Markov traces and Hecke algebras at roots of unity

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Introduction

The Hecke algebra of type B_n, $H_n(q, Q)$, is semisimple for generic values of the parameters q and Q. The simple components are indexed by ordered pairs of Young diagrams. These Hecke algebras can be defined as a finite dimensional quotient of the group algebra of the braid group of type B.

We have constructed a nontrivial homomorphism from a specialization of the Hecke algebra of type B onto a reduced Hecke algebra of type A for q not equal to 1. This homomorphism has turned out to be a very useful tool in reducing questions about the Hecke algebra of type B to the Hecke algebra of type A.

Motivated by their study of link invariants related to the braid group of type B, Geck and Lambropolou [GL] have defined certain linear traces on the Hecke algebra of type B called Markov traces. Their definition is given inductively. But any trace on the Hecke algebra of type B can be written as a weighted linear combination of the irreducible characters (the usual trace), since this algebra is semisimple. The coefficients in this linear expression are called weights. The weights are equal to the values of the trace at the minimal idempotents. It follows that the weights completely determine the trace. The weights are also indexed by ordered pairs of Young diagrams.

We have found the weight formula for the Markov trace defined by Geck and Lambropolou [GL] for the Hecke algebra of type B. The weight formula can be written as a product of Schur functions and a simple factor. Using the above mentioned homomorphism we obtain that the Markov trace on the Hecke algebra of type B appears as a pullback of the Markov trace on the reduced Hecke algebra of type A. These weights provide an alternative way of computing the trace.

In order to find the weights for the Hecke algebra of type D we use the results of Hoefsmit [H] on the inclusion of the Hecke algebra of type D into the Hecke algebra of type B. We also use the results of Geck [G] on obtaining Markov traces of the Hecke algebra of type D from those of type B.

We also show that there is a well-defined surjective homomorphism from the Hecke algebra of type B onto the reduced Hecke algebra of type A at roots of unity. This implies that we can explicitly describe semisimple quotients of the Hecke algebra of type B at roots of unity. These quotients are useful in the theory of subfactors.

Another consequence of the above mentioned homomorphism is the existence of a duality between the quantum group $U_q(\mathfrak{sl}(r))$ and a quotient of the Hecke algebra of type B.

1 Preliminaries

A *partition* is a decreasing sequence of positive integers, $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots]$ with only finitely non-zero λ_i's. $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is called the *weight* of λ. If $|\lambda| = n$ then λ is a partition of n, denoted by $\lambda \vdash n$. It is common to associate partitions with Young diagrams. The *Young diagram* of $\lambda \vdash n$ is an array of n boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on (rows are counted from top to bottom). $l(\lambda)$ denotes the number of nonzero rows. A *standard tableau* of size n is a Young diagram with n boxes, with boxes filled with numbers from 1 to n, in such a manner that the numbers increase from left to right in the rows and from top to bottom in the columns.

A *double partition* of n is an ordered pair of partitions, (α, β) such that $|\alpha| + |\beta| = n$. A double partition is associated with an ordered pair of Young diagrams.

Schur Functions

For background on Schur functions we refer the reader to [M] Ch.I, Sec. 3. We want to remind the reader of the following specialization of the Schur function.

$$s_\alpha(1, q, \ldots, q^{r-1}) = q^{n(\alpha)} \prod_{1 \leq i < j \leq r} \frac{1 - q^{\alpha_i - \alpha_j + j - i}}{1 - q^{j-i}} \tag{1}$$

where $n(\alpha) = \sum_{i=1}^{l(\alpha)} (i-1)\alpha_i$. We define the following Schur function as a normalization of equation (1):

$$s_{\alpha,r}(q) = \frac{s_\alpha(1, q, \ldots, q^{r-1})}{s_{[1]}(1, q, \ldots, q^{r-1})^{|\alpha|}}$$

Notice that $s_{\alpha,r}(q) = 0$ whenever $l(\alpha) > r$.

The Braid Group

The braid group of type A, $\mathcal{B}_n(A)$, can be defined algebraically by generators $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{n-1}$ and relations

$$\tilde{\sigma}_i \tilde{\sigma}_j = \tilde{\sigma}_j \tilde{\sigma}_i \quad \text{if } |i - j| > 1, \tag{2}$$

$$\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1} \quad \text{if } 1 \leq i \leq n-2. \tag{3}$$

Similarly the braid group of type B, $\mathcal{B}_n(B)$, is defined by generators $t, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations given by equations (2) and (3) and

$$\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1, \tag{4}$$

$$t\sigma_i = \sigma_i t \quad \text{if } i > 1. \tag{5}$$

Figure 1 illustrates the generator $\tilde{\sigma}_i \in \mathcal{B}_n(A)$, the full-twist $\Delta_3^2 \in \mathcal{B}_3(A)$, and the generators $t, \sigma_i \in \mathcal{B}_n(B)$.

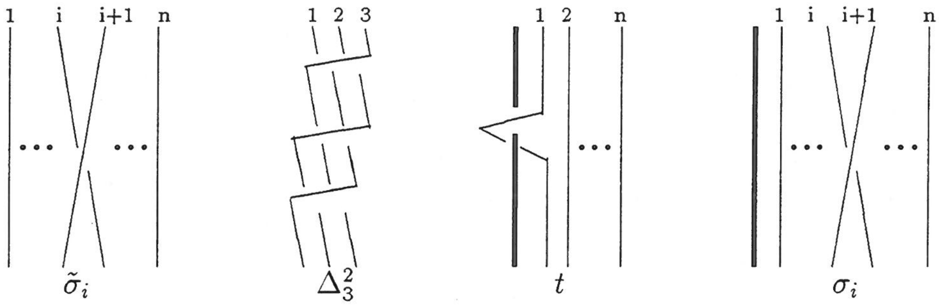

Figure 1

In general, the full-twist Δ_f^2 in f strings, is a central element in $\mathcal{B}_f(A)$. Algebraically, $\Delta_f^2 = (\tilde{\sigma}_{f-1} \ldots \tilde{\sigma}_1)^f$. We now define a map from the generators of $\mathcal{B}_n(B)$ into $\mathcal{B}_{f+n}(A)$. We call the map $\tilde{\rho}_{f,n}$, and we define the image of the generators of $\mathcal{B}_n(B)$ as follows: $\tilde{\rho}_{f,n}(t) = \Delta_f^{-2}\Delta_{f+1}^2$ and $\tilde{\rho}_{f,n}(\sigma_i) = \tilde{\sigma}_{f+i}$ for $i = 1, \ldots, n$. Pictorially, we have the following:

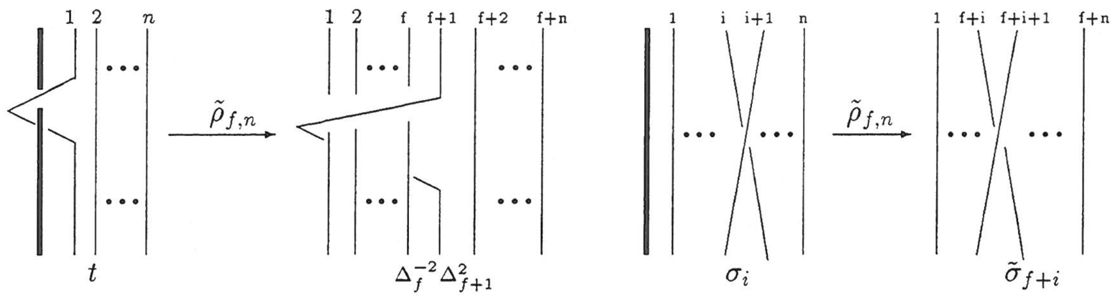

Figure 2

Proposition 1.1 *Let $n, f \in \mathbb{N}$ and $\tilde{\rho}_{f,n}$ be as defined above for the generators of $\mathcal{B}_n(B)$. Then $\tilde{\rho}_{f,n}$ is a well-defined group homomorphism. Furthermore, the representations $\rho_{f,n}$ can be extended in a natural way to representations of the corresponding braid group algebras.*

2 Hecke Algebra of type B_n

The Hecke algebra $H_n(q, Q)$ of type B_n is the free complex algebra with 1 and generators $t, g_1, g_2, \ldots, g_{n-1}$ and parameters $q, Q \in \mathbb{C}$ with defining relations:

(H1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, 2, \ldots, n-2$;

(H2) $g_i g_j = g_j g_i$, whenever $| i - j | \geq 2$;

(H3) $g_i^2 = (q-1) g_i + q$ for $i = 1, 2, \ldots, n-1$;

(H4) $t^2 = (Q-1)t + Q$;

(H5) $t g_1 t g_1 = g_1 t g_1 t$;

(H6) $t g_i = g_i t$ for $i \geq 2$.

Note that for $q = 1$ and $Q = 1$, (H3) and (H4) become $g_i^2 = 1$ and $t^2 = 1$, respectively. In this case the generators satisfy exactly the same relations as a set of simple reflections of the hyperoctahedral group, $\mathcal{H}_n$. It is known that $H_n(q, Q) \cong \mathbb{C}\mathcal{H}_n$ (the group algebra of $\mathcal{H}_n$) if q is not a root of unity and $Q \neq -q^s$ for $-n < s < n$. $\mathbb{C}\mathcal{H}_n$ is semisimple since it is a finite complex group algebra. This implies that $H_n(q, Q)$ is semisimple for generic values of q and Q. Furthermore the simple modules are in 1-1 correspondence with double partitions. Their decomposition rule and their dimensions are the same as for $\mathcal{H}_n$. [See Bourbarki, Groups et algebres de Lie IV, V, VI]

Hoefsmit [H] has written down explicit irreducible representations of $H_n(q, Q)$ for each ordered pair of Young diagrams. This demonstrates that the dimension over $\mathbb{C}(q, Q)$ of $H_n(q, Q)$ is $2^n n!$.

Note that (H1) and (H2) are the defining relations for the braid group $\mathcal{B}_n(A)$. So representations of $H_n(q, Q)$ also yield representations of $\mathcal{B}_n(A)$. Obviously we also obtain representations of the braid group of type B.

The Hecke algebras satisfies the following embedding of algebras $H_0 \subset H_1 \subset H_2 \subset \cdots$. The inductive limit of the Hecke algebra of type B is defined by $H_\infty(q, Q) := \bigcup_{n \geq 0} H_n(q, Q)$.

Double Coset Representatives

The fact that q is invertible in $\mathbb{C}(q, Q)$ implies that the generators g_i are also invertible in $H_n(q, Q)$. In fact, the inverse of the generators is given by

$$g_i^{-1} = q^{-1} g_i + (q^{-1} - 1)1 \in H_n(q, Q)$$

This implies that the following element is well-defined in $H_n(q, Q)$: $t_i' = g_i \cdots g_1 t g_1^{-1} \cdots g_i^{-1}$. We use this elements to define the set $\mathcal{D}_n$ as a subset of $H_n(q, Q)$. If $n = 1$, we let $\mathcal{D}_1 := \{1, t\}$. For $n \geq 2$ we have

$$\mathcal{D}_n := \{1, g_{n-1}, t_{n-1}'\}.$$

These are known as the distinguished double coset representatives of in $H_n(q, Q)$ with respect to $H_{n-1}(q, Q)$.

2.1 A surjective homomorphism

Throughout this section assume that q is not a root of unity. Fix a positive integer n and let $m,\ r_1 \in \mathbb{N}$ be such that $m > n$ and $r_1 > n$. For these integers we choose $\lambda = [m^{r_1}]$.

The representations of $H_n(q, Q)$ defined by Hoefsmit [H] depend on rational functions with denominators $(Qq^d + 1)$ where $d \in \{0, \pm 1, \dots, \pm(n-1)\}$. So if $Q = -q^{r_1+m}$ then $1 - q^{r_1+m+d} \neq 0$ as long as $d \neq -(r_1+m)$, which implies that all representations of $H_n(q, -q^{r_1+m})$ are well-defined as long as $r_1 + m > n$. Thus the specialized algebra, $H_n(q, -q^{r_1+m})$ is well-defined and semisimple.

By the *Hecke algebra of type* A_{n-1}, $H_n(q)$, we mean the free complex algebra with 1 and generators $\tilde{g}_1, \dots, \tilde{g}_{n-1}$ with parameter $q \in \mathbb{C}$ and defining relations given by relations $(H1) - (H3)$ of the Hecke algebra of type B. It is clear that $H_n(q) \subset H_n(q, Q)$.

$H_n(q)$ is semisimple whenever q is not a root of unity. If μ is a Young diagram with n boxes, then (π_μ, V_μ) denotes the irreducible representation of $H_n(q)$ indexed by μ. Here V_μ is the vector space with orthonormal basis given by $\{v_{t^\mu}\}$ where t^μ is a standard tableau of shape μ. These representations can be considered as q-analogs of Young's orthogonal representations of the Symmetric group (see [H] or [W1]).

A special element in $H_f(q)$ is the *full-twist* defined algebraically by $\Delta_f^2 := (\tilde{g}_{f-1} \cdots \tilde{g}_1)^f$. The full-twist is an element in the center of $H_f(q)$. The action of Δ_f^2 on (π_ν, V_ν) is described in the following lemma, the proof of which appears in [W2], pg. 261.

Lemma 2.1 *Let $\nu \vdash f$. Then the full-twist acts by a scalar α_ν on the irreducible representation (π_ν, V_ν) of $H_f(q)$ where $\alpha_\nu = q^{f(f-1)-\sum_{i<j}(\nu_i+1)\nu_j}$.*

In [W1], Cor. 2.3 Wenzl defined a special set of minimal idempotents of $H_f(q)$ indexed by the standard tableaux. The sum of these idempotents is 1. These minimal idempotents are well-defined whenever $H_f(q)$ is semisimple.

If $p \in H_f(q)$ is an idempotent then the *reduced algebra* of $H_f(q)$ with respect to this idempotent is $pH_f(q)p := \{pap \mid a \in H_f(q)\}$. Let $\lambda \vdash f$ and t^λ be a standard tableau of shape λ. Then p_{t^λ} is the minimal idempotent in $H_f(q)$ indexed by t^λ. Thus, the simple modules of the reduced algebra $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ are labeled by Young diagrams which contain the diagram λ.

In particular, if λ is a rectangular diagram, i.e. $[m^{r_1}]$, the reduced algebra $p_{t^\lambda} H_{f+1}(q) p_{t^\lambda}$ has only two nonzero irreducible modules indexed by partitions $[m+1, m^{r_1-1}]$ and $[m^{r_1}, 1]$, since these are the only partitions of $f+1$ which contain $[m^{r_1}]$.

Recall that in Section 1 we showed that there is a homomorphism $\tilde{\rho}_{f,n}$ from the braid group $\mathcal{B}_n(B)$ into the braid group $\mathcal{B}_{f+n}(A)$. The following lemma extends the homomorphism $\tilde{\rho}_{f,n}$ for braid groups (Proposition 1.1), to the corresponding Hecke algebras.

Lemma 2.2 *For a fixed integer n and $m, r_1 \in \mathbb{N}$, let $m > n$ and $r_1 > n$. Set $f = mr_1$ and assume $\lambda = [m^{r_1}]$ and $\gamma = [m^{r_1}, 1]$. Let α_λ and α_γ be as in Lemma 2.1. Choose a* *minimal idempotent* p_{t^λ} *in* $H_f(q)$. *We define a map* $\rho_{f,n}$ *for the generators of* $H_n(q, -q^{r_1+m})$ *as follows:*

$$\rho_{f,n}(1) = p_{t^\lambda}, \qquad \rho_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda} \Delta_f^{-2} \Delta_{f+1}^2 \qquad \text{and} \qquad \rho_{f,n}(g_i) = p_{t^\lambda} \tilde{g}_{i+f}$$

for $i = 1, \ldots, n-1$. *Then* $\rho_{f,n}$ *extends to a well-defined homomorphism of algebras,* $\rho_{f,n} : H_n(q, -q^{r_1+m}) \to p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$.

To prove this lemma it suffices to show that the $\rho_{f,n}$ preserves the relations of the Hecke algebra of type B, (H1)-(H6).

Theorem 2.3 *Let* f, n *be as in Lemma 2.2 and assume that* q *is not a root of unity. Then* $\rho_{f,n}$ *as defined in Lemma 2.2 is an onto homomorphism.*

The idea of the proof is to show that the irreducible representations of the reduced algebra are also irreducible representations of $H_n(q, -q^{r_1+m})$. In particular, we show that the reduced algebra is isomorphic to a quotient of the specialized Hecke algebra of type B.

It is well-known that there exits a duality between the quantum group $U_q(\mathfrak{sl}(r))$ and the Hecke algebra of type A. This duality is the quantum analogue of the Schur-Weyl duality between the general linear group, $GL(n)$, and the symmetric group, S_n, (see [D], [Ji1], and [Ji2]).

Corollary 2.4 *The action of the specialized Hecke algebra of type* B, $H_n(q, -q^{r_1+m})$ *and the diagonal action of* $U_q(\mathfrak{sl}(r))$ *on* $V_\lambda \otimes V^{\otimes n}$ *have the double centralizing property in* $End(V_\lambda \otimes V^{\otimes n})$.

The proof of this corollary follows immediately from the duality between $U_q(\mathfrak{sl}(r))$ and the Hecke algebra of type A and Theorem 2.3.

3 Markov Traces on the Hecke Algebra of type B

In this section we give the necessary background on Markov traces. We refer the reader to [GL] or [G] for the details.

A *trace* function on $H_\infty(q, Q)$ is a $\mathbb{C}(q, Q)$-linear map $\phi : H_\infty(q, Q) \longrightarrow \mathbb{C}(q, Q)$ such that $\phi(hh') = \phi(h'h)$ for all $h, h' \in H_\infty(q, Q)$. The weights we give in this paper correspond to a trace that satisfies the following definition.

Definition: Let $z \in \mathbb{C}(q, Q)$ and $tr : H_\infty(q, Q) \longrightarrow \mathbb{C}(q, Q)$ be an $\mathbb{C}(q, Q)$- linear map. Then tr is called a *Markov trace* (with parameter z) if the following conditions are satisfied:

(1) tr is a trace function on $H_\infty(q, Q)$;

(2) $tr(1) = 1$ (normalization);

(3) $tr(hg_n) = ztr(h)$ for all $n \geq 1$ and $h \in H_n(q,Q)$.

Geck and Pfeiffer in [GP] showed that a trace function on the Hecke algebra is uniquely determined by its value on basis elements corresponding to representatives of minimal length in the various conjugacy classes of the underlying Coxeter group. Also representatives of minimal length in the classes of Coxeter groups of classical types are of the form $d_1 \cdots d_n$, where d_i is a distinguished double coset representative of the $H_i(q,Q)$ with respect to $H_{i-1}(q,Q)$.

Let tr be a Markov trace with parameter z, and let $d_i \in \mathcal{D}_i$ for $i = 1, \cdots, n$. Then

$$tr(d_1 \cdots d_n) = z^a tr(t'_0 t'_1 \cdots t'_{b-1})$$

where a is the number of factors d_i which are equal to g_{i-1} and b is the number of factors which are equal to t'_{i-1}. Thus, tr is uniquely determined by its parameter z and the values on the elements in the set $\{t'_0 t'_1 \cdots t'_{i-1} \mid i = 1, 2, \cdots\}$.

Conversely, given $z, y_1, y_2, \cdots \in \mathbb{C}(q,Q)$ then there exist a unique Markov trace on $H_\infty(q,Q)$ such that $tr(t'_0 t'_1 \cdots t'_{k-1}) = y_k$ for all $k \geq 1$. For details on these results see [GL], Theorem 4.3.

We are particularly interested in the special case when $y_i = y^i$ for all $i \in \mathbb{N}$, in this case there is only two parameters. In particular, if d_i is a distinguished double coset representative then $tr(d_i x) = \xi tr(x)$ where $\xi = y$ or z. The proof of the following proposition is found in [GL].

Proposition 3.1 *Let $z, y \in \mathbb{C}(q,Q)$ and $tr : H_\infty(q,Q) \longrightarrow \mathbb{C}(q,Q)$ be a Markov trace with parameter z such that $tr(t'_0 t'_1 \cdots t'_{k-1}) = y^k$ for all $k \geq 1$ then*

$$tr(ht'_{n,0}) = ytr(h) \text{ for all } n \geq 0 \text{ and } h \in H_n(q,Q)$$

where $t'_{n,0} = g_n \cdots g_1 t g_1^{-1} \cdots g_n^{-1}$ or $g_n^{-1} \cdots g_1^{-1} t g_1 \cdots g_n$.

Notice that the converse is trivially true. We have computed the weight vector for this Markov trace on $H_n(q,Q)$.

3.1 The Weight Formula

In this section we define for every pair of partitions, (α, β), a rational function in q and Q, $W_{(\alpha,\beta)}(q,Q)$. We show that this function gives the weight formula for the Markov trace defined by Geck and Lambropolou [GL] for the Hecke algebra of type B. If we denote the weights by $\omega_{(\alpha,\beta)}$ then the Markov trace, tr, can be written as follows:

$$tr(x) = \sum_{(\alpha,\beta)\vdash n} \omega_{(\alpha,\beta)} \chi^{(\alpha,\beta)}(x), \tag{6}$$

where $x \in H_n(q,Q)$ and $\chi^{(\alpha,\beta)}$ is the character (the usual trace, i.e. sum of diagonal entries) of the irreducible representation of $H_n(q,Q)$ indexed by (α, β).

Let r_1, $r_2 \in \mathbb{N}$. First we define a rational function in q and Q for any arbitrary double partition (α, β) such that $l(\alpha) \leq r_1$ and $l(\beta) \leq r_2$. If $l(\alpha) = s < r_1$ then $\alpha_i = 0$ for $i = s+1, \cdots, r_1$, similarly for β. Let $r = r_1 + r_2$.

$$W_{(\alpha,\beta)}(q,Q) = q^{n(\alpha)+n(\beta)} \left(\frac{1-q}{1-q^r}\right)^{|\alpha|+|\beta|} \prod_{1\leq i<j\leq r_1} \frac{1-q^{\alpha_i-\alpha_j+j-i}}{1-q^{j-i}} \prod_{1\leq i<j\leq r_2} \frac{1-q^{\beta_i-\beta_j+j-i}}{1-q^{j-i}} \times \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{Qq^{\alpha_i-i}+q^{\beta_j-j}}{Qq^{-i}+q^{-j}} \tag{7}$$

Notice that this function can be expressed as a product of Schur functions

$$W_{(\alpha,\beta)}(q,Q) = q^{r_1|\beta|} \frac{s_\alpha(1,q,\cdots,q^{r_1-1}) s_\beta(1,q,\cdots,q^{r_2-1})}{s_{[1]}(1,q,\cdots,q^{r-1})^{|\alpha|+|\beta|}} \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{(1+Qq^{\alpha_i-\beta_j+j-i})}{(1+Qq^{j-i})}. \tag{8}$$

Recall that $s_\alpha(1,q,\cdots,q^{r_1-1}) = q^{n(\alpha)} \prod_{1\leq i<j\leq r_1} \frac{1-q^{\alpha_i-\alpha_j+j-i}}{1-q^{j-i}}$. It is clear from (8) that if $l(\alpha) > r_1$ or $l(\beta) > r_2$ then $W_{(\alpha,\beta)}(q,Q) = 0$.

Observation: Let $\mu = [\alpha_1+m, \cdots, \alpha_{r_1}+m, \beta_1, \cdots, \beta_{r_2}]$. We have the following equality:

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)} = W_{(\alpha,\beta)}(q, -q^{r_1+m}). \tag{9}$$

Notice that $W_{(\alpha,\beta)}(q,-q^{r_1+m})$ is well-defined since $r_1 + m > r_1$ always, and $W_{(\alpha,\beta)}(q,Q)$ is undefined for $Q = -q^{-s}$ where $1 - r_1 < s < r_2 - 1$. So $W_{(\alpha,\beta)}(q,-q^{r_1+m})$ is an analytic rational function.

In [W1] Wenzl showed that the weights of the Markov trace (with parameter $z = q^r \frac{(1-q)}{(1-q^r)}$) on the Hecke algebra of type A are given by the symmetric Schur function $s_{\mu,r}(q)$ as described in Section 1.

The reduced algebra $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ has p_{t^λ} as the identity. The Markov trace for the reduced algebra is given by the renormalized Markov trace of $H_{n+f}(q)$. By renormalization we mean that we must divide the trace by the trace of the identity, i.e. $tr(p_{t^\lambda}) = s_{\lambda,r}(q)$. Therefore, we have that $\frac{s_{\mu,r}(q)}{s_{\lambda,r}(q)}$ are the weights of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$. Notice that this implies that $W_{(\alpha,\beta)}(q,Q)$ becomes to the weights for the reduced algebra when $Q = -q^{r_1+m}$.

Lemma 3.2 *Let $g_{n-1} \in H_n(q,Q)$, $z = \frac{q^r(1-q)}{(1-q^r)}$ and $W_{(\alpha,\beta)}(q,Q)$ as defined in equation (7). Then for any $x \in H_n(q,Q)$*

$$tr(x) = \sum_{(\alpha,\beta)\vdash n} W_{(\alpha,\beta)}(q,Q) \chi^{(\alpha,\beta)}(x) \tag{10}$$

defines a well-defined trace which satisfies the Markov property, i.e. $tr(hg_{n-1}) = ztr(h)$, where $h \in H_{n-1}(q,Q)$.

Lemma 3.2 imply that the function $W_{(\alpha,\beta)}(q,Q)$ defined in equation (7) is a weight function for a Markov trace with parameter $z = q^r(1-q)/(1-q^r)$. At the beginning of this section we noted that a Markov trace on the Hecke algebra of type B is uniquely determined by a parameter z and by the values on the set $\{t'_0 t'_1 \cdots t'_{k-1} \mid k \geq 1\}$. We have the following theorem.

Theorem 3.3 *Let $r_1, r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$. If tr is a Markov trace on the Hecke algebra of type B, with parameter $z = q^r(1-q)/(1-q^r)$, such that $tr(t'_0 t'_1 \cdots, t'_{k-1}) = y^k$ for $k \geq 1$. Then the weights are given by $W_{(\alpha,\beta)}(q,Q)$ as defined in equation (7) with $y = (q^{r_2}Q+1)(1-q^{r_1})/(1-q^r) - 1$.*

The proof of this theorem follows from the fact that every Markov trace on the Hecke algebra of type A induces a Markov trace on the Hecke algebra of type B which satisfies the condition $tr(t'_n x) = ytr(x)$ for every $x \in H_n(q,Q)$. In particular, we show that the trace induced is the one described in Proposition 3.1.

4 Markov Trace for the Hecke algebra of type D

The easiest way to study Markov traces on the Hecke algebras of type D, denoted by H_n^D, is by embedding these algebras into those for type B, denoted in this section by H_n^B.

Hoefsmit [H] observed that in order to obtain an embedding of $H_n^D(q)$ into $H_n^B(q,Q)$ we have to set the parameter Q equal to 1. In this case we have $t^2 = 1$. The Hecke algebra of type D is generated by $u = tg_1t$, $g_1, \cdots, g_{n-1}$ satisfying the relations of the Hecke algebra of type B and u satisfies $u^2 = (q-1)u + q$ and commutes with all other generators.

We have $H_n^D \subset H_n^B$ for all n; then $H_\infty^D = \bigcup_{n>1} H_n^D \subset H_\infty^B$. Geck [G] showed that the restriction of a Markov trace on H_∞^B is a Markov trace on H_∞^D and both have the same parameter. Furthermore, he showed that every Markov trace on H_∞^D can be obtained in this way.

From Hoefsmit [H] we know that the simple components for H_n^D are indexed by double partitions (α,β). If $\alpha \neq \beta$ we have that the H_n^B-modules $V_{(\alpha,\beta)}$ and $V_{(\beta,\alpha)}$ are simple, equivalent H_n^D-modules. And if $\alpha = \beta$ we have that the H_n^B-module $V_{(\alpha,\alpha)}$ decomposes into two simple nonequivalent H_n^D-modules, i.e. $V_{(\alpha,\alpha)_i}$ with $i = 1,2$.

Proposition 4.1 *Let $r_1, r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$. Then the weight formula for the Markov trace on the Hecke algebra of type D with parameters $z = q^r\frac{(1-q)}{(1-q^r)}$ and $y = \frac{(Qq^{r_2}+1)(1-q^{r_1})}{(1-q^r)} - 1$ is given as follows:*

$$W^D_{(\alpha,\beta)}(q) = W_{(\alpha,\beta)}(q,1) + W_{(\beta,\alpha)}(q,1), \quad \text{if } \alpha \neq \beta$$

and

$$W^D_{(\alpha,\alpha)_i}(q) = W_{(\alpha,\alpha)}(q,1), \text{ for } i = 1,2 \quad \text{if } \alpha = \beta$$

where $W_{(\alpha,\beta)}(q,1)$ denote the weight of the Hecke algebra of type B evaluated at $Q = 1$.

5 The Hecke algebra at roots of unity

We observe that the weights, $W_{(\alpha,\beta)}$, for the Hecke algebra of type B defined in (7) are equal to zero whenever $l(\alpha) > r_1$ or $l(\beta) > r_2$, or when $Q = q^k$, $k \in \mathbb{N}$ and q is a root of unity. In this section we want to describe semisimple quotients of the Hecke Algebra of type B when $Q = -q^k$, k a positive integer and q an l-th root of unity. These quotients can be obtained from the Hecke algebra of type B modulo the annihilator of the Markov trace.

In Section 2.1 we defined for $r_1, m \in \mathbb{N}$ such that $r_1 > n$ and $m > n$ an onto homomorphism from the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, onto a reduced Hecke algebra of type A, $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$, where p_{t^λ} is an idempotent indexed by t^λ, a standard tableau corresponding to $\lambda = [m^{r_1}]$.

We have shown that this homomorphism is well-defined and onto when q is a root of unity and $Q = -q^{m+r_1}$ if we map into a well-defined quotient of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$. Notice that if $Q = 1$ we have that the Hecke algebra of type B reduces to the Hecke algebra of type D. So assume that $Q \neq 1$.

Wenzl [W1] has shown that for q an l-th root of unity there exists a quotient of the Hecke algebra of type A which is semisimple, denote this quotient by $H_n^{(l)}(q)$. The simple components of this quotient are indexed by Young diagrams, μ, called *(r,l) diagrams*, these diagrams have at most r rows and must satisfy $\mu_1 - \mu_r \leq l - r$. Wenzl also showed that there exist well-defined minimal idempotents of $H_n^{(l)}(q)$. We denote these idempotents by $p_{t^\lambda}^{(l)}$, where λ is an (r,l) diagram. Throughout the sequel we will be mainly interested in the case when $\lambda = [m^{r_1}]$. Notice that λ is an (r,l)-diagram if $m \leq l - r$. Now we choose a Young tableaux t^λ such that $p_{t^\lambda}^{(l)}$ is well-defined. Define a map, $\hat{\rho}_{f,n}$, from the generators of $H_n(q, -q^{r_1+m})$ into the reduced algebra $p_{t^\lambda}^{(l)} H_{n+f}^{(l)}(q) p_{t^\lambda}^{(l)}$ as follows:

$$\hat{\rho}_{f,n}(1) = p_{t^\lambda}^{(l)}, \quad \hat{\rho}_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda}^{(l)} \Delta_f^{-2} \Delta_{f+1}^2, \quad \text{and } \hat{\rho}_{f,n}(g_i) = p_{t^\lambda}^{(l)} g_{f+i} \quad \text{for } i = 1, \cdots, n-1.$$

Theorem 5.1 *Let m, r_1, r_2, $l \in \mathbb{N}$, $l \geq 4$ and $r = r_1 + r_2 < l$. Assume q is a primitive l-th root of unity and $Q = -q^{m+r_1}$ with $r_1 < m + r_1 \leq l - r_2$. Then $\tilde{\rho}_{f,n}$ as defined above is a nontrivial onto homomorphism.*

This theorem implies that there exists a quotient of $H_n(q, -q^{r_1+m})$ at roots of unity which is semisimple for all n.

Now we define a subset of the set of double partitions. We will show that the quotient of $H_n(q, -q^{r_1+m})$ at roots of unity which is isomorphic to the image of $\hat{\rho}_{f,n}$ is indexed by the ordered pairs of Young diagrams which we now define.

Definition: Let $m, l, r_1, r_2 \in \mathbb{N}$ with $r = r_1 + r_2 \leq l - 1$. A pair of Young diagrams (α, β) such that $l(\alpha) \leq r_1$ and $l(\beta) \leq r_2$ is called a *(m,l,r)-diagram* if

(1) $\alpha_1 - \beta_{r_2} \leq l - r$ and

(2) $\alpha_{r_1} - \beta_1 \geq -m$.

Let $\Gamma_n(l, m, r)$ denote the set of all (m, l, r)-diagrams with n boxes.

Observation: For every (m, l, r) diagram, (α, β), there is an (r, l) diagram $\mu = [m + \alpha_1, \cdots, m + \alpha_{r_1}, \beta_1, \cdots, \beta_{r_2}]$.

The following is a corollary of Theorem 5.1.

Corollary 5.2 *Let $r_1, r_2, m, l \in \mathbb{N}$. Assume q is a primitive l-th root of unity with $l \geq 4$ and let $Q = -q^{r_1+m}$, with $m \leq l - r$. Then there exists for every $(\alpha, \beta) \in \Gamma(l, m, r)$ a semisimple irreducible representation $\pi^{(l)}_{(\alpha,\beta)}$ of $H_n(q, -q^{r_1+m})$. We obtain a semisimple representation which is not faithful in general.*

$$\pi^{(l)}_n : x \in H_n(q, -q^{r_1+m}) \longrightarrow \bigoplus_{(\alpha,\beta)\in\Gamma_n(l,m,r)} \pi^{(l)}_{(\alpha,\beta)}(x) \tag{11}$$

Representations corresponding to different (m, l, r)-diagrams are not equivalent.

To prove this corollary we show that for every (m, l, r) diagram there is a well-defined irreducible representation of $H_n(q, -q^{r_1+m})$. We use the previous observation to show that there is an equivalence between representations $\pi^{(l)}_{(\alpha,\beta)}$ and $\pi^{(l)}_\mu$ where $\mu = [m + \alpha_1, \cdots, m + \alpha_{r_1}, \beta_1, \cdots, \beta_{r_2}]$.

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