MACMAHON SYMMETRIC FUNCTIONS AND THE PARTITION LATTICE

MERCEDES H. ROSAS

ABSTRACT. A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group. We show that the MacMahon symmetric functions are the generating functions for the orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of a symmetric group. As an application of this combinatorial interpretation of the MacMahon symmetric functions we give a proof of a special case of a conjecture of Gessel and we relate MacMahon symmetric functions with Stanley's chromatic symmetric function of a graph. Finally, we show how to compute the transition matrices as well as the scalar product and the Kronecker product of the different bases for the ring of MacMahon symmetric functions.

1. INTRODUCTION

A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group.

We study the relationship between the ring of MacMahon symmetric functions, the partition lattice and Young subgroups of the symmetric group. We provide a combinatorial interpretation for the MacMahon symmetric functions in terms of orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of a symmetric group.

As an application we define the chromatic MacMahon symmetric function of a graph G. It determines the degree sequence of graph G. Moreover, it specializes to the chromatic symmetric function of Stanley [15].

Finally, we show how to compute the transition matrices as well as the scalar product and the Kronecker product of the different bases of the ring of MacMahon symmetric functions.

MacMahon symmetric functions where introduced by MacMahon [10], Vol. II, section XI, p. 281–332. MacMahon applied them to the problem of placing balls into boxes and to the theory of Latin squares.

In [6] Ira Gessel used MacMahon symmetric functions to derive explicit formulas for the number of $2 \times n$ Latin squares, of 0-1 matrices with trace 0, and of words in a partially commuting monoid. Moreover, he extended the concept of P-recursiveness to MacMahon symmetric functions and showed that for fixed k,

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the number of $k \times n$ latin squares is P-recursive as a function of n. Similarly, he showed that for fixed k, the number of $k \times n$ 0–1 matrices with zeros on the diagonal and every row and cloumn sum k is P-recursive as a function of n.

MacMahon symmetric functions appear in other areas of mathematics. For instance, Gelfand and Dikki [4], and Olver and Shakiban [11] used them in connections with the theory of partial differential equations. Futhermore, Adem, Maginis and Milgram [1] used them in the study of the cohomology of the symmetric group.

Let u be a vector in $\mathbf{N}^\infty = \bigcup_{k\geq 1} \mathbf{N}^k$. A vector partition of u is an unordered sequence of vectors λ adding to u. We write $\lambda \vdash u$ to indicate that λ is a vector partition of u. The nonzero vectors are called the parts of λ. Usually, we write

$$\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2)\ldots.$$

We define $\lambda! = b_1!r_1!\cdots w_1!b_2!r_2!\cdots w_2!\cdots$.

Sometimes, it is convenient to write λ using block notation. That is, if vector partition λ has $m(b_i, r_i, \cdots, w_i)$ copies of part $(b_i, r_i, \cdots, w_i)$ for each i, then we write

$$\lambda = \prod_i (b_i, r_i, \ldots, w_i)^{m(b_i, r_i, \ldots, w_i)}.$$

We define $|\lambda| = \prod_i m(b_i, r_i, \ldots, w_i)!$.

The weight of a vector u, written as weight(u), is the sum of the coordinates of u. Let $X = x_1 + x_2 + \cdots$, $Y = y_1 + y_2 + \cdots$, $\cdots$, and $Z = z_1 + z_2 + \cdots$ be infinite alphabets. Given any formal power series in $X, Y, \cdots$ and Z there is a natural action of $\pi \in S_\infty$ on f called the diagonal action of π in f and defined by

$$\pi f(x_1, y_1, \cdots, z_1, x_2, y_2, \cdots, z_2, \cdots) = \\ f(x_{\pi 1}, y_{\pi 1}, \cdots, z_{\pi 1}, x_{\pi 2}, y_{\pi 2}, \cdots, z_{\pi 2}, \cdots).$$

A formal power series f in the k alphabets $X, Y, \cdots, Z$ is called a MacMahon symmetric functions in k systems of indeterminates if it is invariant under the diagonal action of each $\pi \in S_\infty$ and if the multidegree of f is bounded. Let $\mathfrak{M}^{(k)}$ be the ring of MacMahon symmetric functions on k systems of indeterminates. We have that

$$\mathfrak{M}^{(1)} \subseteq \mathfrak{M}^{(2)} \subseteq \cdots \subseteq \mathfrak{M}^{(k)} \subseteq \cdots$$

We define the set of MacMahon symmetric functions, denoted by $\mathfrak{M}$, as

$$\mathfrak{M} = \bigcup_{k\geq 1} \mathfrak{M}^{(k)}.$$

Let f and g be MacMahon symmetric functions. Since $f + g$ and fg are in $\mathfrak{M}$ when both f and g are, it follows that $\mathfrak{M}$ has a ring structure. Moreover, it has a graded ring structure:

$$\mathfrak{M} = \bigcup_{n\geq 0} \bigcup_{\substack{u \\ \text{weight}(u)=n}} \mathfrak{M}_u.$$

where $\mathfrak{M}_u$ is the vector space of MacMahon symmetric functions of homogeneous multidegree u. Akin to the homogeneous pieces of the ring of symmetric functions, the vector spaces $\mathfrak{M}_u$ have bases, indexed by vector partitions of u, called the monomial, elementary, power sum, homogeneous, or forgotten MacMahon symmetric functions, respectively. The first four basis were introduced by MacMahon in [10]. The forgotten MacMahon symmetric functions are defined following Doubilet [3]. Finally, the one-dimensional MacMahon symmetric functions were introduced by Gessel [7]. We follow the notation of Macdonald [6, 9].

The Monomial MacMahon Symmetric Functions. We have that any vector partition $\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2)\ldots$ determines a monomial $\mathbf{x}^\lambda$:

$$\mathbf{x}^\lambda = x_1^{b_1} y_1^{r_1} \cdots z_1^{w_1} x_2^{b_2} y_2^{r_2} \cdots z_2^{w_2} \cdots x_l^{b_l} y_l^{r_l} \cdots z_l^{w_l}.$$

The monomial MacMahon symmetric functions indexed by λ is the sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation π in S_k, where the action of π in $\mathbf{x}^\lambda$ corresponds to the diagonal action. We have that

$$m_\lambda = \sum x_{i_1}^{b_1} y_{i_1}^{r_1} \cdots z_{i_1}^{w_1} x_{i_2}^{b_2} y_{i_2}^{r_2} \cdots z_{i_2}^{w_2} \cdots.$$

where the sum is taken over all different monomials with exponents $(b_1, r_1, \ldots, w_1)$ $(b_2, r_2, \ldots, w_2)\ldots$.

The Elementary MacMahon Symmetric Functions. We define $e_{(b,r,\cdots,w)}$ by

$$\sum_{b,r,\cdots,w} e_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_i (1 + x_i s + y_i t + \cdots + z_i u).$$

Let λ be a vector partition. We set $e_\lambda = e_{(b_1,r_1,\cdots,w_1)} e_{(b_2,r_2,\cdots,w_2)} \cdots$.

The Complete Homogeneous MacMahon Symmetric Functions. We define $h_{(b,r,\cdots,w)}$ by

$$\sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_i \frac{1}{1 - x_i s - y_i t - \cdots - z_1 u}.$$

Let λ be a vector partition. We set $h_\lambda = h_{(b_1,r_1,\cdots,w_1)} h_{(b_2,r_2,\cdots,w_2)} \cdots$.

The Power Sum MacMahon Symmetric Functions. We define $p_{(b,r,\cdots,w)}$ by

$$p_{(b,r,\cdots,w)} = \sum_i x_i^b y_i^r \cdots z_i^w = m_{(b,r,\cdots,w)}.$$

Let λ be a vector partition. We set $p_\lambda = p_{(b_1,r_1,\cdots,w_1)} p_{(b_2,r_2,\cdots,w_2)} \cdots$.

The Forgotten MacMahon Symmetric Functions. Let ω be the involution in $\mathfrak{M}$ defined by $\omega(e_\lambda) = h_\lambda$. The forgotten MacMahon symmetric functions are defined by

$$\omega(m_\lambda) = (\text{sign } \lambda)\, f_\lambda.$$

The One-dimensional MacMahon Symmetric Functions. We define $g_{(b,r,\cdots,w)}$ by

$$1 + \sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_{gcd(b,r,\cdots,w)=1} (1 + \sum_{i=1}^{\infty} g_{(b_i,r_i,\cdots,w_i)} s^{b_i} t^{r_i} \cdots u^{w_i}).$$

Let λ be a vector partition. We set $g_\lambda = g_{(b_1,r_1,\cdots,w_1)} g_{(b_2,r_2,\cdots,w_2)} \cdots$.

A MacMahon symmetric functions is unitary if it is indexed by a partition of $(1)^n = (1, 1, \cdots, 1)$.

We assume the reader to be familiar with the partition lattice and its Möbius function (See [2, 15].) Moreover, we assume the reader to be familiar with the notions of alphabet, words and of Lyndon words (See [8, 12].)

A sentence in A^* is a totally ordered multiset of words. We denote the set of all sentences in A^* by A^S. The evaluation of a word ω, denoted by $\text{ev}(\omega)$, is the monomial $x^b y^r \cdots z^w$ in $\mathbf{Q}[x, y, \cdots, z]$ where $b = |\omega|_X$, $r = |\omega|_Y$, and $w = |\omega|_Z$. The evaluation of a sentence $S = \omega_1\omega_2\cdots\omega_l$, denoted by $\text{ev}(S)$, is the monomial $\text{ev}(\omega_1)\text{ev}(\omega_2)\cdots\text{ev}(\omega_l)$ where $\text{ev}(\omega_i) = x_i^b y_i^r \cdots z_i^w$. If L is a set of words or sentences, then the generating function of L is the sum of the evaluations of the elements of L. For the rest of the paper, we let $A = \{X, Y, \cdots, Z\}$ be a noncommutative alphabet and we let $\{x, y, \cdots, z\}$ be commutative variables.

2. Basic Constructions

Let $u = (b, r, \cdots, w)$ be a vector in $\mathbf{N}^\infty$ of weight n. (That is, the sum of the coordinates of u is n). Let S_u be a Young subgroup of S_n. We have that S_u acts on $[n]$ by restricting the canonical action of S_n on $[n]$ to permutations in S_u. Moreover, this action partitions $[n]$ into equivalence classes. Two elements belong to the same equivalence class if and only if there is a permutation σ in S_u such that $\sigma n_1 = n_2$. We order the equivalence classes using the smallest element in each of them.

Let $[n]_u$ be the set of ordered pairs (i, k) where i belongs to the kth equivalence class of $[n]$ under the action of S_u. Sometimes, we denote the pair (i, k) by putting k dots over i. If $u = (b, r, \cdots, w)$, then we may think of b as the number of blue elements in $[n]_u$, of r as the number of red elements in $[n]_u$, and so on. Similarly, let $(\Pi_n)_u$ be the lattice of partitions of $[n]_u$. Given a partition π in Π_n, the action of S_u on $[n]$ defines a partition π_u in $(\Pi_n)_u$ by replacing element i in π by the ordered pair (i, k). The elements of $(\Pi_n)_u$ are called colored partitions.

Let u be a vector of weight n. The type of π_u, written type(π_u), is the vector partition of u defined by saying that type(π_u) has part $(b_i, r_i, \cdots, w_i)$ with multiplicity m_i if and only if there are exactly m_i blocks of π_u with b_i elements in the first equivalence class, r_i elements in the second equivalence class, and so on until the last equivalence class that has w_i elements.

If π is a partition of $[n]$, then type(π) is a unitary vector partition. Moreover, the partition π can be recovered from type(π). In the rest of this paper, we identify a given partition π with the unitary vector partition type(π).

Example 1 ($S_{(1,2)}$ acts on Π_3). We explore the action of $S_{(1,2)}$ on Π_3. First, we have that [3] is partitioned into two equivalence classes: $\{1\}$ and $\{2,3\}$.

1. Set partition $\pi = 1|2|3$ has type $(1,0,0)(0,1,0)(0,0,1)$.
   Under the action of $S_{(1,2)}$, π is colored as $\pi_{(1,2)} = \dot{1}|\ddot{2}|\ddot{3}$.
   Moreover, type$(\pi_{(1,2)}) = (1,0)(0,1)(0,1)$.
2. Set partition $\pi = 12|3$ has type $(1,1,0)(0,0,1)$.
   Under the action of $S_{(1,2)}$, π is colored as $\pi_{(1,2)} = \dot{1}\ddot{2}|\ddot{3}$.
   Moreover, type$(\pi_{(1,2)}) = (1,1)(0,1)$.

Definition 2. Let F_n to be the set of all functions from $[n]$ to $\mathbf{P}$. That is,

$$F_n = \{f : [n] \to \mathbf{P}\}.$$

We say that ball i is in box j if $f(i) = j$.

Let f be a function from $[n]_u$ to $\mathbf{P}$. We weight f by

$$\gamma(f) = \prod_{d\in[n]} c(d)_{f(d)}$$

where $c(d)$ denotes the color of ball d and we use variables $x, y, \cdots, z$ to denote the colors of the balls. To any set T we associate the generating function:

$$\gamma(T) = \sum_{f\in T} \gamma(f).$$

Given any $f \in F$, we define a partition of $[n]$, denoted $\ker f$, by saying that n_1 and n_2 are in the same block of $\ker f$ if and only if $f(n_1) = f(n_2)$.

Following Doubilet, we define a placing p to be an arrangement of the balls (that is, the elements of $[n]$) into the boxes (that is, the positive numbers, $\mathbf{P}$), in which the balls in each box may be placed in some configuration. Note that any function is a placing where the balls are in no special configuration.

Define P_n^c to be the set of all placings from $[n]$ to $\mathbf{P}$ with prescribed configuration c.

$$P_n^c = \{p : [n] \to \mathbf{P} \text{ with configuration } c\}.$$

We define the underlying function of the placing p to be the function obtained from p if we forget about the extra data given by the configuration of the balls. The weight of a placing is defined as the weight of its underlying function. The kernel of a placing p, written as $\ker p$, is defined as the kernel its underlying function.

Definition 3 (The projection map). Let S_u be a Young subgroup of S_n acting on Π_n. Let $p : [n] \to \mathbf{P}$ be a placing. We define $p_u : [n]_u \to \mathbf{P}$ by

$$p_u((i,k)) = p(i).$$

The map sending $p \mapsto p_u$ is called the projection map and denoted by ρ_u.

3. A combinatorial interpretation of the MacMahon symmetric functions

In this section we define three classes of sets of functions and two classes of sets of placings. Each of these is indexed by partitions in Π_n. We show how their corresponding generating functions are related to the homonymous basis of the space of unitary MacMahon symmetric functions. Moreover, we show that any of monomial, elementary, power sum, homogeneous, or forgotten MacMahon symmetric functions can be obtained as the generating function of the image of one these sets under the action of a Young subgroup of the symmetric group.

Definition 4. Let π be a set partition of $[n]$.

1. Let $\mathcal{M}_\pi$ be the subset of F_n defined by

$$\mathcal{M}_\pi = \{f : f \in F_n, \ker f = \pi\},$$

   and let m_π be its generating function, $m_\pi = \gamma(\mathcal{M}_\pi)$.
2. Let $\mathcal{P}_\pi$ be the subset of F_n defined by

$$\mathcal{P}_\pi = \{f : f \in F_n, \ker f \geq \pi\},$$

   and let p_π be its generating function, $p_\pi = \gamma(\mathcal{P}_\pi)$.
3. Let $\mathcal{E}_\pi$ be the subset of F_n defined by

$$\mathcal{E}_\pi = \{f : f \in F_n, \ker f \wedge \pi = \hat{0}\},$$

   and let e_π be its generating function $e_\pi = \gamma(\mathcal{E}_\pi)$.
4. Let $\mathcal{H}_\pi$ be the set of placings p such that within each box the balls from the same block of π are linearly ordered. Let $h_\pi = \gamma(\mathcal{H}_\pi)$ be its generating function.
5. Let $\mathcal{F}_\pi$ be the set of placings such that balls from the same block of π go into the same box, and within each box the blocks appearing are linearly ordered. Let $f_\pi = \gamma(\mathcal{F}_\pi)$ be its generating function.

Main Theorem. *Let S_u be a Young sugroup of S_n. Let π be a partition of $[n]$ and let λ be the type of π_u. We have that*

$$\begin{aligned}\rho_u : \mathfrak{M}_{(1)^n} &\to \mathfrak{M}_u \\ m_\pi &\mapsto |\lambda| m_\lambda \\ p_\pi &\mapsto p_\lambda \\ e_\pi &\mapsto \lambda! e_\lambda \\ h_\pi &\mapsto \lambda! h_\lambda \\ f_\pi &\mapsto |\lambda| f_\lambda\end{aligned}$$

For a proof of the Main theorem see [13]. In Example 5 we try to give the idea of the proof. (To proof that $h_\pi \mapsto \lambda! h_\lambda$ we first show that $h_{(b,r,\cdots,w)}$ is the generating function for sentences S in A^S such that $|S|_X = b$, $|S|_Y = r$, and $|S|_Z = w$.)

Example 5 ($S_{(4,2)}$ acts on Π_6). Let π be 12|35|46. The type of $\pi_{(4,2)}$ is given by vector partition $\lambda = (2,0)(1,1)^2$.

1. If $f \in \mathcal{M}_\pi$, then $\ker f = \pi$. It follows that $\dot{1}, \dot{2} \mapsto i$, $\dot{3}, \ddot{4} \mapsto j$, and $\dot{5}, \ddot{6} \mapsto k$, and i, j, and k are different indices. Therefore $\gamma(f) = x_i^2 x_j y_j x_k y_k$. Moreover,

$$\gamma(\mathcal{M}_\pi) = \sum_{i,j,k} x_i^2 x_j y_j x_k y_k = 2 \sum_{i,j<k} x_i^2 x_j y_j x_k y_k = 2m_{(2,0)(1,1)^2}.$$

2. If $f \in \mathcal{P}_\pi$, then $\ker f \geq \pi$. Hence, $\dot{1}, \dot{2} \mapsto i$, $\dot{3}, \ddot{5} \mapsto j$, and $\dot{4}, \ddot{6} \mapsto k$, where i, j, k are not necessarily different. Therefore, $\gamma(f) = x_i^2 x_j y_j x_k y_k$. Moreover,

$$\gamma(\mathcal{P}_\pi) = \sum_i x_i^2 \sum_j x_j y_j \sum_k x_k y_k = p_{(2,0)(1,1)^2}.$$

3. If $f \in \mathcal{E}_\pi$ then $\ker f \wedge \pi = \hat{0}$. It follows that $\dot{1}$ and $\dot{2}$ should be in different blocks of $\ker f$, and that $\dot{3}$ and $\ddot{5}$ (and $\dot{4}$ and $\ddot{6}$) also should be in different blocks of $\ker f$. We have that,

$$\begin{aligned}\gamma(\mathcal{E}_\pi) &= \sum_{i \neq j} x_i x_j \sum_{i \neq j} x_i y_j \sum_{i \neq j} x_i y_j \\ &= 2 \sum_{i<j} x_i x_j \sum_{i \neq j} x_i y_j \sum_{i \neq j} x_i y_j = 2e_{(2,0)(1,1)^2}.\end{aligned}$$

4. Let $f \in \mathcal{H}_\pi$. We have that $\gamma(\mathcal{H}_\lambda) = \gamma(\mathcal{H}_{[\dot{1},\dot{2}]})\gamma(\mathcal{H}_{[\dot{3},\ddot{5}]})\gamma(\mathcal{H}_{[\dot{4}\ddot{6}]})$. Therefore,

$$\begin{aligned}\gamma(\mathcal{H}_\lambda) &= 2(\sum_i x_i^2 + \sum_{i<j} x_i x_j)(2 \sum_i x_i y_i + \sum_{i \neq j} x_i y_j)^2 \\ &= 2h_{(2,0)} h_{(1,1)^2} = 2h_\lambda.\end{aligned}$$

Corollary 6. *Let S_n be the symmetric group. Let π be a partition in Π_n and let λ be the type π_n. (Note that λ is a partition of n.) We have that*

$$\rho_n : \mathfrak{M}_{(1)^n} \rightarrow \mathfrak{M}_n$$

where $\mathfrak{M}_n$ is the vector space of symmetric functions of degree n. Moreover, we obtain that m_λ, p_λ, e_λ, h_λ, and f_λ are the usual symmetric functions.

4. The Chromatic symmetric function of a Graph

The chromatic symmetric functions of a graph can be defined in terms of $\mathcal{M}_\pi$. We introduce some terminology. A simple graph is a graph without loops or multiple edges. Let G be a simple graph with vertex set $[n]$ and edge set E. A function $\kappa : [n] \rightarrow \mathbf{P}$ is called a proper coloring of G if $\kappa(u) \neq \kappa(v)$ whenever u and v are vertices of an edge of G.

A stable partition π of G is a partition of $[n]$ such that each block of π is totally disconnected. Let $S(G)$ be the set of all stable partitions of G.

Definition 7 (Stanley [16]). Let G be a simple graph with vertex set $[n]$ and edge set E. The chromatic symmetric function is defined as

$$X_G = \sum_{\kappa} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

where the sum ranges over all proper colorings $\kappa : [n] \rightarrow \mathbf{P}$.

We define the set $\mathcal{F}_n|_G$ in terms of the $\mathcal{M}_\pi$ and show that the generating function of this set is the chromatic symmetric function.

Definition 8. Let G be a simple graph with vertex set $[n]$ and edge set E. Let $\mathcal{F}_n|_G$ be the subset of F_n defined by

$$\begin{aligned} \mathcal{F}_n|_G &= \{f : [n] \rightarrow \mathbf{P} : \text{for any } \{n_1, n_2\} \in E, f(n_1) \neq f(n_2)\} \\ &= \bigcup_{\lambda \vdash d} \bigcup_{\pi \in S(G)} \mathcal{M}_\pi. \end{aligned}$$

Proposition 9. *Let S_n be the symmetric group. Let $\{\mathcal{F}_n|_G\}_n$ be the image of $\mathcal{F}_n|_G$ under ρ_n. We have that*

$$\gamma(\{\mathcal{F}_n|_G\}_n) = X_G.$$

We obtain a MacMahon symmetric function if instead of S_n we have one of its Young subgroup acting on $\mathcal{M}_\pi$. This may be desirable if we have a graph (or directed graph) G and that we have some exceptional vertices that we want to keep track of. Then, may distinguish between the different kinds of vertices by using different kinds variables to weight them. For instance, suppose that we are interested in the degree sequence of graph G. Then, we define the MacMahon chromatic symmetric function as follows.

Definition 10. Let G be a simple graph with vertex set $[n]$ and edge set E. The chromatic MacMahon symmetric function (in alphabets $X_1, X_2, \cdots$) is defined by

$$\bar{X}_G = \sum_{\kappa} x_{d(1),\kappa_1} x_{d(2),\kappa_2} \cdots x_{d(n),\kappa_n}.$$

Proposition 11. *Let $\bar{X}_G$ be the chromatic MacMahon symmetric function of graph G. Then we have that*

1. *There is a vector $u = (u_1, u_2, \cdots, u_n)$ such that $\bar{X}_G \in \mathfrak{M}_u$.*
2. *The degree sequence of G is $(1^{u_1}2^{u_2}\cdots n^{u_n})$.*
3. *We can recover the chromatic symmetric function X_G from $\bar{X}_G$:*

$$\rho_u(\bar{X}_G) = X_G$$

Example 12. In [15] Stanley computed the chromatic symmetric function for graphs G and H and showed that $X_G = X_H$. See Figure 1.

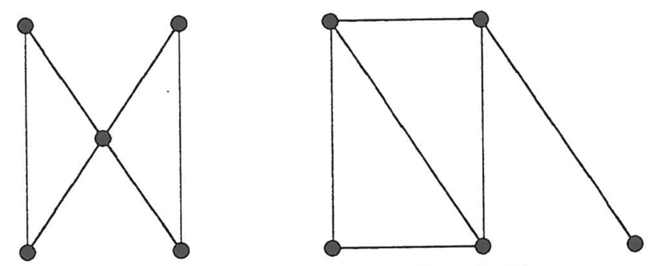

FIGURE 1.

The chromatic MacMahon symmetric function of G and H are given by

$$\bar{X}_G = 2m_{(0,2,0,0)^2(0,0,0,1)} + 4m_{(0,2,0,0)(0,1,0,0)^2(0,0,0,1)} + m_{(0,1,0,0)^4(0,0,0,1)}.$$

$$\begin{aligned}\bar{X}_H = 2m_{(0,1,1)(1,0,1)(0,0,1)} &+ 2m_{(1,0,1)(0,0,1)^2(0,1,0)} \\ &+ m_{(0,1,1)(0,0,1)^2(1,0,0)} + m_{(1,1,0)(0,0,1)^3} \\ &+ m_{(0,0,1)^3(0,1,0)(1,0,0)}.\end{aligned}$$

That is, the chromatic MacMahon symmetric function distinguishes between these two graphs.

We have that $X_G \in \mathfrak{M}_{(0,4,0,1)}$. Therefore, the degree sequence of G is $(2,2,2,2,4)$. Similarly, we have that $X_H \in \mathfrak{M}_{(1,1,3)}$. Therefore, the degree sequence of H is $(1,2,3,3,3)$.

We have that $\rho_{(0,4,0,1)}(\bar{X}_G) = \rho_{(1,1,3)}(\bar{X}_H)$ is the chromatic symmetric function of both graph G and graph H.

5. The one-dimensional MacMahon symmetric functions and an special case of a conjecture of Gessel.

We prove a special case of a conjecture of Ira Gessel using our combinatorial interpretation for the MacMahon symmetric functions.

Conjecture 13 (Gessel). *Let λ be a vector partition of u. The sign of the coefficient of h_λ in g_u is $(-1)^{l(\lambda)-1}$.*

We obtain that the following proposition holds.

Proposition 14. 1. *Let π be a partition of $(1)^n$. Then,*

$$[h_\pi]g_{(1)^n} = (-1)^{(l(\pi)-1)}(l(\pi)-1)!.$$

2. *Let λ be a vector parition of $(1,n)$. Then*

$$[h_\pi]g_{(1,n)} = \frac{(-1)^{(l(\lambda)-1)}(l(\lambda)-1)!}{|\lambda|}.$$

In the proof of Proposition 14 we use the projection map and following characterization of the one-dimensional MacMahon symmetric functions.

Proposition 15 (Gessel). *Let $b, r, \cdots$, and w be coprime numbers. Let*

$$G_{(nb,nr,\cdots,nw)} = \{M : M \text{ is a multiset of Lyndon words in } A^S \text{ and } |M|_X = nb, |M|_Y = nr, \cdots |M|_Z = nw.\}$$

Then, $\gamma(G_{(nb,nr,\cdots,nw)}) = g_{(nb,nr,\cdots,nw)}$.

6. The scalar and Kronecker products of MacMahon symmetric functions.

Following Gessel [6], we define a scalar product on $\mathfrak{M}$ by $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$. We have that for any MacMahon symmetric functions f and for any vector partition λ, the inner product $\langle h_\lambda, f\rangle$ gives the coefficient of $x_1^{b_1}\cdots z_1^{w_1}x_2^{b_2}\cdots z_2^{w_2}\cdots$ in f.

We define the Kronecker product in the ring of MacMahon symmetric functions by $p_\lambda * p_\mu = \langle p_\lambda, p_\mu\rangle p_\lambda$ and extend by linearity.

We define a lifting map $\hat{\rho}_u : \mathfrak{M}_u \to \mathfrak{M}_{(1)^n}$. This map allows us to compute the Kronecker product and the scalar product of Macmahon symmetric functions. To prove proposition 16 and 17 we use the fact that the number of partition $\pi \in \Pi_n$ such that π_u has type λ is $\binom{u}{\lambda} = \frac{u!}{\lambda!|\lambda|}$.

Definition 16 (The lifting map). Define $M_\lambda = \binom{u}{\lambda}|\lambda| m_\lambda$. Then, we define the lifting map $\hat{\rho}_u : \mathfrak{M}_u \to \mathfrak{M}_{(1)^n}$ by

$$\hat{\rho}_u(M_\lambda) = \sum_{\pi\in\lambda} m_\pi$$

where the sum is taken over all π of type λ.

(Similarly, we define $P_\lambda = \binom{u}{\lambda} p_\lambda$, $E_\lambda = \binom{u}{\lambda}\lambda!\, e_\lambda$, $H_\lambda = \binom{u}{\lambda}\lambda!\, h_\lambda$, and $F_\lambda = \binom{u}{\lambda}|\lambda| f_\lambda$. Then, we obtain that $\hat{\rho}_u(P_\lambda) = \sum_{\pi\in\lambda} p_\pi$, $\hat{\rho}_u(E_\lambda) = \sum_{\pi\in\lambda} e_\pi$, $\hat{\rho}_u(H_\lambda) = \sum_{\pi\in\lambda} h_\pi$, and that $\hat{\rho}_u(F_\lambda) = \sum_{\pi\in\lambda} f_\pi$.)

Proposition 17. *The lifting map $\hat{\rho}_u$ has the property that*

$$\rho_u \hat{\rho}_u = 1$$

Moreover, we have that for all $f, g \in \mathfrak{M}_u$

$$\langle f, g\rangle = u!\langle \hat{\rho}_u(f), \hat{\rho}_u(g)\rangle.$$

Proposition 18. *Let f and g be functions in $\mathfrak{M}_u$. Then*

1. *The map $\hat{\rho}_u$ satisfies*
$$\hat{\rho}_u(f * g) = u!\hat{\rho}_u(f) * \hat{\rho}_u(g).$$
2. *The Kronecker products on $\mathfrak{M}_{(1)^n}$ and $\mathfrak{M}_u$ are related by*
$$f * g = u!\rho(\hat{\rho}_u(f) * \hat{\rho}_u(g)).$$
3. *The homomorphism ω is an algebra homomorphism. That is,*
$$\omega(f) * \omega(g) = f * g.$$

for all $f, g \in \mathfrak{M}_u$.

Propositions 17 and 18 together with the Main Theorem and [3] allow us to compute the scalar product of any two functions in $\mathfrak{M}$.

Example 19 (The inner product $\langle p_\lambda, p_\mu\rangle$). To compute $\langle p_\lambda, p_\mu\rangle$ we proceed as follows:

$$\begin{aligned}\langle P_\lambda, P_\mu\rangle &= u!\langle \hat{\rho}_u(P_\mu), \hat{\rho}_u(P_\lambda)\rangle = u!\Big\langle \sum_{\pi\in\lambda} p_\pi, \sum_{\sigma\in\mu} p_\sigma\Big\rangle \\ &= u!\sum_{\substack{\pi\in\lambda\\ \sigma\in\mu}}\langle p_\pi, p_\sigma\rangle = u!\sum_{\pi\in\lambda}\frac{\delta_{\pi,\sigma}}{|\mu(\hat{0},\pi)|} = u!\binom{u}{\lambda}\frac{\delta_{\pi,\sigma}}{|\mu(\hat{0},\pi)|}\end{aligned}$$

Therefore, we have that $\langle p_\lambda, p_\lambda\rangle = u!\dfrac{\delta_{\pi,\sigma}}{\binom{u}{\lambda}|\mu(\hat{0},\pi)|}$.

Example 20 (The Kronecker product $h_\lambda * h_\mu$). To compute $h_\lambda * h_\mu$ we proceed as follows:

$$\begin{aligned}h_\lambda * h_\mu &= \frac{1}{\lambda!\binom{u}{\lambda}\mu!\binom{u}{\mu}} H_\lambda * H_\mu = \frac{1}{\lambda!\binom{u}{\lambda}\mu!\binom{u}{\mu}}\rho_u\Big(\hat{\rho}_u(H_\lambda) * \hat{\rho}_u(H_\mu)\Big) \\ &= \frac{|\lambda||\mu|}{u!}\rho_u\Big(\sum_{\pi\in\lambda} h_\pi * \sum_{\sigma\in\mu} h_\sigma\Big) = \frac{|\lambda||\mu|}{u!}\sum_{\substack{\pi\in\lambda\\ \sigma\in\mu}}\rho_u(h_{\pi\wedge\sigma})\end{aligned}$$

$$= \frac{|\lambda||\mu|}{u!} \sum_{\substack{\pi \in \lambda \\ \sigma \in \mu}} \operatorname{type}(\pi \wedge \sigma)! h_{\operatorname{type}(\pi \wedge \sigma)}$$

References

[1] A. Adem, J. Maginnis and R.J. Milgram, *Symmetric invariants and cohomology of groups,* Math. Ann. **287** (1990), 391–491.

[2] G.E. Andrews, *The theory of partitions,* Cambridge Mathematical Library, Cambridge University Press, 1998.

[3] P. Doubilet, *On the Foundations of Combinatorial Theory. VII: Symmetric Functions through the Theory of Distribution and Occupancy,* Stud. Appl. Math. Vol. LI No. 4 (1972), 377-396. Reprinted in *Gian-Carlo Rota in Combinatorics,* J.P. Kung Ed., Birhäuser, (1995), 403–422.

[4] I.M. Gelfand and L.A. Dikii, *Asymptotic behaviour of the resolvent of Sturm-Lioville equations and the algebra of Korteweg-deVries equations,* Russian Math. Surveys **30** (1975), 77–113.

[5] I.M. Gessel, *Symmetric Functions and P-recursiveness,* J. of Comb. Th. **A. 53** (1988), 257–285.

[6] I.M. Gessel, *Enumerative applications of Symmetric Functions,* Actes 17[e] Séminaire Lotharingien, pp 5–21, Publ. I.R.M.A. Strasbourg, 348/ 5-17, 1988.

[7] I.M. Gessel, *Private Comunication,* 1997.

[8] M. Lothaire, *Combinatorics on words,* Cambridge Mathematical Library, Cambridge University Press, 1997.

[9] I.G. Macdonald, *Symmetric Functions and Hall Polynomials,* second edition, Oxford Science Publications, 1995.

[10] P.A. MacMahon, *Combinatory Analysis,* Vol 1-2, Cambridge University Press (1915, 1916); reprinted by Chelsea, New York (1960).

[11] P.J. Olver and C. Shakibar, *Dissipative decomposition of partial equations,* Rocky Mountain J. Math. **4** (1992), 1483–1503.

[12] C. Reutenauer, *Free Lie Algebras,* London Math. Soc. Monographs. New Series. **7**. Oxford University Press, 1993.

[13] M.H. Rosas, *MacMahon Symmetric Functions, the Partition Lattice and Young Subgroups,* preprint 1998.

[14] G.-C. Rota and J.A. Stein, *Plethystic algebras and vector symmetric functions,* Proc. Natl. Acad. Sci. USA, **91**, 13062–13066, December 1994. Mathematics.

[15] R. Stanley, *Enumerative Combinatorics,* Vol I, Cambridge Studies in Advance Mathematics **49**, Cambridge University Press, 1997.

[16] R. Stanley, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph,* Adv. in Math., **111**, 166–194 (1995)

Department of Mathematics, Brandeis University, Waltham, MA 02254
E-mail address: rosas@math.brandeis.edu