SYMMETRIC CHAIN DECOMPOSITION AND EL-SHELLABILITY OF DING ORDER

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ABSTRACT. We show that Ding's partial order on maximal rook placements on any Ferrers board has a symmetric chain decomposition and is EL-shellable. As a consequence the partial order is Peck, and we show that it has Möbius function values of $-1, 0$ or $+1$.

1. INTRODUCTION

A *Ferrers board* corresponds to an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m > 0)$. The board can be drawn as rows of squares of lengths λ_i, such that the rows are right justified and the row of length λ_i is directly below the row of length λ_{i-1}. Abusing terminology, we will denote the Ferrers board itself by λ.

A subset σ of λ is a *rook placement* on λ if no two squares (rooks) in σ occupy the same row or column.

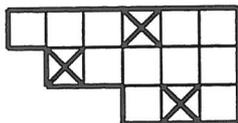Figure 1 gives an example of a rook placement on a Ferrers board.

FIGURE 1. A representation of a Ferrers board with 3 rooks on it.

Let R_λ^r be the set of all rook placements σ with r rooks on board λ. Let ω denote the maximal number of rooks that may be placed on board λ. The set of all maximal rook placements on board λ is then denoted R_λ^ω.

In the case $\omega = m$, Ding [5, Definition 4.28] placed an ordering on R_λ^m, which we denote $\leq_D$. In Section 2, we extend this to an order on R_λ^ω in general. The two main theorems of this paper are as follows:

Theorem 1. $(R_\lambda^\omega, \leq_D)$ *has a symmetric chain decomposition.*

Theorem 2. $(R_\lambda^\omega, \leq_D)$ *is EL-shellable.*

The motivation for these results is that both are known to hold for the strong Bruhat order on the symmetric group, which may be viewed as the special case of Ding's order for rook placements on an $n \times n$ board. Ding [6] showed that R_λ^m is the inclusion ordering on the cells in a cell decomposition of a certain complex projective variety which he called the *partition variety* M_λ/B. This generalizes the fact that the strong Bruhat order gives the ordering of cells in the classical flag manifold G/B. Theorems 1 and 2 further tighten the analogy between Ding's order in general and the special case of strong Bruhat order.

Section 2 recalls Ding's partial order on rook placements as well as further definitions. It also shows that we may restrict ourselves to examining boards where $\omega = m$ and m is the number of rows in λ. Section 3 proves Theorem 1. Section 4 proves Theorem 2, and as a consequence that the Möbius function, $\mu(x, y) = 0$ or ± 1.

2. DEFINITIONS

Definition 3. Given two rook placements σ and τ on a Ferrers board λ we say that $\sigma \leq_D \tau$ if for $\sigma = s_1, s_2, ..., s_m$ and $\tau = t_1, t_2, ..., t_m$ the relation $s_{i,k} \leq t_{i,k}$ holds for all $1 \leq i \leq k \leq m$ where

- s_i is the column index of the rook in row i of rook placement σ and columns are labeled right to left. If row i of rook placement σ contains no rook, then $s_i = 0$.
- $s_{i,k}$ is the i-th entry in the increasing rearrangement of $s_m, s_{m-1}, ..., s_{m-k+1}$.

The rules are similar for t_i and $t_{i,k}$ in τ.
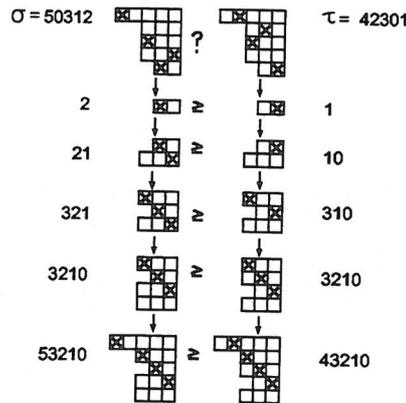
Figure 2 illustrates an example of $\sigma >_D \tau$.

FIGURE 2. Two boards σ and τ of shape λ are compared as in Definition 3

Definition 4. The symmetric group S_n is in an obvious bijection with all maximal rook placements on an $n \times n$ board. *Strong Bruhat order uses the same comparison order as Definition 3. So, $(S_n, \leq_B)$ is Ding order on $n \times n$ boards, i.e. $(S_n, \leq_B)$ is $(R_{n \times n}^n, \leq_D)$.*

In writing π as a permutation $\pi_1 \pi_2 ... \pi_n$, each π_i is the column index of the rook in row i, where the column indices increase from left to right and the row numbers increase from top to bottom. This is opposite to the column index given above for the comparison method.

For $\pi = \pi_1 \pi_2 \, ... \, \pi_n \in S_n$, let $Inv_k(\pi) := |\, \{j \mid \pi_j > \pi_k, \; j < k\} \,|$. Then,

$$(2.1) \qquad\qquad Inv(\pi) := \sum_{k=1}^{n} Inv_k(\pi)$$

is well known to be the rank of π in strong Bruhat order on S_n.

The order relations given in Definition 4 are defined by a comparison. An order relation based on rook swaps is defined as follows:

Definition 5. The covering relation $\sigma \lessdot_{swap} \tau$ holds if τ and σ differ by a swap of 2 rooks and $Inv(\sigma) + 1 = Inv(\tau)$. Then $\leq_{swap}$ is the transitive closure of $\lessdot_{swap}$.

The following lemma is well known.

Lemma 6. $\leq_B$ and $\leq_{swap}$ on S_n coincide.

See [4] for the proof and other references.

Ding defined an embedding of R_λ^r in S_n where $n = m + \lambda_1 - r$. This will be used later to show $(R_\lambda^r, \leq_D)$ is an induced subposet of $(S_n, \leq_B)$. The embedding, $f : R_\lambda^r \to S_n$, is done by extending σ on λ to a rook placement (permutation) π on an $n \times n$ board. Extending the board is done by locating the empty rows and columns in σ. For each empty column add a new row above the initial board. For each empty row, add a new column to the right of the row extended board. Place rooks in the new empty rows added above, by finding the leftmost empty column and putting a rook in that column of the first row, then finding the next empty column and putting a rook in that column of the second row. Repeat for all of the new rows. Place rooks in the new empty columns in a similar fashion. Find the topmost empty row and put a rook in that row of the leftmost empty column. Find the new topmost empty row and put a rook in that row of the now leftmost empty column. Repeat until there are no empty rows. This is illustrated in Figure 3.
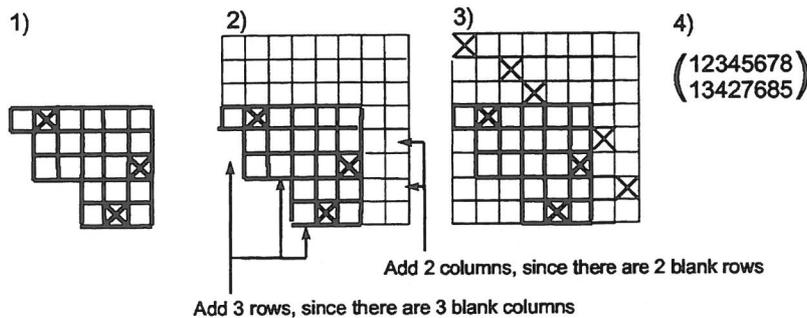

FIGURE 3. Board λ is extended to a permutation.

Lemma 7. *The map f embeds $(R_\lambda^r, <_D)$ as an induced subposet of $(S_n, <_B)$, in the sense that $\sigma \leq_D \tau$ for $\sigma, \tau \in R_\lambda^r$ iff $f(\sigma) \leq_B f(\tau)$ in S_n.*

Proof. Bjorner and Brenti [4, corollary 5(iii)] showed that when doing comparisons in Bruhat order, checking if $f(\sigma) \leq_B f(\tau)$ requires checking $r_{i,k} \leq s_{i,k}$ as given in Definition 3, but only doing so for i where $r_i > r_{i-1}$ in $f(\tau)$. Note that for $i = 1$ to $= n - m$ this cannot occur. Also note that values $\lambda_1 + 1$ through n will be in increasing order in $f(\tau)$ and are the largest $n - \lambda_1$ values. Thus, these values cannot be r_i where $r_i > r_{i-1}$. Therefore, all comparisons which need to be checked are at rows where the rook lies in the board λ. □
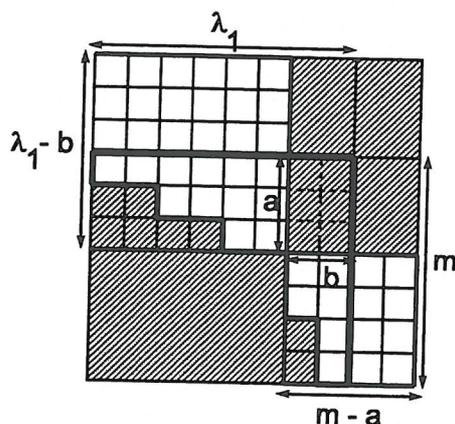

FIGURE 4. $R_\lambda^\omega \subset S_{m-a} \times S_{\lambda_1-b} \subset S_{\lambda_1+m-\omega}$ where $\omega = a + b$ and a and b are defined below.

Let ω be the maximal number of rooks that may be placed on board λ. We next describe a reduction which shows that we can work with the special case of $\omega = m$.

Let a be the smallest integer such that $\lambda_{a+1} = \omega - a$. Given board λ, it can be separated into three components, board α with a rows, board β with b columns and an $a \times b$ rectangular board, where a is as defined above and $b = \omega - a$. See Figure 4.

Proposition 8.

$$R_\lambda^\omega \cong R_\alpha^a \times R_{\beta^T}^b$$

as posets, where here β^T denotes the transpose operation on a board β, which is flipping the board across the diagonal cutting through the upper right corner.

Proof. We have the following sequence of poset isomorphisms, explained below.

$$
\begin{aligned}
R_\lambda^\omega &\cong f(R_\lambda^\omega) &&\subseteq S_n, \text{where } n = \lambda_1 + m - \omega \\
&\cong f(R_\alpha^a) \times f(R_\beta^b) &&\subseteq S_{\lambda_1-b} \times S_{m-a} \\
&\cong f(R_\alpha^a) \times f(R_{\beta^T}^b) &&\subseteq S_{\lambda_1-b} \times S_{m-a} \\
&\cong R_\alpha^a \times R_{\beta^T}^b.
\end{aligned}
$$

The first isomorphism follows from Lemma 7.

The second isomorphism follows since a placement $\sigma \in R_\lambda^\omega$ can have no rook in the $(m - a) \times (\lambda_1 - b)$ rectangle as we now explain. If a rook were in the smaller $a \times b$ rectangle, as shown in Figure 4, then it would occupy both one of the top a

rows and one of the rightmost b columns, thus prohibiting all ω rooks from being placed. In $f(\sigma) \in f(R_\lambda^\omega)$ there will be no rooks in the extended board in the region above λ's rightmost b columns or to the right of λ's topmost a rows, since those rows and columns are already occupied in λ. Thus, the $(m - a) \times (\lambda_1 - b)$ corner of $f(\sigma)$ has no rooks.

There is no ambiguity about working with the Cartesian product Bruhat order on $S_{\lambda_1 - b} \times S_{m-a}$ in place of the Bruhat order on S_n as we now explain. The rooks in S_{m-a} which are the rooks in the lower $m - a$ rows of S_n all lie to the right of the rooks in $S_{\lambda_1 - b}$ which are the rooks in the upper $\lambda_1 - b$ rows of S_n. Thus, when the rows are compared as in Definition 3 for Bruhat order on S_n, the lower $m - a$ rows will always remain the lower $m - a$ rows when doing a comparison of the bottom $m - a + k$ rows. This allows separate comparisons for the top $\lambda_1 - b$ rows and the bottom $m - a$ rows as in $S_{\lambda_1 - b} \times S_{m-a}$.

The third isomorphism follows since the transpose operation corresponds to $\pi \mapsto \pi^{-1}$, it is an automorphism of Bruhat order and $f(R_{\beta T}^b) \cong f(R_\beta^b)$.

The final isomorphism follows again from Lemma 7. $\qquad \square$

Definition 9. For σ a rook placement on λ, define
$$l_{\lambda,k}(\sigma) := \alpha_{\lambda,k}(\sigma) + \beta_{\lambda,k}(\sigma) + \gamma_{\lambda,k}(\sigma)$$
where r_k is the rook in the row k of λ and

$\alpha_{\lambda,k}(\sigma)$ is the number of blank rows above row k

$\beta_{\lambda,k}(\sigma)$ is the number of rooks both above and to the right of r_k and

$\gamma_{\lambda,k}(\sigma)$ is the number of blank columns to the right of r_k's column.

If there is no rook in row k, then $l_{\lambda,k}(\sigma) = 0$.

Define $l_\lambda(\sigma) := \sum_{k=1}^m l_{\lambda,k}(\sigma)$.

Corollary 10. *For any rook placement σ on any board λ, we have*
$$l_\lambda(\sigma) = Inv(f(\sigma)).$$

Proof.
$$Inv_k(f(\sigma)) = \begin{cases} l_{\lambda,k}(\sigma) & \text{if row } k \text{ has a rook of } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Now compare the definition of $l_\lambda(\sigma)$ and equation (2.1) and the corollary follows. $\qquad \square$

Ding proved the following:

Lemma 11. *R_λ^m is ranked with rank function $l_\lambda(\sigma)$.*

3. SYMMETRIC CHAIN DECOMPOSITION

In this section we show that R_λ^ω has a symmetric chain decomposition. We begin by recalling some definitions.

Let P be a finite ranked poset of rank n and P_i the set of elements at rank i.

A subset A of P is an *antichain* if for all $a, b \in A$, $a \not\geq b$ and $b \not\geq a$.

P is *Sperner* if the size of the largest antichain in P is equal to the size of its maximal rank.

P is *rank symmetric* if $|P_i| = |P_{n-i}|$.

P is *rank unimodal* if $|P_o| \leq |P_1| \leq ... \leq |P_k| \geq |P_{k+1}| \geq ... \geq |P_n|$ for some k, $0 \leq k \leq n$.

P is *strongly Sperner* if the size of the largest union of k antichains in P is the sum of the sizes of the k largest ranks.

A *symmetric chain decomposition* (SCD) of P is a disjoint set of unrefinable chains which cover P such that each chain that starts at rank i of the poset ends at rank $n - i$.

It is obvious that if P has an SCD then it is rank symmetric and rank unimodal. It is also not hard to show that if P has an SCD then it is strongly Sperner. Posets which are rank symmetric, rank unimodal and strongly Sperner are called *Peck*. So, having an SCD implies Peck.

The following proposition is obvious:

Proposition 12. *Assume* $g : P_1 \to P_2$, *is order preserving, rank preserving, bijective and P_1 has an SCD. Then P_2 has an SCD.*

Recall that m is the number of rows in λ.

Theorem 13. *The Ding order on R_λ^m has an SCD and thus is Peck.*

Proof. We shall show that R_λ^m has an encoding mapping it to a product of chains. A product of chains has an SCD and Proposition 12 will be shown to apply to the inverse mapping.

We will define a map $h : R_\lambda^m \to M = C(\lambda_m) \times C(\lambda_{m-1} - 1) \times ... \times C(\lambda_1 - m + 1)$ where σ is a rook placement on board λ, and the poset $C(n)$ is the set $\{0, 1, 2, ..., n\}$ linearly ordered as usual. Let $h(\sigma) = (i_1, i_2, ..., i_m)$ where $i_k = l_{\lambda,k}(\sigma)$ as defined in Definition 9 to be the contribution to the length function of the rook in the kth row of board λ. Note that $0 \leq i_k \leq \lambda_k - (m - k + 1)$.

We will show that Proposition 12 applies to h^{-1} by showing it to be bijective, rank preserving and order preserving.

$h^{-1} : M \to R_\lambda^m$ is defined by the following algorithm: given $(i_1, i_2, ..., i_m) \in M$, in the bottom row of λ, a rook is placed in the $(i_m + 1)^{st}$ blank column from the right edge. In the next row a rook is placed in the $(i_{m-1} + 1)^{st}$ blank column from the right edge. Continue for rooks in rows $k = m - 2$ through 1 by putting a rook in the $(i_k + 1)^{st}$ blank column from the right edge.

$$h^{-1}(1435) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & \times & \\ \hline \times & & & & & \\ \hline & & \times & \times & & \\ \hline & \times & & & & \\ \hline \end{array}$$

FIGURE 5. $h^{-1} : M \to R_{m \times n}$.

Figure 5 gives an example of h^{-1} acting on an element in M.

Since $(i_1, i_2, ..., i_m)$ has a rank of $\sum_k i_k = \sum_k l_{\lambda,k}(\sigma)$, the map h^{-1} is rank preserving.

To show h^{-1} is order preserving it suffices to show that a cover relation in M is mapped by h^{-1} to a Ding order relation in R_λ^m.

Let $\sigma = h^{-1}(i_1, i_2, ..., i_k, i_{k+1}, ..., i_m)$ and $\tau = h^{-1}(i_1, i_2, ..., i_k + 1, i_{k+1}, ..., i_m)$ We shall show that $\sigma <_D \tau$.

By Lemma 7 it is sufficient to show that $f(\sigma) <_B f(\tau)$, where f is the extension function given in Section 2. Note that since the map f extending a rook placement σ to an $n \times n$ board does not depend on the shape λ, it suffices to show h^{-1} meets the necessary criteria when λ is rectangular of shape $m \times n$.

Since h^{-1} is rank-preserving, we will have $Inv(f(\tau)) = Inv(f(\sigma)) + 1$. Hence by Lemma 6, if $f(\sigma)$ and $f(\tau)$ differ by a swap of 2 rooks, then $f(\tau)$ covers $f(\sigma)$ in strong Bruhat order.
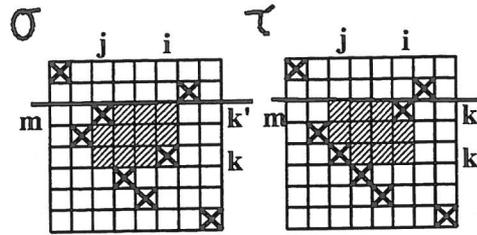

FIGURE 6. Increasing a single i is a swap.

By definition $h(f(\sigma))$ equals $(\overbrace{0, 0, \cdots, 0}^{n-m}, i_1, i_2, ... i_k, i_{k+1}, ... i_m)$ and $h(f(\tau))$ equals $(\overbrace{0, 0, \cdots, 0}^{n-m}, i_1, i_2, ... i_k + 1, i_{k+1}, ... i_m)$. Increasing i_k moves r_k to the next available column to the left at the $m - k + 1^{th}$ step of creation, by the description of h^{-1}. Call the column r_k was originally in i and the one it moved to j. Since $f(\sigma)$ and $f(\tau)$ are permutations, column j must contain a rook. Call the row this rook occupies k'. The row must be above row k by the description of h^{-1}.

We want to show that placements $f(\sigma)$ and $f(\tau)$ differ by the column swap (ij). Let $f(\sigma')$ be the permutation obtained from $f(\sigma)$ by swapping the rooks in columns i and j. Let $f(\sigma) = s_1 s_2 ... s_n$ and $f(\sigma') = s_1' s_2' ... s_n'$ such that s_i and s_i' are the column indexes of the rooks in row i of $f(\sigma)$ and $f(\sigma')$ respectively. Note that $s_i = s_i'$ for all $i \neq k$ or k' and $s_k = s_{k'}', s_{k'} = s_k'$. It suffices for us to show $h(f(\sigma')) = h(f(\tau))$, since then $f(\sigma') = f(\sigma)$.

It is easy to see that for $p \leq k'$ or $p > k$, we have

$$l_{n \times n, p}(f(\sigma')) = l_{n \times n, p}(f(\sigma)) = l_{n \times n, p}(f(\tau))$$

For $k' < p < k$, by construction, $r_p < r_{k'}$ or $r_p > r_k$ so we have

$$l_{n \times n, p}(f(\sigma')) = l_{n \times n, p}(f(\sigma)) = l_{n \times n, p}(f(\tau))$$

For $p = k$ we have

$$l_{n \times n, p}(f(\sigma')) = l_{n \times n, p}(f(\sigma)) + 1 = l_{n \times n, p}(f(\tau))$$

Thus $h(f(\sigma')) = h(f(\tau))$ as desired. $\square$

Recall that ω is the maximum number of rooks that may be placed on λ. We may now prove

Theorem 1($R_\lambda^\omega, \leq_D$) *has a symmetric chain decomposition.*

Proof. [1, Theorem 3.6.1] states that if two posets P_1 and P_2 each have an SCD then $P_1 \times P_2$ has an SCD. Therefore by Proposition 8 and Theorem 13, R_λ^ω has an SCD. $\qquad\square$

Remark.

In the very special case of a rectangular $m \times n$ board, Peckness of $R_{m \times n}^m$ can be deduced from a result of Stanley. For notation, let (W, S) be a Coxeter system for which W is a Weyl group. Let $J \subset S$ so each coset wW_J of W_J in W contains a unique element w_J of minimal length. Let W^J be the set of minimal length coset representatives w_J.

Theorem 14. [10, Theorem 3.1] *In the above situation, the Bruhat order restricted to W^J is Peck.*

In our case, if $W = S_n$ and $J = \{(n-1, n), (n-2, n-1), ..., (n-m-1, n-m)\}$ then $W^J = R_{m \times n}^m$.

Stanley's method uses the fact that Bruhat ordering on W^J gives the inclusion ordering on the cells in a cell decomposition of a smooth complex projective variety, namely a partial flag manifold G/P_J. Peckness then follows from the Hard Lefschetz Theorem for smooth varieties.

As was mentioned in the introduction, Ding has also shown that his ordering R_λ^m is the ordering on the cells in a cell decomposition of a complex projective variety which he calls the *partition variety* M_λ/B. In deducing Peckness via the hard Lefschetz Theorem, Stanley has pointed out (see e.g. [11]) that it is not strictly necessary for the variety to be smooth. It need only satisfy the weaker condition of being a *V-variety*, that is, it looks locally like $\mathbb{C}^n/G$ for some finite subgroup $G \subset GL(n, \mathbb{C})$. In light of this and Theorem 1, we conjecture the following:

Conjecture 15. *Ding's partition varieties M_λ/B are always V-varieties.*

4. EL-SHELLABILITY

In this section we show that R_λ^ω is EL-shellable. We begin by recalling some terminology from [3].

An *edge labeling* of P is an assignment of labels from some linearly ordered set Λ to the edges of its Hasse diagram. Given a, b two unrefinable chains with the same end points in P, we say that $a <_L b$ if the first edges in the rising chains a and b where the edge $i \in a$ differs from the edge $j \in b$, has $i <_\Lambda j$. A chain is *rising* if each edge in the chain precedes the edge above it in the linear order.

The definition of *EL-labeling* as given by Björner [3], is: For every interval $[x, y]$ in P,

    i there exists a unique rising chain c in $[x, y]$, and
    ii $c <_L c'$ for all maximal chains c' in $[x, y]$.

A finite graded poset is *lexicographically shellable (EL- shellable)* if it has an EL-labeling.

Edelman in [8] showed that the Bruhat order of the symmetric group is EL-shellable. He did this by defining a labeling with a natural order on the edges of the poset. Label the edge $\pi <_B \pi'$ by the transposition (i, j), $i < j$, which satisfies $(i, j)\pi = \pi'$. For example, 3214 is covered by 3412 and their common edge would be labeled 24. The labels are ordered as follows: label ij precedes label kl if $i < k$ or if $i = k$ then $j < l$. For example in the strong Bruhat order on S_4 with $\pi = 2134$

and $\pi' = 4231$, $2134 < 2314 < 2341 < 3241 < 4231$ is a rising chain since its edge labeling is $(13, 14, 23, 34)$.

Björner [3] discusses how EL-shellability can be inherited by subposets.

Proposition 16. *Assume P is a ranked poset with an EL-labeling and $Q \subset P$ has the property that for all $x < y \in Q$ the unique rising chain from x to y in P is completely contained in Q. Then the EL-labeling of P restricted to Q is an EL-labeling of Q.*

Recall m is the number of rows in λ.

Theorem 17. *The poset R_λ^m under Ding's order is EL-Shellable.*

Proof. In this proof we identify R_λ^m with its image $f(R_\lambda^m) \subset S_n$. By Proposition 16, it would suffice to exhibit an EL-labeling for S_n where the unique rising chain between x and y with both in R_λ^m is contained in the subposet R_λ^m. The labeling described above on S_n [8] can be used on the subposet; however, the same ordering of the labels will not work. A similar ordering, Λ can be applied which is an EL-labeling on both S_n and the subposet R_λ^m: edge ij precedes edge kl if $j > l$ or if $j = l$ then $i > k$.


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2578 |3154  τ
1678 |3254
1678 |3245      ↑
1378 |6245
1378 |4265
1378 |4256
1348 |7256  σ'
1348 |5276  σ
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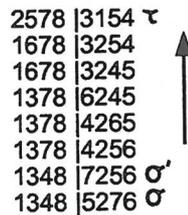
FIGURE 7. Unique rising chain in the ordering Λ.

We next describe the unique rising chain in S_n. (The correctness of this description follows from the same verification as in [8]. Given σ and τ such that $\sigma <_B \tau$, we must describe the next step σ' in the unique rising chain from σ to τ. Let $\sigma(i)$ be the ith value in σ and $\sigma^{-1}(a)$ be the position in σ of the value a and let $\tau(i)$ and $\tau^{-1}(a)$ be defined similarly. Move up from σ to σ' by the following method. Let a be the largest value in σ such that $\sigma^{-1}(a) \neq \tau^{-1}(a)$, and let b be the largest value such that $\sigma(\tau^{-1}(a)) \leq b < a$ and $\tau^{-1}(a) \leq \sigma^{-1}(b) < \sigma^{-1}(a)$. Then $\sigma' = (a, b)\sigma$. In Figure 7 the permutation associated with the extended board is written in one line notation with the values to the left of the bar in the extended board and the values to the right of the bar in the shape λ. As shown in Figure 7, if $\tau = 2678|3154$ and $\sigma = 1348|5276$, then the first edge label on the unique rising chain from σ to τ would be 57, i.e. $\sigma' = (57)\sigma$. This is because $a = 7$, $\sigma^{-1}(a) = 7$ and $\tau^{-1}(a) = 3$ gives $b = 5$ by $\sigma(\tau^{-1}(a)) = 4 \leq b < a = 7$ and $\tau^{-1}(a) = 3 \leq \sigma(b) < \sigma(a) = 7$. If instead we had *sigma* $= 1678|3254$ and τ as before, then 12 would be the next transposition since 1 is the largest value between 1 and 2 in a position between 1 and 6.

There are two conditions on a permutation π in S_n required for it to lie in $f(R_\lambda^m)$:

(1) The extended board $f(\lambda)$ must contain all of the rooks in the non-extended part of the board (the bottom m rows) within the original shape λ at each swap and

(2) for the rooks in the extended part of the board (the top $n - m$ rows,)

$$\pi(1) < \pi(2) < ... < \pi(n - m).$$

We need to show that a swap from σ to σ' given by the algorithm above will not create a permutation σ' which violates either of these conditions.

First we show that σ' will not violate condition (1). Let a be the largest value such that $\sigma^{-1}(a) \neq \tau^{-1}(a)$. Let $b = \sigma(\tau^{-1}(a))$. It follows that $b < a$. Label values $c_i < a$ where $\tau^{-1}(a) \leq \sigma^{-1}(c_i) < \sigma^{-1}(a)$ such that $c_1 < c_2 < ... < c_p$. The algorithm produces a swap between a and c_p to get from rook placement σ to rook placement σ'. In order for σ' to be a legal rook placement on board $f(\lambda)$ we need λ to contain a cell at row $\sigma^{-1}(a)$ and column c_p.
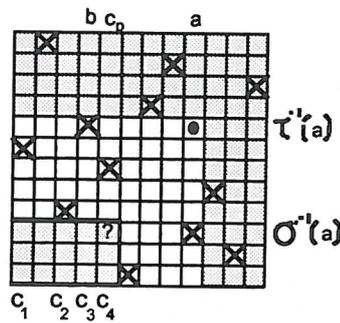

FIGURE 8. For τ to be greater than σ in Ding order, cell $(\sigma^{-1}(a), c_p)$ must exist in board λ. The X's are rooks in σ and the dot is a rook in τ.

Assume board λ does not contain cell $(\sigma^{-1}(a), c_p)$, the cell in row $\sigma^{-1}(a)$ and column c_p. Then λ must be contained within the $n \times n$ board with the lower left $c_p \times \sigma^{-1}(a)$ corner removed.

We shall show that when the comparisons are done as in Definition 3,

$$s_{p,n-\tau^{-1}(a)+1} \not\leq t_{p,n-\tau^{-1}(a)+1}.$$

We shall show that the failure is due to σ having p rooks in rows weakly below $\tau^{-1}(a)$ that are weakly to the left of column c_p, and τ having at most $p - 1$ such rooks.

There are p rooks weakly to the left of column c_p in σ. These were labeled $c_1, c_2, ..., c_p$. Any row at or below $\sigma^{-1}(a)$ cannot have rooks in columns weakly left of c_p. Since all rooks in rows $\tau^{-1}(a)$ through $\sigma^{-1}(a)$ of σ to the left of column a were accounted for by the c_i's, all other rooks in this region are in columns strictly to the right of a. They then occupy the same positions in σ and in τ, because a was the smallest value such that $\sigma^{-1}(a) \neq \tau^{-1}(a)$. Since $\sigma(\tau^{-1}(a)) = c_i$ for some $i \leq p$ and $\tau(\tau^{-1}(a)) = a$ with $a > c_p$, there can only be $p - 1$ rooks weakly below row $\tau^{-1}(a)$ in columns at or to the left of column c_p.

It follows that a rook in τ must be in the removed $c_p \times \sigma^{-1}(a)$ corner. Therefore the cell $(\sigma^{-1}(a), c_p)$ must exist on board λ. Hence, condition (1) holds.

Next we show that σ' does not violate condition (2), i.e.

$$\sigma'(1) < \sigma'(2) < ... < \sigma'(n-m).$$

Both σ and τ obey condition (2). We shall consider two cases:

Case 1: $(\sigma')^{-1}(a) \geq n - m$. Then $\sigma'(i) = \sigma(i)$ for all $i < n - m$.

Case 2: $(\sigma')^{-1}(a) < n - m$. Then for $i < (\sigma')^{-1}(a)$ it follows that $\sigma'(i) = \sigma(i)$, and for $a > \sigma(i)$ and if $i > (\sigma')^{-1}(a)$ it follows that $\sigma'(i) = \tau(i)$ and $a < \tau(i)$.

In both cases $\sigma'(1) < \sigma'(2) < ... < \sigma'(n-m)$. $\square$

We may now prove

Theorem 2 : *the poset R_λ^ω under Ding's order is EL-shellable.*

Proof. [3, Theorem 4.3] states that if P and Q are bounded finite posets, then $P \times Q$ is EL-shellable if and only if both P and Q are EL-shellable. Therefore by Proposition 88 and Theorem 17, R_λ^ω is shellable. $\square$

We shall now explain how Theorem 2 allows us to compute the Möbius function of $(R_\lambda, \leq_D)$.

Lemma 18. *[9, Prop 3.8.6] The Möbius function, $\mu(x,y)$ equals the reduced Euler characteristic, $\tilde{\chi}(\Delta(x,y))$, of the simplicial complex $\Delta(x,y)$ of chains in the open interval (x,y).*
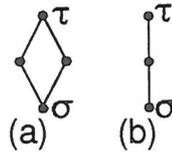

FIGURE 9. Intervals of length 2.

Corollary 19. *For $\sigma, \tau \in R_\lambda^m$, the Möbius function*

$$\mu(\sigma, \tau) = \begin{cases} (-1)^{l(\tau) - l(\sigma)} & \textit{if } [\sigma, \tau]_D \cong [f(\sigma), f(\tau)]_B \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Björner showed [2] that in an EL-shellable poset with all intervals of length 2 as in Figure 9 (a) and (b), if $\sigma < \tau$ with $l(\tau) - l(\sigma) \geq 2$, then $\Delta(x,y)$ is homeomorphic to:

$$\begin{cases} \mathbf{S}^{l(\tau)-l(\sigma)-2} & \text{if all length 2 intervals in } [\sigma, \tau] \text{ look like (a)} \\ \mathbf{B}^{l(\tau)-l(\sigma)-2} & \text{if any length 2 interval in } [\sigma, \tau] \text{ looks like (b)} \end{cases}$$

where $\mathbf{S}^d$ is the d-dimensional sphere, and $\mathbf{B}^d$ is the d-dimensional ball.

The poset R_λ^m will have length 2 intervals all of type (a) or (b) of Figure 9, since $(S_n, \leq_B)$ has all length two intervals of type (a). Type (b) occurs when one of the permutations in the interval $[\sigma, \tau]$ does not have its rooks all on the board λ. The reduced Euler characteristic of $\mathbf{S}^{l(\tau)-l(\sigma)-2}$ is $(-1)^{l(\tau)-l(\sigma)}$ and of $\mathbf{B}^{l(\tau)-l(\sigma)-2}$ is 0. $\square$

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Acknowledgements

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