qquad\text{for}\quad 1\le i<j\le n+m.\end{aligned}$$

Consider the subalgebra $\mathfrak{gl}(m) \subset \mathfrak{gl}(n+m)$ spanned by the basis elements E_{ij} with $i, j = 1, \dots, m$. Given a $\mathfrak{gl}(m)$-highest weight $\mu = (\mu_1, \dots, \mu_m)$ we denote by $L(\lambda)^+_\mu$ the subspace of $\mathfrak{gl}(m)$-highest vectors in $L(\lambda)$ of weight μ:

$$L(\lambda)^+_\mu = \{\eta \in L(\lambda) \mid E_{ii}\eta = \mu_i \eta \quad \text{for} \quad i = 1, \dots, m, \\ E_{ij}\eta = 0 \quad \text{for} \quad 1 \le i < j \le m\}.$$

The dimension of $L(\lambda)^+_\mu$ coincides with the multiplicity of the $\mathfrak{gl}(m)$-module $L(\mu)$ in the restriction of $L(\lambda)$ to $\mathfrak{gl}(m)$. The multiplicity space $L(\lambda)^+_\mu$ admits a natural structure of an irreducible representation of the centralizer algebra

$$\mathrm{A} = \mathrm{U}(\mathfrak{gl}(n+m))^{\mathfrak{gl}(m)},$$

see [3, Section 9.1]. On the other hand, we have an algebra homomorphism

$$\mathrm{Y}(n+m) \to \mathrm{U}(\mathfrak{gl}(n+m)), \qquad T(u) \mapsto 1 + Eu^{-1}, \tag{2.1}$$

where E denotes the $(n+m)\times(n+m)$-matrix (E_{ij}); see e.g. [10]. Take the composition of (2.1) with the homomorphism $\mathrm{Y}(n) \to \mathrm{Y}(n+m)$ provided by Theorem 1.2. Then the image of the series $t_{kl}(u)$ is given by

$$t_{kl}(u) \mapsto \mathrm{qdet}\,(1 + Eu^{-1})_{\mathcal{C}_k\mathcal{C}_l}, \tag{2.2}$$

where $\mathcal{C}_k = \{1, \dots, m, m+k\}$ for $k \in \{1, \dots, n\}$. Proposition 1.1 implies that the image in (2.2) is contained in the centralizer A and so, we obtain an algebra homomorphism $\mathrm{Y}(n) \to \mathrm{A}$. One can show (see [13]) that the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ defined via this homomorphism is irreducible. Following [11] we call it *elementary*.

2.2 Highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$

A representation L of the Yangian $\mathrm{Y}(n)$ is called *highest weight* if it is generated by a nonzero vector ζ (the *highest vector*) such that

$$t_{kk}(u)\zeta = \lambda_k(u)\zeta \quad \text{for} \quad k = 1, \dots, n, \\ t_{kl}(u)\zeta = 0 \quad \text{for} \quad 1 \le k < l \le n$$

for certain formal series $\lambda_k(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The set $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ is called the *highest weight* of L; cf. [4, 1]. Every finite-dimensional irreducible representation of the Yangian $\mathrm{Y}(n)$ is highest weight. It contains a unique, up to scalar multiples, highest vector. An irreducible representation of $\mathrm{Y}(n)$ with the highest weight $\lambda(u)$ is finite-dimensional if and only if there exist monic polynomials $P_1(u), \dots, P_{n-1}(u)$ in u (called the *Drinfeld polynomials*) such that

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u+1)}{P_k(u)}, \qquad k = 1, \dots, n-1.$$

These results are contained in [4]; see also [1, 8].

For all $a \in \{m+1, \ldots, m+n\}$ introduce the following elements of $\mathrm{U}(\mathfrak{gl}(n+m))$

$$s_{ia} = \sum_{i > i_1 > \cdots > i_s \geq 1} E_{i i_1} E_{i_1 i_2} \cdots E_{i_{s-1} i_s} E_{i_s a} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}),$$

$$s_{ai} = \sum_{i < i_1 < \cdots < i_s \leq m} E_{i_1 i} E_{i_2 i_1} \cdots E_{i_s i_{s-1}} E_{a i_s} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}),$$

where $s = 0, 1, \ldots$ and $\{j_1, \ldots, j_r\}$ is the complementary subset to $\{i_1, \ldots, i_s\}$ respectively in the set $\{1, \ldots, i-1\}$ or $\{i+1, \ldots, m\}$; $h_i = E_{ii} - i + 1$. The s_{ia} and s_{ai} are respectively called *raising* and *lowering operators*. They act in the subspace of the $\mathfrak{gl}(m)$-highest vectors in $L(\lambda)$ so that

$$s_{ia} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu+\delta_i}, \qquad s_{ai} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu-\delta_i},$$

see [16] for more details.

For $k, l = 1, \ldots, n$ set

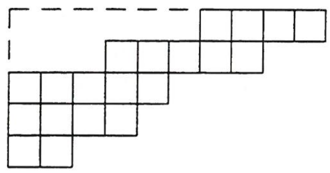
$$T_{m+k, m+l}(u) = u(u-1) \cdots (u-m)\, t_{kl}(u).$$

Proposition 2.1 *For $a, b \in \{m+1, \ldots, m+n\}$ the action of the element $T_{ab}(u)$ in $L(\lambda)^+_\mu$ is given by*

$$T_{ab}(u) \mapsto (\delta_{ab} u + E_{ab}) \prod_{i=1}^m (u + h_i - 1) - \sum_{i=1}^m s_{ib} s_{ai} \prod_{j=1,\, j \neq i}^m \frac{u + h_j - 1}{h_i - h_j}.$$

From now on we shall assume that the highest weight λ is a partition, that is, the λ_i are nonnegative integers. This does not lead to a real loss of generality because the formulas and arguments below can be easily adjusted to be valid in the general case. Given a general λ one can add a suitable complex number to all entries of λ to get a partition.

As it follows from the branching rule for the general linear Lie algebras (see [15]) the space $L(\lambda)^+_\mu$ is nonzero only if μ is a partition such that $\mu \subset \lambda$ and each column of the skew diagram λ/μ contains at most n cells. The figure below illustrates the skew diagram for $\lambda = (10, 8, 5, 4, 2)$ and $\mu = (6, 3)$:



Introduce the *row order* on the cells of λ/μ corresponding to reading the diagram by rows from left to right starting from the top row. For a cell $\alpha \in \lambda/\mu$ denote by $r(\alpha)$

the row number of α and by $l(\alpha)$ the (increased) leglength of α which equals 1 plus the number of cells of λ/μ in the column containing α which are below α. Consider the following element of $L(\lambda)$:

$$\zeta = \prod_{\alpha\in\lambda/\mu,\ r(\alpha)\leq m} s_{m+l(\alpha),r(\alpha)}\,\xi, \tag{2.3}$$

where ξ is the highest vector of $L(\lambda)$ and the product is taken in the row order. For the above example of λ/μ we have $m=2$, $n=3$ and

$$\zeta = (s_{41})^2\,(s_{31})^2\,s_{52}\,s_{42}\,(s_{32})^3\,\xi.$$

Given three integers i,j,k we shall denote by middle$\{i,j,k\}$ that of the three which is between the two others.

Theorem 2.2 *The vector ζ defined by (2.3) is the highest vector of the* Y(n)*-module $L(\lambda)^+_\mu$. The highest weight of this module is $(\lambda_1(u),\dots,\lambda_n(u))$ where*

$$\lambda_a(u) = \frac{(u+\nu_a^{(1)})(u+\nu_a^{(2)}-1)\cdots(u+\nu_a^{(m+1)}-m)}{u(u-1)\cdots(u-m)}$$

and

$$\nu_a^{(i)} = \text{middle}\{\mu_{i-1},\mu_i,\lambda_{a+i-1}\}$$

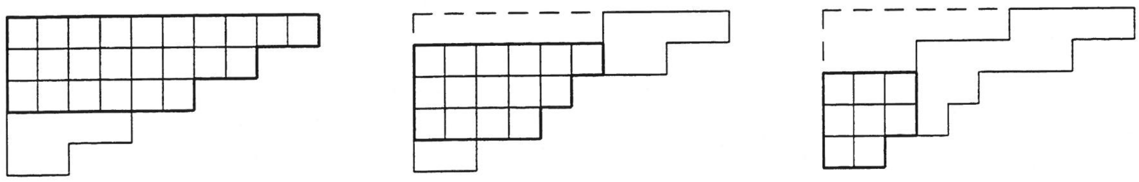
with $\mu_{m+1}=0$, and μ_0 is considered to be sufficiently large.

For the proof we find first a quantum minor representation for the raising and lowering operators; cf. [7]. Then we use Propositions 1.1 and 2.1.

Note that for each i the n-tuple $\nu^{(i)} = (\nu_1^{(i)},\dots,\nu_n^{(i)})$ is a partition which can be obtained from λ/μ as follows. Consider the subdiagram of λ of the form $\lambda^{(i)} = (\lambda_i,\lambda_{i+1},\dots,\lambda_{i+n-1})$. Replace the rows of $\lambda^{(i)}$ which are longer than μ_{i-1} by μ_{i-1} while those which are shorter than μ_i replace with μ_i and leave the remaining rows unchanged. The resulting partition is $\nu^{(i)}$. For the above example with $\lambda=(10,8,5,4,2)$ and $\mu=(6,3)$ we have

$$\nu^{(1)} = (10,8,6), \qquad \nu^{(2)} = (6,5,4), \qquad \nu^{(3)} = (3,3,2),$$

as illustrated:



By the *content* of a cell $\alpha = (i, j) \in \lambda/\mu$ we mean the number $j - i$.

Corollary 2.3 *The Drinfeld polynomials for the* $\mathrm{Y}(n)$*-module* $L(\lambda)^+_\mu$ *are given by*

$$P_a(u) = \prod_c (u + c), \qquad a = 1, \dots, n-1,$$

where c *runs over the contents of the top cells of columns of height* a *in the diagram* λ/μ.

If $\lambda = (10, 8, 5, 4, 2)$ and $\mu = (6, 3)$ (see the example above) then we have

$$P_1(u) = (u+4)(u+8)(u+9), \qquad P_2(u) = u(u+3)(u+6)(u+7).$$

The corollary shows that the $\mathrm{Y}(\mathfrak{sl}(n))$-module $L(\lambda)^+_\mu$ is isomorphic to that considered by Nazarov and Tarasov; see [11].

Remark. The approach to study the elementary representation of the Yangian $\mathrm{Y}(n)$ based on a quantum analogue of Sylvester's theorem can be applied to other series of classical Lie algebras, where the Yangian is replaced by the *twisted Yangians* corresponding to the orthogonal or symplectic Lie algebras; see [14, 10]. In particular, it can be used to construct an analogue of the Gelfand–Tsetlin basis for representations of the symplectic Lie algebras [9].

References

[1] V. Chari and A. Pressley, *Yangians and R-matrices*, L'Enseign. Math. **36** (1990), 267–302.

[2] I. V. Cherednik, *A new interpretation of Gelfand–Tzetlin bases*, Duke Math. J. **54** (1987), 563–577.

[3] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.

[4] V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[5] I. M. Gelfand and V. S. Retakh, *Determinants of matrices over noncommutative rings*, Funct. Anal. Appl. **25** (1991), 91–102.

[6] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), no. 1, 1–23.

[7] A. I. Molev, *Gelfand–Tsetlin basis for representations of Yangians*, Lett. Math. Phys. **26** (1992), 211–218.

[8] A. I. Molev, *Finite-dimensional irreducible representations of twisted Yangians*, J. Math. Phys. **39** (1998), 5559–5600.

[9] A. I. Molev, *A basis for representations of symplectic Lie algebras*, Comm. Math. Phys., to appear; math.QA/9804127.

[10] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Surveys **51**:2 (1996), 205–282.

[11] M. Nazarov and V. Tarasov, *Representations of Yangians with Gelfand–Zetlin bases*, J. Reine Angew. Math. **496** (1998), 181–212.

[12] G. I. Olshanski, *Extension of the algebra $U(g)$ for infinite-dimensional classical Lie algebras g, and the Yangians $Y(gl(m))$*. Soviet Math. Dokl. **36** (1988), no. 3, 569–573.

[13] G. I. Olshanski, *Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians*, in 'Topics in Representation Theory' (A. A. Kirillov, Ed.), Advances in Soviet Math. **2**, AMS, Providence RI, 1991, pp. 1–66.

[14] G. I. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in 'Quantum Groups' (P. P. Kulish, Ed.), Lecture Notes in Math. **1510**, Springer, Berlin-Heidelberg, 1992, pp. 103–120.

[15] D. P. Želobenko, *Compact Lie groups and their representations*, Transl. of Math. Monographs **40** AMS, Providence RI 1973.

[16] D. P. Zhelobenko, *Z-algebras over reductive Lie algebras*, Soviet. Math. Dokl. **28** (1983), 777–781.