Acyclic sets and colourings in digraphs

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Abstract

The *dichromatic number* of a digraph is the minimum number of colours needed to colour its vertices so that no monochromatic directed cycle appears. In this article we will give a view of the present state of this invariant.

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1 Introduction.

Many fundamental concepts and invariants of Graph Theory are related to connectedness. The chromatic number is one of such invariants. In this article we will give a view of the present state of knowledge of the dichromatic number, an invariant which generalizes the chromatic number.

The *dichromatic number* $dc(D)$ of a digraph D is the least number of colours needed to colour the vertices of D in such a way that each chromatic class is acyclic ([4,13,14]). So $dc(D) = 1$ if and only if D is acyclic and $dc(D^{op}) = dc(D)$ where D^{op} is obtained from D by reversing each one of its arcs. If G^* denotes the digraph obtained from a graph G by directing each edge in the two opposite directions then $dc(G^*) = \chi(G)$.

The dichromatic number has been used to prove the existence of objects such as kernel perfect digraphs and kernel imperfect critical digraphs having extremely complex cyclic structure [10] and for a similar purpose in Continuum Theory [20]. Another application has been given in [11].

The *acyclic disconnection* of a digraph, which is the maximum number of weak components which can be obtained in a digraph after deleting an acyclic set of arcs, gives also a (decreasing) measure of the complexity of the cyclic structure of the digraph. In [18,21] relations between the dichromatic number and the acyclic disconnection are studied.

2 Preliminaries.

Let $D = (V(D), A(D))$ be a digraph. $\Delta^+(D)$ and $\Delta^-(D)$ (resp: $\delta^+(D)$ and $\delta^-(D)$) will denote the maximum (resp: minimum) outdegree of D and maximum (resp: minimum) indegree of D respectively; $\vec{\beta}(D)$ will be the maximum cardinality of an acyclic set of vertices in D.

D is called *r-dichromatic* if $dc(D) = r$, *vertex-critical* (v.c.) if $dc(D-u) < dc(D)$ for every $u \in V(D)$; *arc-critical* (resp: *minimal*) if $dc(D-uw) < dc(D)$ for every $uw \in A(D)$ (resp: $dc(D_0) < dc(D)$ for every proper subdigraph D_0 of D). Obviously, a digraph without isolated vertices is minimal if and only if it is arc-critical.

A digraph obtained from a graph G by assigning to each edge just one direction is called an *orientation* of G.

In what follows, $I_n = \{1,\ldots,n\}$, Z_n is the ring of integers $\mathrm{mod}\, n$ and for any nonempty set $J \subseteq Z_n - \{0\}$, $\vec{C}_n(J)$ is the digraph defined by $V(\vec{C}_n(J)) = Z_n$ and $A(\vec{C}_n(J)) = \{(i,j): i,j \in Z_n \text{ and } j-i \in J\}$. Notice that $\vec{C}_n(\{1\})$ is the directed cycle $\vec{C}_n$ and that $\vec{C}_{2m+1}(J)$ is a circulant tournament if and only if $|\{j,-j\} \cap J| = 1$ for every $j \in Z_{2m+1} \setminus \{0\}$. Finally we define $I_{m,j} = I_m \cup \{-j\} \setminus \{j\}$ for $j \in I_m$.

For general terminology we refer the reader to [1,2].

3 The dichromatic number of digraphs.

Theorem 3.1 [14] $dc(D) \leq \min\{\Delta^-(D), \Delta^+(D)\} + 1$.

Theorem 3.2 *If D is vertex-critical then* $dc(D) \geq \min\{\delta^-(D), \delta^+(D)\} - 1$.

Let $c_0(s,m)$ denote the maximum number of edge-disjoint cycles of length m in K_s passing by a given vertex and define $c(s,m) = 2c_0(s,m)$ for $2 < m \leq s$ and $c(s,2) = c_0(s,2)$. Notice that $c_0(s,m) \geq \lfloor (s-1)/(m-1) \rfloor \lfloor (m-1)/2 \rfloor$ [14].

Theorem 3.3 [14] *If D is a minimal $(k+1)$-dichromatic digraph, $k \geq 2$ and m is an integer such that $2 \leq m \leq k$. Then*

(i) *For any two adjacent vertices u, v in D, there exists a set of $c(k,m)$ mutually arc-disjoint directed uv-paths of length $\equiv 0 \pmod{m}$.*

(ii) *Any arc uw of D is contained in $c(k,m)$ directed cycles of length $\equiv 1 \pmod{m}$ such that any two of them share only one arc, namely: uw.*

(iii) *Every vertex u of D is contained in $c(k,m)$ pairwise arc-disjoint directed cycles of length $\equiv 0 \pmod{m}$.*

In [6], Erdös and Hajnal proved that if $\chi(G) \geq 3$ then G contains an odd cycle of length at least $\chi(G)-1$. Taking $m = \lfloor k/2 \rfloor$ in Theorem 3.3 (iii), we obtain the following version for digraphs.

Theorem 3.4 [14] *If D is a minimal $(k+1)$-dichromatic digraph with $k \geq 2$, then every arc belongs to an odd directed cycle of length at least k.*

As a direct consequence of Theorem 3.3 we also obtain the following

Theorem 3.5 [14] *If D is a minimal $(k+1)$-dichromatic digraph then D is strongly k-arc connected.*

For the composition $D[H]$ of D and H holds

Theorem 3.6 [14] $dc(D[H]) \geq dc(D) + dc(H) - 1$.

4 Lexicographical sums and dichromatic number.

Let D be a digraph and $\alpha = (\alpha_i)_{i \in V(D)}$ a family of nonempty (mutually disjoint) digraphs. The lexicographical sum $\sigma(\alpha, D)$ of α over D is defined by $V(\sigma(\alpha, D)) = \bigcup_{i \in V(D)} V(\alpha_i)$;

$$A(\sigma(\alpha, D)) = \bigcup_{i \in V(D)} A(\alpha_i) \cup \{uw \colon u \in V(\alpha_i),\ w \in V(\alpha_j)\ \&\ ij \in A(D)\}.$$

If the members of the family α are not mutually disjoint, we replace each of them by one isomorphic copy so that the new family α' becomes one of mutually disjoint digraphs. Notice that the resulting digraph $\sigma(\alpha', D)$ is defined up to isomorphism and that $\sigma(\alpha, D)$ is just $D[W]$ whenever $\alpha_i \cong W$ for every $i \in V(D)$.

Let H be an hypergraph without isolated vertices and suppose a positive integer ξ_u has been assigned to each vertex u of H (such an assignement ξ will be called a *weight function on* H).

The *covering number* $\tilde{n}(H,\xi)$ is the minimum cardinality of a family of non necessarily different edges of H such that each vertex u belongs to at least ξ_u edges of the family. The covering number $\tilde{n}(H,\mathbf{1})$, where $\mathbf{1}$ denotes the constant function of value 1, has been extensely studied (see [1]). In what follows, $\mathbf{k}$ will denote a constant function of value k whenever k is a positive integer. We define $|\xi| = \sum_{u\in V(H)} \xi(u)$.

A weight function ξ on H is said to be *$\tilde{n}$-subcritical* (resp: *$\tilde{n}$-upcritical*) if for every weight function ξ' such that $\xi' \leq \xi$ and $|\xi'| = |\xi| - 1$ (resp: $\xi \leq \xi'$ and $|\xi'| = |\xi| + 1$), we have $\tilde{n}(H,\xi') = \tilde{n}(H,\xi) - 1$ (resp: $\tilde{n}(H,\xi') = \tilde{n}(H,\xi) + 1$).

Let D be a digraph and $H_1(D)$ the hypergraph whose vertex set is $V(D)$ and has the maximal acyclic subsets of $V(D)$ as hyperedges.

Theorem 4.1 [19] *Let $Q = (Q_u)_{u\in V(D)}$ be a family of digraphs and ξ_Q the weight function defined by $\xi_Q(u) = dc(Q_u)$. Then $dc(\sigma(D,Q)) = \tilde{n}(H_1(D),\xi_Q)$. Moreover $\sigma(D,Q)$ is vertex-critical if and only if Q_u is vertex-critical for every $u \in V(D)$ and ξ is $\tilde{n}$-subcritical.*

Theorem 4.2 [19] *Every acyclic $\tilde{n}(H_1(D),\xi_Q)$-colouring of $\sigma(D,Q)$ induces in each Q_u an optimal acyclic colouring if and only if ξ_Q is $\tilde{n}$-upcritical.*

The instances of Theorems 4.1 and 4.2 given by $D = \vec{C}_3$ were implicitly considered in [22] to prove the existence of an infinite family of vertex-critical r-dichromatic regular tournaments for $r \geq 3$, $r \neq 4$ and in [23] to construct uniquely colourable r-dichromatic oriented graphs. An infinite family of vertex-critical 4-dichromatic circulant tournaments (namely, $\vec{C}_{6m+1}(I_{3m,2m})$ for $m \geq 2$) was given in [17].

An application of Theorem 4.1 allows the construction of an infinite set of mutually non isomorphic v.c. r-dichromatic tournaments of even order for every integer $r \geq 4$ [19] solving a question of [22].

Some properties and the behaviour of the function $\tilde{n}(H_1(G^*),\mathbf{k})$ have been studied in several papers [8,9,12,24].

Theorem 3.6 can be extended as follows:

Corollary 4.3 [19] *If $dc(\alpha) = k$ then $dc(D[\alpha])) = \tilde{n}(H_1(D),\mathbf{k})$.*

Theorem 4.1 shows that the problem of computing the dichromatic number of a lexicographical sum of digraphs over a digraph D reduces to that of computing the covering number of $H_1(D)$ with respect to an adequate assignement of weights.

The function $\tilde{n}$ has the following simple properties.

P_1: $\tilde{n}(H,\xi+\xi') \le \tilde{n}(H,\xi)+\tilde{n}(H,\xi')$ and $\tilde{n}(H,k\xi) \le k\tilde{n}(H,\xi)$ for every positive integer k.

P_2: $\tilde{n}(H,\xi) \le \tilde{n}(H,\xi')$ whenever $\xi \le \xi'$.

P_3: $\tilde{n}(H,\xi) \ge \lceil |\xi|/\rho(H) \rceil$ where $\rho(H)$ is the maximal cardinality of an edge in H.

P_4: If H_0 is a spanning subhypergraph of H then $\tilde{n}(H,\xi) \le \tilde{n}(H_0,\xi)$.

Moreover if H' is the spanning subhypergraph of H whose edges are the maximal edges of H, then $\tilde{n}(H,\xi)=\tilde{n}(H',\xi)$.

If $r \le n$, let $\Lambda_{n,r}$ be the circulant r-graph such that $V(\Lambda_{n,r}) = Z_n$, $E(\Lambda_{n,r}) = \{\alpha_j : j \in Z_n\}$ where $\alpha_j = \{j, j+1, \ldots, j+r-1\}$ for $j \in Z_n$.

Using the previous properties and applying Corollary 4.3 it is easy to prove the following

Lemma 4.4 [19] *Let D be a digraph of order n and α a k-dichromatic digraph. If $H_1(D)$ contains an isomorphic copy of $\Lambda_{n,r}$ where $r = \vec{\beta}(D)$ then $dc(D[\alpha])) = \lceil k.n/r \rceil$.*

Lemma 4.4 yields the next result.

Theorem 4.5 [19] *If $dc(\alpha) = k$ then*

(i) $dc(\vec{C}_{2m+1}(I_m)[\alpha]) = \lceil k.(2m+1)/(m+1) \rceil$ *for* $m \ge 2$.

(ii) $dc(\vec{C}_{2m+1}(I_{m,m})[\alpha]) = \lceil k.(2m+1)/m \rceil$ *for* $m \ge 3$.

(iii) $dc(\vec{C}_{6m+1}(I_{3m,2m})[\alpha]) = \lceil k.(6m+1)/2m \rceil$ *for* $m \ge 2$.

(iv) $dc(\vec{C}_{17}(I_{8,5})[\alpha]) = \lceil 17k/5 \rceil$, $dc(\vec{C}_{17}(I_{8,7})[\alpha]) = \lceil 17k/7 \rceil$ *and* $dc(\vec{C}_{17}(I_{8,6})[\alpha]) = \lceil 17k/6 \rceil$.

From Theorems 4.1 and 4.5 we obtain the next

Theorem 4.6 [19] *Let α be a vertex-critical k-dichromatic digraph. Then*

(i) $\vec{C}_{2m+1}(I_m)[\alpha]$ *is v.c. if and only if* $k \equiv m (\bmod m+1)$ *and* $m \geq 2$.

(ii) $\vec{C}_{2m+1}(I_{m,m})[\alpha]$ *is v.c. if and only if* $k \equiv 1 (\bmod m)$ *and* $m \geq 3$.

(iii) $\vec{C}_{6m+1}(I_{3m,2m})[\alpha]$ *is v.c. if and only if* $k \equiv 1 (\bmod 2m)$ *and* $m \geq 2$.

(iv) $\vec{C}_3[\alpha]$ *is v.c. if and only if* k *is odd;*
$\vec{C}_{17}(I_{8,5})[\alpha]$ *is v.c. if and only if* $k \equiv 3 (\bmod 5)$;
$\vec{C}_{17}(I_{8,7})[\alpha]$ *is v.c. if and only if* $k \equiv 5 (\bmod 7)$;
$\vec{C}_{17}(I_{8,6})[\alpha]$ *is v.c. if and only if* $k \equiv 5 (\bmod 6)$.

Finally, applying Theorems 4.5 and 4.6 we can obtain

Theorem 4.7 [19] *For every integer* $k \geq 3$, $k \neq 7$ *there exists an infinite family* $\mathcal{F}_k$ *of pairwise non isomorphic vertex critical* k*-dichromatic circulant tournaments.*

Considering Lemma 4.4 it is worth introducing the next definition:

A tournament T is said to be a *Λ-tournament of index* r whenever $H_1(T)$ contains an isomorphic copy of $\Lambda_{n,r}$ where n is the order of T and $r = \vec{\beta}(T)$. Thus $\tilde{n}(H_1(T), \mathbf{k}) = \lceil kn/r \rceil$ for every Λ-tournament of index r and order n. Moreover $\mathbf{k}$ is subcritical whenever $kn \equiv 1 (\bmod r)$ and upcritical whenever $kn \equiv 0 (\bmod r)$.

Theorem 4.8 *If* $r \leq m-1$, $2r \geq m+2 \geq 5$ *and* $2m+1 \neq 3r$ *then* $\vec{C}_{2m+1}(I_{m,r})$ *is a* Λ*-tournament of index* r.

5 The dichromatic number of a graph.

The *dichromatic numbers of a graph* G is the maximum of the dichromatic number of all its orientations [4,7].

For complete graphs we have the following results:

Let W, W_0 and W_1 be the tournaments such that
$V(W) = \{w_0, w_1^-, w_2^-, w_3^-, w_1^+, w_2^+, w_3^+\}$; $A(W) = \{w_i^+ w_j^- : 1 \leq i,j \leq 3\} \cup \{w_0 w_i^+ : i = 1,2,3\} \cup \{w_j^- w_0 : j = 1,2,3\} \cup \{w_i^+ w_{i+1}^+ : i = 1,2,3\} \cup \{w_j^- w_{j+1}^- : j = 1,2,3\}$ (the sum taken $\bmod 3$).

$W_0 = W + \{w_1^- w_1^+, w_2^- w_2^+, w_3^- w_3^+\} - \{w_1^+ w_1^-, w_2^+ w_2^-, w_3^+ w_3^-\}$ and $W_1 = W + \{w_2^- w_2^+, w_3^- w_3^+\} - \{w_2^+ w_2^-, w_3^+ w_3^-\}$.

There are exactly four 3-dichromatic tournaments of order 7 (7 is the minimum order of a 3-dichromatic tournament): $\vec{C}_7(I_{3,2})$, W, W_0 and W_1, see Figures. Two of them ($\vec{C}_7(I_{3,2})$ and W_0) are minimal, the others have just one unessential arc. There is only one 4-dichromatic oriented graph of order at most 11: $\vec{C}_{11}(I_{5,2})$ which is minimal [16].

Using Theorem 4.1 we can construct a 5-dichromatic tournament of order 19. The minimum order of a 5-dichromatic tournament is not known, but it can be proved that it is at least 17.

Theorem 5.1 [7] *There are positive constants c_1 and c_2 such that $c_1.n/\log_2 n \leq dc(K_n) \leq c_2.n/\log_2 n$ with $c_1 \geq 1/3$, $c_2 \leq 8/3$.*

Let $f(n)$ be the smallest integer for which there is a graph $G_{f(n)}$ of size $f(n)$ and dichromatic number n. It is obvious that $G_{f(n)}$ is edge-critical. Moreover $G_{f(n)}$ is not always complete, for instance $dc(K_7-$ one edge$) = 3$ so $f(3) \leq 19$ and $G_{f(3)}$ is not complete [7].

Theorem 5.2 [7] *The quotient $f(n)/n^2$ tends to ∞ as $n \to \infty$.*

Lemma 5.3 [7] *The number of acyclic orientations of $K_{m,m}$ is not bigger than $14^{\lceil m/2 \rceil^2}$.*

Theorem 5.4 [7] *There is a positive constant c and an orientation of $K_{n,n}$ such that every induced subgraph of $K_{n,n}$ isomorphic to $K_{m,m}$, such that $m \geq c\log_2 n$, contains a cyclically oriented square. (We can take $c = \frac{2}{1-(\log_2 14)/4}$).*

Denote by $K_n(n)$ the complete n-partite graph with independent sets of cardinality n.

Theorem 5.5 [7] *For n large enough, there is an orientation $K_n^{\rightarrow}(n)$ of $K_n(n)$ such that $\vec{\beta}(K_n^{\rightarrow}(n) = n + 1$.*

Corollary 5.6 [7] *For n large enough, $dc(K_n(n)) = n$.*

Theorem 5.7 [7] *For every k and $r \geq 3$ there exist k-dichromatic oriented graphs with girth at least r.*

Let $G[H]$ be the composition of G and H.

Theorem 5.8 [7] *Let G be a graph. There exists an integer n_0 depending only on $\chi(G)$ such that if $n \geq n_0$, $dc(G[\overline{K}_n]) = \chi(G)$.*

Other results have been obtained in [5].

Recently, Th. Davoine and Neumann-Lara [3] proved that $dc(\vec{C}_3[\overline{K}_3]) = 3$,

$dc(\vec{C}_5[\overline{K}_3]) = 2$, $dc(\vec{C}_5[\overline{K}_4]) = 3$. Let $\eta(G) = \min\{n: dc(G[\overline{K}_n] = \chi(G)\}$. We have $\eta(\vec{C}_3) = 3$, $\eta(\vec{C}_5) = 4$ and in general $\eta(\vec{C}_{2m+1}) = 4$ for $m \geq 2$. Moreover $\eta(K_4) \leq 7$, $\eta(K_6) \leq 16$, $\eta(K_7) \leq 19$ and in general, $\eta(K_n) \leq n^2 - 3n + 3$.

Open Problems.

1.) Is there a function $f(m)$ such that $dc(G) \geq m$ whenever $\chi(G) \geq f(m)$? [7].

2.) Is $dc(G) \leq 2$ for every planar graph? It is easy to see that $dc(G) \leq 3$ (Neumann-Lara, Urrutia).

3.) Which is the minimum order of a 5-dichromatic tournament? [16].

4.) If every ex-neighbourhood of a tournament T is acyclic, then $dc(T) \leq 2$. Is it true that if $dc(N^+(u,T))$ is at most k-dichromatic then $dc(T) \leq c_k$ for some constant depending only of k? It is not known even for $k = 2$.

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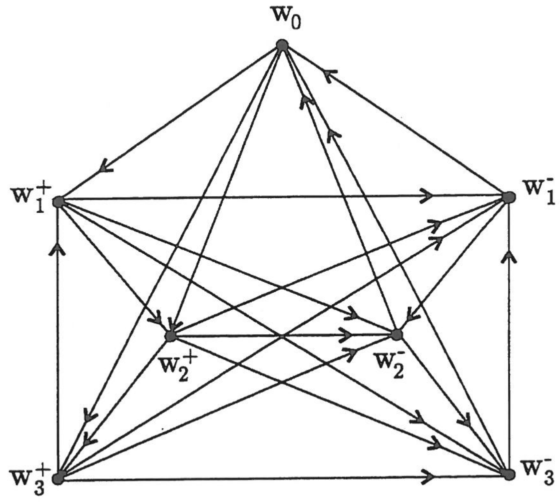

Fig. 1: W.

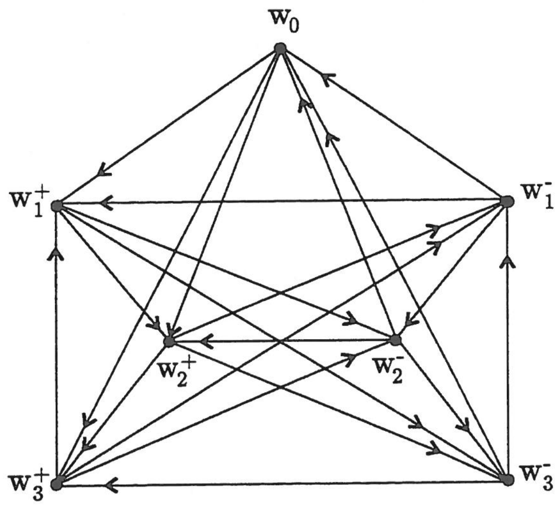

Fig. 2: W_0.

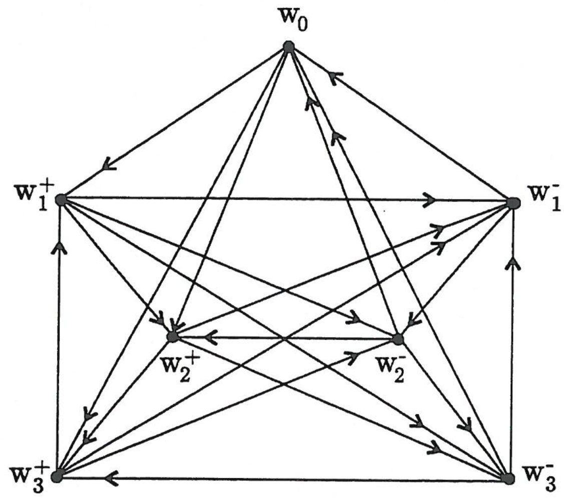

Fig. 3: W_1.