On generating functions for subalgebras of free Lie algebras

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Abstract

Let G be a free group of rank n and let $H \subset G$ be a subgroup of a finite index. Then H is also a free group and the rank m of H is determined by Schreier's formula $m-1=(n-1)\cdot|G:H|$.

Any subalgebra of a free Lie algebra is free (Shirshov-Witt). But a straightforward analogue of Schreier's formula for free Lie algebras does not exist, it is easy to see that any subalgebra of a finite codimension has an infinite number of generators.

The goal of this talk is to show that the appropriate formula exists in terms of formal power series. We also consider its version for exponential generating functions. These formulae have various applications and they can be generalized in many ways. For example, they allow to write down explicit formulas for generating functions (ordinary and exponential) for free solvable (or more generally, polynilpotent) Lie algebras; this helps in the study of asymptotics.

1 Schreier's formula for free Lie algebras

By K we denote the ground field. Let X be at most countable set with a weight function $\mathrm{wt} : X \to \mathbb{N}$ such that

$$X = \bigcup_{i=1}^{\infty} X_i, \quad X_i = \{x \in X \mid \mathrm{wt}\, x = i\}; \quad |X_i| < \infty,\ i \in \mathbb{N}.$$

We call such a set *finitely graded.* For any monomial $y = x_{i_1} \dots x_{i_n}$, $x_{i_j} \in X$, we set $\mathrm{wt}\, y = \mathrm{wt}\, x_{i_1} + \dots + \mathrm{wt}\, x_{i_n}$. Suppose that A is a homogeneous algebra generated by X. Then we define the weight $\mathrm{wt}\, a$, $a \in A$ as the maximum of weights of monomials in decomposition of a. Suppose that Y is a set of monomials on X, then we define the Hilbert-Poincaré series of Y as

$$\mathcal{H}_X(Y) = \mathcal{H}_X(Y,t) = \sum_{i=1}^{\infty} |Y_i| t^i; \qquad Y_i = \{y \in Y \mid \mathrm{wt}\, y = i\},\ i \in \mathbb{N}.$$

If $V \subset A$ is a subspace then by $\mathcal{H}_X(V)$ we mean the series of a homogeneous basis of $\mathrm{gr}\, V$; where $\mathrm{gr}\, V$ is the associated graded space.

Next, we consider the operator $\mathcal{E}$ on power series $\phi(t) = \sum_{n=1}^{\infty} b_n t^n$, $b_n \in \{0,1,2,\dots\}$:

$$\mathcal{E} : \phi(t) = \sum_{n=1}^{\infty} b_n t^n \longrightarrow \mathcal{E}(\phi(t)) = \sum_{n=0}^{\infty} a_n t^n = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{b_n}}. \tag{1}$$

Let L be a Lie algebra generated by X and $U(L)$ be its universal enveloping algebra and suppose that

$$\mathcal{H}_X(L,t) = \sum_{n=1}^{\infty} b_n t^n, \quad \mathcal{H}_X(U(L),t) = \sum_{n=0}^{\infty} a_n t^n.$$

It is well known that $\mathcal{H}_X(U(L)) = \mathcal{E}(\mathcal{H}_X(L))$ [18].

The following is the analogue of Schreier's formula.

Theorem 1.1 *Let L be a free Lie algebra generated by a finitely graded set X. Suppose that H is a subalgebra and Y is the set of free generators for H. Then*

$$\mathcal{H}(Y) - 1 = (\mathcal{H}(X) - 1) \cdot \mathcal{E}(\mathcal{H}(L/H)).$$

Proof is based on the construction of the free generating set Y for H. Let us describe main ideas of the proof given in [9].

First, we consider the case $L = H \oplus \langle z \rangle$, where $z \in X$, let $\operatorname{wt} z = a$. In this case H is freely generated by the following set [2]

$$\begin{aligned} Y &= \{[x, z^m] \mid x \in X \backslash \{z\},\ m = 0,1,2,\ldots\}. \\ \mathcal{H}(Y) &= \mathcal{H}(X\backslash\{z\}) \cdot (1 + t^a + t^{2a} + \ldots + t^{ma} + \ldots) = \\ &= (\mathcal{H}(X) - t^a)\frac{1}{1-t^a} = (\mathcal{H}(X) - 1 + (1 - t^a))\frac{1}{1-t^a} = \\ &= 1 + (\mathcal{H}(X) - 1)\frac{1}{1-t^a} = 1 + (\mathcal{H}(X) - 1) \cdot \mathcal{E}(\mathcal{H}(L/H)), \end{aligned}$$

and the desired formula is valid for this particular case.

Second, the operator $\mathcal{E}$ is multiplicative: $\mathcal{E}(\phi_1(t) + \phi_2(t)) = \mathcal{E}(\phi_1(t)) \cdot \mathcal{E}(\phi_2(t))$. If the formula is valid for transitions $L \supset H_1$, $H_1 \supset H_2$, then this property allows to prove that the formula is also valid for the transition $L \supset H_2$.

Now we suppose that $H \subset L$ is a homogeneous subalgebra in terms of our weight function. The free generating set Y for H can be constructed by infinitely many steps of type $\dim L/H = 1$ (see e.g. [2]). Namely, there exist subalgebras $L = L_0 \supset L_1 \supset \cdots \supset L_n \supset \cdots$, where $\dim L_n/L_{n+1} = 1$, and $H = \cap_{n \geq 0} L_n$. These subalgebras have free generating sets $X = Y_0, Y_1, \ldots, Y_n, \ldots$, and Y is constructed from Y_i, $i \geq 0$. We consider generating functions for these sets, and two remarks above are applied to prove the result.

The case of an arbitrary subalgebra is reduced to the case of a homogeneous subalgebra by standard arguments [2]. □

We apply this result to study varieties of Lie algebras. Recall that the *variety* of (Lie) algebras is a class of all (Lie) algebras that satisfy some set of identical relations. If L is a Lie algebra then the lower central series is defined by iteration $L^1 = L$, $L^{i+1} = [L, L^i]$, $i = 1,2,\ldots$. Now L is called *nilpotent* of class s iff $L^{s+1} = \{0\}$, $L^s \neq \{0\}$. All Lie algebras nilpotent of class s form the variety denoted by $\mathbf{N}_s$. Recall that L is *polynilpotent* with a tuple $(s_q, \ldots, s_2, s_1)$ iff there exists a chain of ideals $0 = L_{q+1} \subset L_q \subset \ldots \subset L_2 \subset L_1 = L$ with $L_i/L_{i+1} \in \mathbf{N}_{s_i}$. All polynilpotent Lie algebras with the fixed tuple $(s_q,\dots,s_2,s_1)$ form the variety denoted by $\mathbf{N}_{s_q}\dots\mathbf{N}_{s_2}\mathbf{N}_{s_1}$. In the case $s_q=\dots=s_1=1$ one has the variety $\mathbf{A}^q$ of *solvable* Lie algebras of length q.

If $\mathbf{M}$ is a variety of Lie algebras then by $F(\mathbf{M},k)$ we denote its free algebra of rank k (this is the algebra generated by k elements $x_1,\dots,x_k$ and such that for all $H\in\mathbf{M}$ and any $y_1,\dots,y_k\in H$ there exists a homomorphism $\phi: F\to H$ with $\phi(x_i)=y_i$, $i=1,\dots,k$). For the theory of varieties of Lie algebras see the monograph [2].

This formula allows to find the Hilbert-Poincaré series for free polynilpotent Lie algebras. Denote by $\mu(*)$ the Möbius function.

Theorem 1.2 *Suppose that $L=F(\mathbf{N}_{s_q}\dots\mathbf{N}_{s_1},k)$. We set $\beta_0(z)=0$, $\alpha_0(z)=z$, and recursively define functions*

$$\begin{aligned}\beta_i(z) &= \beta_{i-1}(z)+\sum_{m=1}^{s_i}\frac{1}{m}\sum_{a|m}\mu\left(\frac{m}{a}\right)\left(\alpha_{i-1}(z^{m/a})\right)^a; \quad 1\le i\le q;\\ \alpha_i(z) &= 1+(kz-1)\cdot\mathcal{E}(\beta_i(z)), \quad 1\le i\le q;\end{aligned}$$

Then $\mathcal{H}(L,z)=\beta_q(z)$.

The variety for tuple $(1,d)$ is also denoted by $\mathbf{AN}_d$.

Corollary 1.1 *Let $L=F(\mathbf{AN}_d,k)$. Then*

$$\mathcal{H}(L,z)=\psi_k(1)z+\psi_k(2)z^2+\dots+\psi_k(d)z^d+1+\frac{kz-1}{(1-z)^{\psi_k(1)}(1-z^2)^{\psi_k(2)}\cdots(1-z^d)^{\psi_k(d)}},$$

where $\psi_k(m)=\frac{1}{m}\sum_{a|m}\mu(m/a)k^a$.

Also, there exist Schreier's formulae for free Lie p-algebras and free Lie superalgebras [9].

2 Complexity functions and Exponential Schreier's formula

Let $\mathbf{V}$ be a variety of Lie algebras, and suppose that $F(\mathbf{V},X)$ is its free algebra generated by a countable set $X=\{x_i\mid i\in\mathbb{N}\}$. Denote by $P_n(\mathbf{V})\subset F(\mathbf{V},X)$ the subspace of all multilinear elements in $\{x_1,\dots,x_n\}$ and consider the *codimension growth sequence* $c_n(\mathbf{V})=c_n(F(\mathbf{V},X),X)=\dim_K P_n(\mathbf{V})$, $n=1,2,\dots$, which is an important characteristic of the variety $\mathbf{V}$. Yu. P. Razmyslov introduced the so called *complexity function* which is defined as

$$\mathcal{C}(\mathbf{V},z)=\sum_{n=1}^{\infty}\frac{c_n(\mathbf{V})}{n!}z^n, \quad z\in\mathbb{C}.$$

This notion enabled Yu. P. Razmyslov to reformulate the upper bound on the codimension growth of varieties of Lie algebras due to A. N. Grishkov in the following nice way.

Theorem 2.1 ([12], Razmyslov) *Let* $\mathbf{V}$ *be a nontrivial variety of Lie algebras. Then* $\mathcal{C}(\mathbf{V}, z)$ *is an entire function of the complex variable.*

The complexity function is one of many examples of *exponential generating functions* [4]. We consider complexity functions in a more general situation. Suppose that we are given a set A of monomials Y in letters $X = \{x_i \mid i \in \mathbb{N}\}$. For any set of distinct elements $\widetilde{X} = \{x_{i_1}, \dots, x_{i_n}\} \subset X$ we denote by $P_n(A, \widetilde{X})$ the set of all multilinear elements of degree n on $\widetilde{X}$ belonging to A. Suppose that the number of elements $c_n(A, \widetilde{X})$ does not depend on the choice of $\widetilde{X}$, but depends only on n. In this case we write $c_n(A) = c_n(A, \widetilde{X})$ and say that A is *X-uniform* and define the complexity function with respect to X;

$$\mathcal{C}_X(A, z) = \sum_{n=1}^{\infty} \frac{c_n(A)}{n!} z^n, \quad z \in \mathbb{C}.$$

(where the sum is taken from $n = 0$, $c_0 = 1$ for associative algebras and groupoids with unity). Remark that A need not necessarily consist of multilinear elements. Often we omit the variable z and (or) the set X (if it is clear what set is used) and write $\mathcal{C}_X(A, z) = \mathcal{C}(A)$.

Let us consider some examples. Suppose that X is a countable set and $A = A(X)$ is a free associative algebra and $L = L(X)$ is a free Lie algebra. One can easily compute

$$\mathcal{C}_X(A, z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z},$$

$$\mathcal{C}_X(L, z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1 - z),$$

$$\mathcal{C}(\mathbf{N}_s, z) = \sum_{n=1}^{s} \frac{z^n}{n}.$$

The following statement is an exponential version of the Schreier's formula for free Lie algebras.

Theorem 2.2 *Let L be a free Lie algebra with a countable generating set X. Suppose that H is an X-uniform subalgebra. Then H has a uniform set of free generators, and for any such set Y one has*

$$\mathcal{C}_X(Y, z) - 1 = (z - 1) \cdot \exp(\mathcal{C}_X(L/H, z));$$

3 Explicit formulae for complexity functions

By exponential Schreier's formula we can easily derive the following two results. Suppose that $\mathbf{M}$, $\mathbf{V}$ are varieties of Lie algebras. Then $\mathbf{M} \cdot \mathbf{V}$ is the class of all Lie algebras L such that there exists an ideal $H \subset L$ with $H \in \mathbf{M}$, $L/H \in \mathbf{V}$ [2].

Theorem 3.1 *Suppose that* $\mathbf{M} \cdot \mathbf{V}$ *is the product of varieties of Lie algebras, where* $\mathbf{M}$ *is multihomogeneous. Then*

$$\mathcal{C}(\mathbf{M} \cdot \mathbf{V}, z) = \mathcal{C}(\mathbf{V}, z) + \mathcal{C}(\mathbf{M}, 1 + (z - 1)\exp(\mathcal{C}(\mathbf{V}, z))).$$

For example, $\mathbf{V} = \mathbf{N}_{s_q} \dots \mathbf{N}_{s_1}$ is the product $\mathbf{V} = \mathbf{N}_{s_q} \cdot \dots \cdot \mathbf{N}_{s_2} \cdot \mathbf{N}_{s_1}$, in particular, the variety $\mathbf{A}^q$ of solvable Lie algebras of length q is presented as $\mathbf{A}^q = \underbrace{\mathbf{A} \cdot \dots \cdot \mathbf{A}}_{q \text{ times}}$.

Theorem 3.2 *Suppose that* $\mathbf{V} = \mathbf{N}_{s_q} \dots \mathbf{N}_{s_1}$. *We define functions*

$$\begin{aligned} \beta_1(z) &= \sum_{m=1}^{s_1} \frac{z^m}{m}; \\ \beta_i(z) &= \beta_{i-1}(z) + \sum_{m=1}^{s_i} \frac{(1 + (z-1)\exp(\beta_{i-1}(z)))^m}{m}; \qquad 2 \le i \le q. \end{aligned}$$

Then $\mathcal{C}(\mathbf{V}, z) = \beta_q(z)$.

Corollary 3.1

$$\mathcal{C}(\mathbf{AN}_d, z) = z + \frac{z^2}{2} + \dots + \frac{z^d}{d} + 1 + (z-1)\exp\left(z + \frac{z^2}{2} + \dots + \frac{z^d}{d}\right). \qquad (2)$$

Corollary 3.2 ([6]) *The variety of solvable Lie algebras* $\mathbf{A}^q$ *has the following complexity function*

$$\begin{aligned} &\mathcal{C}(\mathbf{A}^q, z) = \beta_q(z), \quad \text{where} \\ &\beta_1(z) = z, \quad \beta_{i+1}(z) = \beta_i(z) + 1 + (z-1)\exp(\beta_i(z)), \ i = 1, 2, \dots. \end{aligned}$$

These explicit formulas allows us to obtain asymptotics for the codimension growth sequence $c_n(\mathbf{V})$ for solvable varieties $\mathbf{A}^q$, or more generally, for polynilpotent varieties of Lie algebras. These asymptotics are better than those found in [6].

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