A ROOT SYSTEM APPROACH TO MINIMAL LEFT COSET REPRESENTATIVES (EXTENDED ABSTRACT)

PAOLO PAPI

ABSTRACT. We provide a criterion affording the minimal left coset representative w^I of an element w of a Coxeter group W through the combinatorics of the reflection representation of W. We analyze in greater detail the case of finite and affine Weyl groups of the $ABCD$-series, realized as groups of $\mathbb{Z}$-permutations, obtaining explicit descriptions of the inversions, inversion tables and minimal left coset representatives.

§1 NOTATION AND STATEMENT OF THE MAIN RESULTS

Let (W, S) be a Coxeter system with length function ℓ; for $I \subseteq S$ denote as usual by W_I the parabolic subgroup corresponding to I and by W^I the set of minimal left coset representatives:

$$W^I = \{w \in W \mid \ell(ws) > \ell(w) \ \forall\ s \in I\}.$$

It is well known [5, 1.10] that any element $w \in W$ can be uniquely decomposed as $w = w^I w_I$, $w^I \in W^I$, $w_I \in W_I$.

In this paper we provide a simple algorithm affording w^I. Our criterion follows easily from well-known facts about W^I, and it can be recast in a nice combinatorial fashion, in the framework of the geometric representation of W.

We then analyze in detail the case of finite and affine Weyl groups of classical type, discussing the relationships between the permutation realization and the geometric representation. This analysis affords an explicit description of the class inversions and the inversion table of an element $w \in W$ (cf. [4]).

Finally, as an application of our general criterion and of the previous analysis, we describe W^I in the permutation realization and we give concrete algorithms to obtain w^I from w. In this extended abstract we confine ourselves to deal with cases $\tilde{A}$, $\tilde{B}, \tilde{C}$. Details will appear elsewhere.

Let V be the space of the geometric representation of W [5, 5.3]; introduce the canonical root system Δ of W by taking a basis $\Pi = \{\alpha_s \mid s \in S\}$ of V (the *simple* roots) and by considering the set of W- orbits of Π: $\Delta = W\Pi$. Then any root is positive or negative, i.e. it can be written as a linear combination of simple roots with (real) coefficients of the same sign: $\Delta = \Delta^+ \cup \Delta^-$. We write $\alpha > 0$ (resp. $\alpha < 0$) to mean $\alpha \in \Delta^+$ (resp. $\alpha \in \Delta^-$); moreover, for $\beta \in \Delta$ we denote by

1991 *Mathematics Subject Classification.* Primary: 20F55; Secondary 17B20.
Key words and phrases. Coxeter group, Weyl group, minimal left coset representative.

s_β the reflection in β . Notice that in the context of the geometric representation roots correspond to reflections (W-conjugate of the elements of S); we can therefore consider the analog of the set of (left) descents for $w \in W$:

$$L(w) = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) < 0\}.$$

Recall that if $w = s_1 \cdots s_n,\ s_i \equiv s_{\beta_i},\ \beta_i \in \Pi$ is a reduced expression, then

$$L(w) = \{\beta_1, s_1(\beta_2), \dots, s_1 \cdots s_{n-1}(\beta_n)\}.$$

The sets $L(w)$ are connected with the weak Bruhat order $\preceq$ [3, §3] as follows: $u \preceq v$ iff $L(u) \subseteq L(v)$, iff there exists $w \in W$ such that $v = uw,\ \ell(v) = \ell(u) + \ell(w)$. Moreover they can be characterized from a combinatorial point of view.

Proposition 1. *A finite subset $X \subseteq \Delta^+$ is of the form $L(w),\ w \in W$ if and only if it satisfies the following two conditions:*

(1) *if $\alpha,\ \beta \in X,\ q\alpha + r\beta \in \Delta, q, r \in \mathbb{R}_{\geq 0}$, then $q\alpha + r\beta \in X$.*
(2) *if $q\alpha + r\beta \in X$ and $\alpha,\ \beta \in \Delta^+,\ q, r \in \mathbb{R}_{\geq 0},\ \beta \notin X$, then $\alpha \in X$.*

Moreover, w is unique.

This result can be found in [8; §2, Remark] and in [3, Prop.3] (in a slightly different formulation). For finite and affine Weyl groups not of type $\tilde{A}_1$ a simpler statement holds: we can obtain this statement from the previous one replacing $q,\ r$ by 1 in conditions (1), (2) (see [7; §3, Theorem]). In [7] it is also shown how to recover w from the combinatorial conditions stated in the proposition: it is easy to prove that $\xi\ \in \Pi$ if and only if ξ is *indecomposable*, i.e. there do not exist $\gamma, \beta \in \Delta^+,\ q, r \in \mathbb{R}_{\geq 0}$ such that $\xi = q\gamma + r\beta$; then if X verifies the two conditions of the proposition, by (2) it contains a simple root α. One proves that $s_\alpha(X \setminus \{\alpha\})$ is a set of positive roots which still verifies (1), (2): therefore a reduced expression for $w = s_\alpha \cdots$ is inductively determined. In particular, describing $L(w)$ allows us to recover concretely w.

Now we can state the main result. For $I \subseteq S$ denote by Δ_I the root subsystem of Δ generated by $\Pi_I = \{\alpha_s \mid s \in I\}$.

Theorem 1. *Let (W, S) be a Coxeter system and $I \subseteq S$. Given $w \in W$, let $w = w^I w_I,\ w^I \in W^I,\ w_I \in W_I$ be its decomposition; then*

$$L(w^I) = L(w) \setminus \left(w(\Delta_I^-) \cap \Delta^+\right)$$
$$L(w_I^{-1}) = L(w^{-1}) \cap \Delta_I.$$

Moreover

$$w \in W^I \iff L(w^{-1}) \cap \Pi_I = \emptyset. \qquad (*)$$

Example. Consider, for Δ of type H_4 and $I = \{2, 3\},\ w = s_1 s_2 s_4 s_1 s_2 s_3 s_1$; then

$$L(w) = \{\alpha_1,\ \tfrac{1+\sqrt{5}}{2}\alpha_1 + \alpha_2,\ \alpha_4,\ \tfrac{1+\sqrt{5}}{2}(\alpha_1 + \alpha_2),\ \alpha_1 + \tfrac{1+\sqrt{5}}{2}\alpha_2,$$
$$\tfrac{3+\sqrt{5}}{2}(\alpha_1 + \alpha_2) + \alpha_3 + \alpha_4,\ \alpha_2\},$$
$$L(w^I) = \{\alpha_2,\ \tfrac{1+\sqrt{5}}{2}\alpha_1 + \alpha_2,\ \alpha_4,\ \tfrac{1+\sqrt{5}}{2}(\alpha_1 + \alpha_2),\ \alpha_1 + \tfrac{1+\sqrt{5}}{2}\alpha_2\},$$
$$L(w_I^{-1}) = \{\alpha_3, \alpha_2 + \alpha_3\}.$$

Therefore $w^I = s_2 s_1 s_2 s_1 s_4,\ w_I = s_2 s_3$.

§2 Affine Weyl groups of classical type as permutation groups

In this section we analyze in detail the relationships between the permutation realization and the geometric representation of the affine Weyl groups of classical type. The permutation realization was introduced by Lusztig in [6]; we remind these facts following the approach of [4]. Then we describe the action of these permutation groups on the affine root system. This analysis affords an easy and explicit description of the "affine inversions" and of the inversion table.

Denote by T_n the translation by $n \in \mathbb{Z}$ and by R_m the reflection with respect to $m \in \frac{\mathbb{Z}}{2}$ (regarded as mappings $\mathbb{Z} \to \mathbb{Z}$). The following facts are well-known.

(1) A group W of mappings $\mathbb{Z} \to \mathbb{Z}$ consisting of translations or reflections (a *rigid group* [4, §4]) is generated by either one translation or one reflection or two reflections.

(2) Any Weyl group (possibly affine) of classical type can be realized as a group of locally finite $\mathbb{Z}$-permutations[1] commuting with a rigid group G.

Indeed type A is obtained taking G trivial, type B is obtained when G is generated by one reflection (we take $G = \langle R_0 \rangle$).

We say that a locally finite $\mathbb{Z}$-permutation w is *locally even at* $m \in \mathbb{Z}$ if the set $\{k \in \mathbb{Z} \mid k < m, w(k) > m\}$ has an even number of elements. Then type D is obtained from type B by considering the permutations which are locally even at 0.

We now describe in greater detail the remaining cases, i.e. the affine Weyl groups.

$\mathbf{\tilde{A}_{n-1}}$. W is obtained when $G = \langle T_n \rangle$; it is easy to prove that W is faithfully represented as the group of *affine permutations* [2]:

$$W \cong \left\{ \pi : \mathbb{Z} \leftrightarrow \mathbb{Z} \mid \pi(t+n) = \pi(t) + n \ \forall t \in \mathbb{Z}, \ \ \sum_{t=1}^{n} \pi(t) = \frac{n(n+1)}{2} \right\}.$$

$\mathbf{\tilde{C}_n}$. W can be realized as the group of $\mathbb{Z}$-permutations commuting with the reflections w.r.t. $0, n+1$ (i.e. $G = \langle R_0, R_{n+1} \rangle$); explicitly:

$$W \cong \{ w : \mathbb{Z} \leftrightarrow \mathbb{Z} \mid w(i) + w(-i) = 0, \ w(i) + w(2n+2-i) = 2n+2 \ \forall i \in \mathbb{Z} \}.$$

$\mathbf{\tilde{B}_n}$, $\mathbf{\tilde{D}_n}$. W is the subgroup of $\tilde{C}_n$ formed by the permutations which are locally even at 0 (for $\tilde{B}_n$) or at 0 and $n+1$ (for $\tilde{D}_n$).

In any case, an element $w \in W$ is determined by the images of $[n] \equiv \{1, \dots, n\}$; we call $w([n])$ the *fundamental n-tuple* of w and we denote it by $[w(1), \dots, w(n)]$. Let us check the previous statement. For type $\tilde{A}$ take $i \in \mathbb{Z}$ and write it as $i = kn + j$, $j \in [n]$; then $w(i) = kn + w(j)$. In the other cases remark that points corresponding to multiples of the mirror positions are fixed points; if instead $m \neq k(n+1)$, m can be uniquely written as $m = (2n+2)h \pm r$ for $h \in \mathbb{Z}$ and $r \in [n]$: then $w(m) = (2n+2)h \pm w(r)$.

[1] i.e., moving a finite number of values from the negative to the positive $\mathbb{Z}$-axis.

Our choice of Coxeter generators is displayed in the following list.

$\tilde{A}_{n-1}$ $s_i = [1, \dots, i+1, i, \dots, n], i = 1, \dots, n-1,\ s_n = [0, 2, \dots, n-1, n+1]$.

$\tilde{C}_n$ $s_1 = [-1, 2, \dots, n],\ s_i = [1, \dots, i, i-1, \dots, n]\ i = 2, \dots, n,$
$s_{n+1} = [1, 2, \dots, n-1, n+2]$.

$\tilde{B}_n$ $s_1 = [-2, -1, 3, \dots, n],\ s_i = [1, \dots, i, i-1, \dots, n]\ i = 2, \dots, n,$
$s_{n+1} = [1, 2, \dots, n-1, n+2]$.

$\tilde{D}_n$ $s_1 = [-2, -1, 3, \dots, n],\ s_i = [1, \dots, i, i-1, \dots, n]\ i = 2, \dots, n,$
$s_{n+1} = [1, 2, \dots n-2, n+2, n+3]$.

We want to express the rules for the action of an element w in the permutation realization of W on its root system. Since we are considering only the case of the classical groups, we prefer to adopt the usual realization of finite root systems with multiple lengths rather than the one used in section §1.

Recall that if the affine root system Δ_{aff} is obtained by extending an irreducible finite root system Δ, the choice of a positive system $\Delta^+ \subset \Delta$ induces a corresponding choice for Δ^+_{aff}:

$$\Delta^+_{aff} = (\Delta^+ + \mathbb{N}\delta) \cup (-\Delta^+ + \mathbb{Z}_+\delta),$$

where $\delta = \theta + \alpha_*$, θ is the highest root of Δ and α_* is the simple root corresponding to the "new" vertex $*$ in the extended Dynkin diagram of Δ. For $\alpha \in \Delta$ set:

$$\underline{\alpha} := \begin{cases} \{\alpha + n\delta \mid n \in \mathbb{N}\} & \text{if } \alpha \in \Delta^+, \\ \{\alpha + m\delta \mid m \in \mathbb{Z}^+\} & \text{if } -\alpha \in \Delta^+. \end{cases}$$

Since δ is fixed by W, the action of W on Δ_{aff} is completely determined by the action on Δ; it will be convenient for our goals to display the positive roots of Δ in a $n \times n$ square matrix (β_{ij}) (with entries in $\Delta^+ \cup \{0\}$). The indexing of this matrix is chosen according to the euclidean realization of the root system; we use the following convention about Δ: the long (resp. short) simple root of $\Delta \cong C_n$ (resp. $\Delta \cong B_n$) is α_1.

Type A:

$$\beta_{ij} = \begin{cases} \varepsilon_i - \varepsilon_j = \sum_{i \le k \le j-1} \alpha_k & \text{if } i < j \\ 0 & \text{if } i \ge j. \end{cases}$$

Type C:

$$\beta_{ij} = \begin{cases} \varepsilon_j - \varepsilon_i = \sum_{i < k \le j} \alpha_k & \text{if } i < j \\ 2\varepsilon_i = \alpha_1 + 2\sum_{2 \le k \le i} \alpha_k & \text{if } i = j \\ \varepsilon_j + \varepsilon_i = \alpha_1 + 2\sum_{2 \le k \le j} \alpha_k + \sum_{j < k \le i} \alpha_k & \text{if } i > j. \end{cases}$$

Type B:

$$\beta_{ij} = \begin{cases} \varepsilon_j - \varepsilon_i = \sum_{i < k \le j} \alpha_k & \text{if } i < j \\ \varepsilon_i = \sum_{1 \le k \le i} \alpha_k & \text{if } i = j \\ \varepsilon_j + \varepsilon_i = 2\sum_{1 \le k \le j} \alpha_k + \sum_{j < k \le i} \alpha_k & \text{if } i > j. \end{cases}$$

Type D:

$$\beta_{ij} = \begin{cases} \varepsilon_j - \varepsilon_i = \sum_{i<k\leq j} \alpha_k & \text{if } 1 \leq i < j \\ 0 & \text{if } i = j \\ \varepsilon_1 + \varepsilon_i = \alpha_1 + \widehat{\alpha_2} + \sum_{3\leq k\leq i} \alpha_k & \text{if } i > j = 1 \\ \varepsilon_i + \varepsilon_j = \alpha_1 + \alpha_2 + 2\sum_{3\leq k\leq j} \alpha_k + \sum_{j<k\leq i} \alpha_k & \text{if } i > j > 1. \end{cases}$$

Now we are able to express the W-action on Δ_{aff}; this kind of analysis has already been done in [9] for type $\tilde{A}$. Let us recall this result (in a slightly different formulation) before writing down the action in the other cases.

Suppose $w = [x_1, \ldots, x_n] \in W$: then $w = [\pi(1) + k_1 n, \ldots, \pi(n) + k_n n]$ where $\pi \in S_n$ (S_n being the symmetric group on n letters) and $k_i \in \mathbb{Z}$, $i = 1, \ldots, n$; indeed $\pi = [\overline{x}_1, \ldots, \overline{x}_n]$, where $\overline{x}_i$ denotes the residue class of x_i modulo n ($1, \ldots, n$ having been chosen as representatives for $\mathbb{Z}/n\mathbb{Z}$). For $\pi \in S_n$ and $i < j$, set $a = min\{|\pi(i)|, |\pi(j)|\}$, $b = max\{|\pi(i)|, |\pi(j)|\}$; then we have $\pi(\beta_{ij}) = \pm\beta_{ab}$, where we take $+$ iff $w(i) < w(j)$. Therefore the action of $w \in W$ on Δ_{aff} is given as follows. For $1 \leq i < j \leq n$, let q_i, r_i, q_j, r_j be the unique integers such that $w(i) = q_i n + r_i$, $w(j) = q_j n + r_j$, $(q_i, q_j \in \mathbb{Z}, r_i, r_j \in [n])$. Then

$$w(\pm\beta_{ij} + k\delta) = \begin{cases} \pm\beta_{r_i r_j} + (\pm(q_j - q_i) + k)\,\delta & \text{if } r_i < r_j \\ \mp\beta_{r_j r_i} + (\pm(q_j - q_i) + k)\,\delta & \text{if } r_i > r_j. \end{cases}$$

where we take simultaneously the upper or the lower sign.

Let now W be a Weyl group of type $\tilde{B}_n$; for $\sigma \in H_n$, the hyperoctahedral group, set $a = min\{|\sigma(i)|, |\sigma(j)|\}$, $b = max\{|\sigma(i)|, |\sigma(j)|\}$. Then

$$\sigma(\beta_{ii}) = \pm\beta_{|\sigma(i)|\,|\sigma(i)|} \quad \begin{cases} + & \text{if } \sigma(i) > 0 \\ - & \text{if } \sigma(i) < 0 \end{cases}$$

$$i < j: \qquad \sigma(\beta_{ij}) = \begin{cases} \pm\beta_{ab} & \text{if } \sigma(i)\sigma(j) > 0 \quad \begin{cases} + & \text{if } \sigma(i) < \sigma(j) \\ - & \text{if } \sigma(i) > \sigma(j) \end{cases} \\ \beta_{ba} & \text{if } \sigma(j) > 0 > \sigma(i) \\ -\beta_{ba} & \text{if } \sigma(i) > 0 > \sigma(j). \end{cases}$$

$$i > j: \qquad \sigma(\beta_{ij}) = \begin{cases} \beta_{ba} & \text{if } \sigma(i) > 0, \sigma(j) > 0 \\ -\beta_{ba} & \text{if } \sigma(i) < 0, \sigma(j) < 0 \\ \pm\beta_{ab} & \text{if } \sigma(i)\sigma(j) < 0 \quad \begin{cases} + & \text{if } \sigma(i) + \sigma(j) > 0 \\ - & \text{if } \sigma(i) + \sigma(j) < 0. \end{cases} \end{cases}$$

As in the previous case, this suffices to recover the action of the corresponding affine Weyl group on Δ_{aff}. Indeed take $w = [x_1, \ldots, x_n] \in W$ and recall from [4, 8.1.4] that $w = [\sigma(1) + (2n+2)k_1, \ldots, \sigma(n) + (2n+2)k_n]$ with $\sigma = [\sigma(1), \ldots, \sigma(n)]$ in H_n and $k_i \in \mathbb{Z}$, $1 \leq i \leq n$. In fact $\sigma = [\overline{x}_1, \ldots, \overline{x}_n]$, where $\overline{x}_i$ denote the residue class of x_i modulo $2n+2$ ($\pm 1, \ldots, \pm n$ having been chosen as (part of a) system of representatives for $\mathbb{Z}/(2n+2)\mathbb{Z}$).

Take $i, j \in [n]$ and write $w(i) = (2n+2)q_i + r_i$, $w(j) = (2n+2)q_j + r_j$, $r_i, r_j \in \pm[n]$; then

$$\begin{aligned} w(\pm\beta_{ii} + k\delta) &= \pm\sigma(\beta_{ii}) + (\pm q_i + k)\delta \\ w(\pm\beta_{ij} + k\delta) &= \pm\sigma(\beta_{ij}) + (\pm(q_j - q_i) + k)\,\delta && \text{if } i < j \\ w(\pm\beta_{ij} + k\delta) &= \pm\sigma(\beta_{ij}) + (\pm(q_j + q_i) + k)\,\delta && \text{if } i > j. \end{aligned}$$

For cases $\tilde{C}, \tilde{D}$ the formulas for the action of the finite Weyl group on Δ are the same as above, whereas those expressing the action of the affine Weyl group on Δ_{aff} become

$$\begin{aligned} w(\pm\beta_{ij} + k\delta) &= \pm\tau(\beta_{ij}) + (\pm(q_j - q_i) + k)\,\delta && \text{if } i < j \\ w(\pm\beta_{ij} + k\delta) &= \pm\tau(\beta_{ij}) + (\pm(q_j + q_i) + k)\,\delta && \text{if } i \geq j \end{aligned}$$

for type $\tilde{C}_n$ $(\tau \in H_n)$ and

$$\begin{aligned} w(\pm\beta_{ij} + k\delta) &= \pm\gamma(\beta_{ij}) + (\pm(q_j - q_i) + k)\,\delta && \text{if } i < j \\ w(\pm\beta_{ij} + k\delta) &= \pm\gamma(\beta_{ij}) + (\pm(q_j + q_i) + k)\,\delta && \text{if } i > j \end{aligned}$$

for type $\tilde{D}_n$ ($\gamma \in H_n^{even}$, where H_n^{even} is the subgroup of H_n consisting of the elements which move an even number of elements from $[n]$ to $-[n]$).

Let W be a Weyl group of classical type, realized as above as group of $\mathbb{Z}$-permutations commuting with a rigid group G.

Definitions. 1. *Consider $w \in W$. A (generalized) inversion is a pair $(k, i) \in \mathbb{Z} \times \mathbb{Z}$ with $k > i$, $w^{-1}(k) < w^{-1}(i)$.*
2. *Let (k, i) be an inversion for w such that neither k nor i are fixed points for G. A class inversion $\langle k, i \rangle$ (cf. [4, 4.2]) is the orbit of an such an inversion (k, i) under the diagonal G-action on $\mathbb{Z} \times \mathbb{Z}$.*

Theorem 2. *Denote by $\mathcal{I}_w$ the set of class inversions for w.*

(1) *Let $(Inv_w(i, j))$ the inversion table of w [4, §8]. Then*

$$Inv_w(i, j) = \mp|L(w) \cap \underline{\pm\beta_{ij}}|.$$

(2) *The map $\phi : L(w) \to \mathcal{I}_w$*

$$\begin{aligned} \beta_{ij} + m\delta &\mapsto \langle mn' + j, i \rangle && \text{if } i < j \\ \beta_{ij} + m\delta &\mapsto \langle mn' + i, -j \rangle && \text{if } i > j \\ \beta_{ii} + m\delta &\mapsto \langle mn' + i, -i \rangle \\ -\beta_{ij} + m\delta &\mapsto \langle mn' + i, j \rangle && \text{if } i < j \\ -\beta_{ij} + m\delta &\mapsto \langle mn' - j, i \rangle && \text{if } i > j \\ -\beta_{ii} + m\delta &\mapsto \langle mn' - i, i \rangle \end{aligned}$$

establishes a canonical bijection between $L(w)$ and $\mathcal{I}_w$. Here $n' = n$ for type $\tilde{A}_{n-1}$, whereas $n' = 2n + 2$ for types $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$.

Remark. Note that the second statement of the Theorem affords a canonical choice of representatives for class inversions.

Sketch of proof. The first part follows from a direct computation; in turn this computation is an easy consequence of our previous analysis. For the second statement it suffices to prove that ϕ is well-defined and surjective. The first fact follows from the definition. For the other we have to chose a suitable normalization among representatives of the same class inversion. This can be done in the following way. Take $\langle\bar{k},\bar{i}\rangle \in \mathcal{I}_w$ and select a representative (k,i), by requiring that $i = max\{h \in \pm[n] \mid (k,h) \in \langle\bar{k},\bar{i}\rangle$ for some $k\}$; this is always possible since, by assumption, $\bar{i}$ is not a mirror position, therefore its residue class mod n' belongs indeed to $\pm[n]$. Write k as $k = mn' + j$; note that either j or i is positive. Since $i,j \in \pm[n]$, we can build up $\xi \in \Delta^+_{aff}$ such that $\phi(\xi) = (k,i)$: e.g., if $0 < i < j$, then $\xi = \beta_{ij} + m\delta$ works. $\square$

§3 Minimal left coset representatives in affine Weyl groups of classical type

In the following we describe minimal left coset representatives for affine Weyl groups of classical type using the permutation realization. For related results, see [1], [10]. As announced in the introduction, we omit the analysis of the $\tilde{D}_n$ case, which can however be developed along the same lines of the other affine cases.

$\boldsymbol{\Delta_{\mathrm{aff}} \cong \tilde{\mathrm{A}}_{n-1}}$.

Label by n the extra vertex $*$ in the extended Dynkin diagram of A_{n-1}. Assume that $I = I_1 \sqcup \ldots \sqcup I_k$ is the irreducible decomposition of I, with $I_r = [i_r, j_r]$, $1 \le i_r \le j_r \le n-1$ for $r = 2,\ldots,k$ and $j_r < i_{r+1}$ for $r = 2,\ldots,k-1$. Moreover $I_1 = [1,j_1] \sqcup [i_1, n-1] \sqcup \{n\}$, $1 \le j_1 < i_2$, $j_k < i_1 \le n-1$ ($I_1 \neq \emptyset$ iff $n \in I$).

Proposition 2.

$$\begin{aligned} W^I = \{w \in W \mid & w(i_1) < \cdots < w(n),\ w(1) < \cdots < w(j_1+1), \\ & w(n) - w(1) < n, \\ & w(i_r) < \cdots < w(j_r+1),\ 2 \le r \le k\} \end{aligned}$$

The first two conditions should be considered only if $I_1 \neq \emptyset$. Given $w \in W$, w^I is obtained in the following way: $w^I(i_r),\ldots,w^I(j_r+1)$, $2 \le r \le k$ are given by the increasing arrangement of the sets $\{w(i_r),\ldots,w(j_r+1)\}$, $2 \le r \le k$. If $I_1 \neq \emptyset$, to get $w^I(i_1),\ldots,w^I(n),w^I(1),\ldots,w^I(j_1+1)$ proceed as follows:

(1) *Consider $\{w(i_1),\ldots,w(n)\}$, $\{w(1)\ldots w(j_1+1)\}$ and let $(y_{i_1},\ldots,y_n)$, $(z_1,\ldots,z_{j_1+1})$ be the corresponding increasing arrangements.*
(2) *If $y_n - z_1 < n$, then*

$$(w^I(i_1),\ldots,w^I(n),w^I(1),\ldots,w^I(j_1+1)) = (y_{i_1},\ldots,y_n,z_1,\ldots,z_{j_1+1}).$$

Otherwise replace y_n, z_1 by z_1+n, y_n-n respectively and go back to step (1).

Remark. For the case of the maximal parabolic corresponding to $J = [n-1]$, w^J is obtained by arranging $\{w(1), \dots, w(n)\}$ in increasing order: e.g., $w = [5, -2, 3, 4] = s_2 s_3 s_4 s_3 s_2 s_1$, $w^J = [-2, 3, 4, 5] = s_2 s_3 s_4$. In particular we obtain, as in [2, Prop. 3.5], $W^J = \{w \in | \, w(1) < w(2) < \dots < w(n)\}$.

$\Delta_{\mathbf{aff}} \cong \tilde{\mathbf{C}}_{\mathbf{n}}$

Suppose that $I = I_1 \sqcup \dots \sqcup I_k$ is the irreducible decomposition of I, $I_r = [i_r, j_r]$, $1 = i_1 \leq j_1 < i_2 \leq j_2 < \dots < i_k \leq j_k = n+1$ with I_1, I_k empty if $1 \notin I$, $* = n+1 \notin I$ respectively. Consider $a \in \mathbb{Z}$, $a \neq k(n+1)$, $k \in \mathbb{Z}$, and set $\bar{a} = min\{a, R_{n+1}(a)\}$; more explicitly, if $a = (2n+2)q + r$ (with the conventions of the previous section about representatives for $\mathbb{Z}/(2n+2)\mathbb{Z}$), we have $\bar{a} = min\{(2n+2)q + r, (2n+2)(1-q) - r\}$.

Proposition 3.

$$\begin{aligned} W^I = \{w \in W : 0 < w(1) < w(2) < \dots < w(j_1), \\ w(i_r - 1) < \dots < w(j_r),\ 2 \leq r \leq k-1, \\ w(i_k - 1) < \dots < w(n) < n+1\}. \end{aligned}$$

The first (resp. third) condition should be considered only if $I_1 \neq \emptyset$ (resp. $I_k \neq \emptyset$). Moreover w^I is obtained from w by arranging in increasing order the sets $\{|w(1)|, |w(2)|, \dots, |w(j_1)|\}$, $\{w(i_r - 1), \dots, w(j_r)\}$, $2 \leq r \leq k-1$, $\{\overline{w(i_k - 1)}, \dots, \overline{w(n)}\}$.

$\Delta_{\mathbf{aff}} \cong \tilde{\mathbf{B}}_{\mathbf{n}}.$

Suppose that $I = I_1 \sqcup \dots \sqcup I_k$ is the irreducible decomposition of I, displayed in the following way: $I_r = [i_r, j_r]$, $1 = i_1 \leq j_1 < i_2 \leq j_2 < \dots < i_{k-1} \leq j_{k-1} \leq n$ and I_k containing $n+1$ (so either $I_k = \emptyset$ or $I_k = \{n+1\}$ or $I_k = [i_k, j_k] \sqcup \{n+1\}$, $i_k \leq j_k \leq n$; in particular either $j_k = n-1$ or $j_k = n$). As above I_1 (resp. I_k) is assumed to be empty if $1 \notin I$ (resp. $n+1 \notin I$).

Proposition 4.

$$\begin{aligned} W^I = \{w \in W : 0 < w(1) < w(2) < \dots < w(j_1), \\ w(i_r - 1) < \dots < w(j_r),\ 2 \leq r \leq k, \\ w(n-1) + w(n) < 2n+2\}. \end{aligned}$$

where the first (resp. third) condition should be considered only if $I_1 \neq \emptyset$ (resp. $I_k \neq \emptyset$). To get w^I proceed as follows. If $I_1 \neq \emptyset$ then $w^I(1), \dots, w^I(j_1)$ are given by the increasing arrangement of $|w(1)|, \dots, |w(j_1)|$; $w^I(i_r - 1), \dots, w^I(j_r)$, $2 \leq r \leq k-1$, are given by the increasing arrangement of $w(i_r - 1), \dots, w(j_r)$. Finally, if $I_k \neq \emptyset$, $w^I(i_k), \dots, w^I(j_k)$ are determined by the following inductive procedure.

(1) *Suppose $j_k = n$. Arrange $w(i_k), \dots, w(n)$ in increasing order; let $(x_{i_k}, \dots, x_n)$ be the resulting s-tuple ($s = n - i_k + 1$).*
Suppose $j_k = n-1$. Arrange $w(i_k), \dots, w(n-1)$ in increasing order; let $(x_{i_k}, \dots, x_{n-1})$ be the resulting $(s-1)$-tuple; set $x_n = w(n)$.
(2) *If $x_{n-1} + x_n < 2n+2$ stop; otherwise replace x_{n-1} with $R_{n+1}(x_n)$ and x_n with $R_{n+1}(x_{n-1})$.*
(3) *Go back to step (1).*

References

[1] R.Bedard, *Cells for two Coxeter groups representations of semisimple p-adic groups*, Comm. Algebra **14** (1986), 1253–1286.

[2] A. Björner, F. Brenti, *Affine permutations of type A*, The electronic journal of combinatorics **3(2)** (1995).

[3] A. Björner, *Orderings of Coxeter groups*, Combinatorics and Algebra, Contemp. Math., vol. **34**, 1984, pp. 175–195.

[4] H. Eriksson, K. Eriksson, *Affine Weyl groups as infinite permutations*, The electronic journal of combinatorics **5** (1998).

[5] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[6] G.Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer.Math. Soc. **277** (1983), 623–653.

[7] P. Papi, *A characterization of a special ordering in a root system*, Proc. Amer. Math. Soc. **120** (1994), 661–665.

[8] ______, *Convex orderings in affine root systems II*, J. Algebra **186** (1996), 72–91.

[9] ______, *Affine permutations and inversion multigraphs*, The electronic journal of combinatorics **4** (1997).

[10] Shi Jian-Yi, *The Kazhdan-Lusztig cells in certain affine Weyl groups, Lect.Notes in Math. 1179*, Springer, Berlin, 1986.

Dipartimento di Matematica Istituto G. Castelnuovo
Università di Roma "La Sapienza"
Piazzale Aldo Moro 5
00185 Rome — ITALY
e-mail address: papi@mat.uniroma1.it