On combinatorial equalities of coordination sequences *

Joan Serra-Sagristà †

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Abstract

The coordination sequence $\{S(m)\}$ of a lattice gives the number of lattice points that are m bonds away from a given point. This sequence may be useful in applications of lattices as vectors quantizers, for instance, in still image and video coding. We show here several combinatorial equalities between explicit formulae for the coordination sequences of two important lattices. A combinatorial explanation for these formulae is also provided.

Abstract

La séquence de coordination $\{S(m)\}$ d'un réseau donne le numéro de points qui sont à distance m d'un certain point. Cette séquence s'avère utile lors de l'application des réseaux en la quantification vectorielle des signaux, par exemple, dans la compression d'images et de video. On montre ici quelques égalités combinatoriques entre explicites formules pour la séquence de coordination de deux réseaux importants. On fournit aussi une explication combinatorique de ces formules.

Keywords: Coordination sequences; combinatorial equalities; lattice enumeration.

1 Introduction

The *coordination sequence* of an infinite graph G is the infinite sequence $\{S(0), S(1), S(2), \ldots\}$, where $S(m)$ is the number of vertices at distance m from some fixed vertex of G.

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†The author is with the Computer Science Department, Universitat Autònoma de Barcelona, Spain. E-mail: jserra@ccd.uab.es

A d-dimensional *lattice* Λ_d in $\mathbb{R}^d$, is a set of points such that

$$\Lambda_d = \{x \in \mathbb{R}^d \ : \ x = \xi M\},$$

where $\xi = (\xi_1, \xi_2, .., \xi_d) \in \mathbb{Z}^d$ and M is a generator matrix whose rows are the basis vectors $\nu_1, \nu_2, \ldots, \nu_d$, each of dimension d. The definition of the different lattice types is postponed to the following sections.

Throughout this paper, G is the *contact graph* of a d-dimensional lattice packing, formed by taking the vertices to be the points of the lattice and joining each point to its closest neighbors. Notice that G is a regular and a distance-regular graph.

If G is a distance-transitive graph with some fixed choice of origin, and v is a vertex of G, then $ht(v)$, the height of v, is the number of edges in the shortest path from v to the origin. Then

$$S(m) = \#\{v \in G \ : \ ht(v) = m\}.$$

In case of subband coding applications, as still images or video, lattices are commonly used as vectors quantizers, thus reducing the exponential increase in computational search and storage that most of the VQ techniques suffers when the vector dimension grows [4], and taking advantage of the optimality conjecture of lattices in the high rate [5]. To this goal, lattice points should be enumerated, and due to the statistics of images, the l_1 norm is preferable to the l_2 norm [3], therefore making useless the Θ series of a lattice [1].

From now on let $N_\Lambda(d, m)$, the *contour points*, denote the number of points $x \in \Lambda_d$ such that $\|x\|_1 = m$,

$$N_\Lambda(d, m) = \#\{x \in \Lambda_d \ : \ \|x\|_1 = \sum_{i=1}^{d} |x_i| = m\}.$$

Usefulness of $N_\Lambda(d, m)$ becomes now clear: given an available finite bit rate per dimension R, the maximum number of lattice points to be encoded is $T = 2^{dR}$ $(R = \frac{1}{d}\log_2 T)$, which is equivalent to finding the maximum l_1 norm L such that

$$\sum_{m=0}^{L} N_\Lambda(d, m) \leq T.$$

In 1997 Conway and Sloane [2], extending work of O'Keeffe and others, propose explicit formulae for the coordination sequences of some of the root lattices.

In 1998 we have proposed combinatorial expressions for computing the contour points for some of the most used lattices [9].

In this paper the equivalency of coordination sequences and contour points for the $\mathbb{Z}^d$ and A_d lattices is shown by giving combinatorial proofs. Moreover, a combinatorial explanation of the coordination sequences formulae is now provided.

2 Integer Cubic Lattice

The cubic lattice can be defined as follows

$$\mathbb{Z}^d = \{x = (x_1, x_2, ..., x_d) \in \mathbb{Z}^d\}.$$

Proposition 1 From [2],

$$S_{\mathbb{Z}}(m) = \sum_{i=0}^{d} \binom{d}{i} \binom{d-i+m-1}{d-1}.$$

Proposition 2 From [9],

$$N_{\mathbb{Z}}(d,m) = \sum_{i=1}^{\min\{d,m\}} 2^i \binom{d}{i} \binom{m-1}{i-1}.$$

Lemma 3 Since the height of any point $x \in \mathbb{Z}^d$ is $ht(x) = \sum_{i=1}^{d} |x_i|$, then

$$S_{\mathbb{Z}}(m) = N_{\mathbb{Z}}(d,m).$$

Proof: We shall use here the 'Snake Oil' method [11]. The ordinary power series generating function (opsgf) of the identity left hand side is first computed:

$$
\begin{aligned}
F_1(x) &= \sum_m x^m \sum_{i=0}^{d} \binom{d}{i} \binom{d-i+m-1}{d-1} \\
&= \sum_i \binom{d}{i} \sum_m \binom{d-i+m-1}{d-1} x^m \\
&= \sum_i \binom{d}{i} x^{-(d-i-1)} \\
&\qquad \cdot \sum_m \binom{d-i+m-1}{d-1} x^{m+(d-i-1)} \\
&= \sum_i \binom{d}{i} x^{-(d-i-1)} \frac{x^{d-1}}{(1-x)^d} \\
&= \frac{1}{(1-x)^d} \sum_i \binom{d}{i} x^{-(d-i-1)+(d-1)} \\
&= \frac{1}{(1-x)^d} \sum_i \binom{d}{i} x^i \\
&= \frac{1}{(1-x)^d} (1+x)^d = \left(\frac{1+x}{1-x}\right)^d .
\end{aligned}
$$

For the identity right hand side, proceeding as before:

$$
\begin{aligned}
F_2(x) &= \sum_m x^m \sum_{i=1}^{\min\{d,m\}} 2^i \binom{d}{i} \binom{m-1}{i-1} \\
&= \sum_i 2^i \binom{d}{i} \sum_m \binom{m-1}{i-1} x^m \\
&= \sum_i 2^i \binom{d}{i} x \sum_m \binom{m-1}{i-1} x^{m-1} \\
&= \sum_i 2^i \binom{d}{i} x \frac{x^{i-1}}{(1-x)^i} \\
&= \sum_i 2^i \binom{d}{i} \frac{x^i}{(1-x)^i} \\
&= \sum_i \binom{d}{i} \left(\frac{2x}{1-x}\right)^i \\
&= \left(1 + \frac{2x}{1-x}\right)^d = \left(\frac{1+x}{1-x}\right)^d .
\end{aligned}
$$

Since the two generating functions are the same, the expressions must be equal. ∎

Example 4 For $\mathbb{Z}^3$, $S_{\mathbb{Z}}(2) = N_{\mathbb{Z}}(3,2) = 18$. The points are

$$\begin{array}{lll} (0,0,2),(0,0,-2) & (0,1,1),(0,-1,-1) & (0,1,-1),(0,-1,1) \\ (0,2,0),(0,-2,0) & (1,0,1),(-1,0,-1) & (1,0,-1),(-1,0,1) \\ (2,0,0),(-2,0,0) & (1,1,0),(-1,-1,0) & (1,-1,0),(-1,1,0). \end{array}$$

For the sake of clarity, the first values are tabulated:

$\mathbb{Z}^d$	0	1	2	3	4	5
1	1	2	2	2	2	2
2	1	4	8	12	16	20
3	1	6	18	38	66	102
4	1	8	32	88	192	360
5	1	10	50	170	450	1002
8	1	16	128	688	2816	9424
10	1	20	200	1340	6800	28004
16	1	32	512	5472	44032	285088
24	1	48	1152	18448	221952	2141808
32	1	64	2048	43712	700416	8991552

The combinatorial interpretation of $N_{\mathbb{Z}}(d,m)$ was given in [9]; now we provide it for $S_{\mathbb{Z}}(m)$.

When the sign of each particular coordinate in a point is taken into account, $N_{\mathbb{Z}}(d,m)$ can also be expressed as

$$\begin{aligned} N'_{\mathbb{Z}}(d,m) = & \binom{m-1}{d-1} + \binom{d+m-1}{d-1} \\ & + \sum_{p=1}^{d-1} \binom{d}{p} \sum_{k=1}^{m} \binom{k-1}{p-1} \binom{(d-p)+(m-k)-1}{(d-p)-1}, \end{aligned}$$

where the first binomial coefficient stands for those points in which all coordinates are strictly positive; the second binomial coefficient stands for those points with no strictly positive coordinate; and the last term stands for those points with both positive coordinates (p) and negative or zero coordinates.

Now $N'_{\mathbb{Z}}(d,m) =$

$$=\binom{m-1}{d-1}+\binom{d+m-1}{d-1}+\sum_{p=1}^{d-1}\binom{d}{p}\sum_{k=1}^{m}\binom{k-1}{p-1}\binom{(d-p)+(m-k)-1}{(d-p)-1}$$

$$=\binom{m-1}{d-1}+\binom{d+m-1}{d-1}+\sum_{p=1}^{d-1}\binom{d}{p}\sum_{j=0}^{m-1}\binom{j}{p-1}\binom{(d-p)+(m-(j+1))-1}{(d-p)-1}$$

$$=\binom{m-1}{d-1}+\binom{d+m-1}{d-1}+\sum_{p=1}^{d-1}\binom{d}{p}\sum_{j=0}^{m-1}\binom{j}{p-1}\binom{d-p+m-j-2}{d-p-1}$$

$$=\binom{m-1}{d-1}+\binom{d+m-1}{d-1}+\sum_{p=1}^{d-1}\binom{d}{p}\binom{(d-p+m-2)+(0)+1}{(d-p-1)+(p-1)+1} \text{ by [6, 5.26]}$$

$$=\binom{m-1}{d-1}+\binom{d+m-1}{d-1}+\sum_{p=1}^{d-1}\binom{d}{p}\binom{d-p+m-1}{d-1}$$

$$=\binom{m-1}{d-1}+\sum_{p=0}^{d-1}\binom{d}{p}\binom{d-p+m-1}{d-1}$$

$$=\sum_{p=0}^{d}\binom{d}{p}\binom{d-p+m-1}{d-1}$$

$= S(m)$.

3 A_d Lattice

The A_d lattice can be defined as follows

$$A_d = \{x = (x_0, x_1, x_2, ..., x_d) \in \mathbb{Z}^{d+1} \ : \ (\textstyle\sum_{i=0}^{d} x_i) = 0\} \quad (d \geq 2).$$

Proposition 5 From [2],

$$S_A(m) = \sum_{i=0}^{d}\binom{d}{i}^2\binom{d-i+m-1}{d-1}.$$

Proposition 6 From [9],

$$N_A(d,m) = \sum_{i=1}^{d}\binom{d+1}{i}\binom{\frac{m}{2}-1}{i-1}\binom{d+1-i+\frac{m}{2}-1}{\frac{m}{2}}.$$

Lemma 7 Since the height of any point $x \in A_d$ is $ht(x) = \frac{1}{2}\sum_{i=1}^{d} |x_i|$, then

$$S_A(m) = N_A(d, 2m).$$

Proof: As in the previous proof, one would like to use the 'Snake Oil' method, but while the ordinary power series generating function (opsgf) of the identity left hand side can be computed:

$$\begin{aligned}
F_3(x) &= \sum_m x^m \sum_{i=0}^{d} \binom{d}{i}^2 \binom{d-i+m-1}{d-1} \\
&= \sum_i \binom{d}{i}^2 \sum_m \binom{d-i+m-1}{d-1} x^m \\
&= \sum_i \binom{d}{i}^2 x^{-(d-i-1)} \\
&\qquad \cdot \sum_m \binom{d-i+m-1}{d-1} x^{m+(d-i-1)} \\
&= \sum_i \binom{d}{i}^2 x^{-(d-i-1)} \frac{x^{d-1}}{(1-x)^d} \\
&= \sum_i \binom{d}{i}^2 \frac{x^i}{(1-x)^d} \\
&= \frac{1}{(1-x)^d} \sum_i \binom{d}{i}^2 x^i;
\end{aligned}$$

the ordinary power series generating function of the identity right hand side can not, since no matter which variable one takes, d or m, they always appear in two binomial coefficients and the method does not work.

We shall then use the 'WZ' method [7].

The identity left hand side is first studied: let

$$H_1(m, i) = \binom{d}{i}^2 \binom{d-i+m-1}{d-1},$$

then $H_1(m, i)$ satisfies the recurrence

$$\begin{aligned}
m^2 H_1(m, i) + (-2m^2 - d^2 - 4m - d - 2) H_1(m+1, i) + (m+2)^2 H_1(m+2, i) \\
= G_1(m, i+1) - G_1(m, i), \qquad (1)
\end{aligned}$$

where

$$G_1(m,i) = \frac{(-d+i-m)i^2(d-1)}{(i-1-m)(i-2-m)} H_1(m,i).$$

The recurrence (1) is now summed over all integers i,

$$\sum_{i=-\infty}^{\infty} \left[m^2 H_1(m,i) + (-2m^2 - d^2 - 4m - d - 2)H_1(m+1,i) + (m+2)^2 H_1(m+2,i)\right] = \sum_{i=-\infty}^{\infty} \left[G_1(m,i+1) - G_1(m,i)\right], \tag{2}$$

which, in its turn, and letting

$$h_1(m) = \sum_{i=-\infty}^{\infty} \binom{d}{i}^2 \binom{d-i+m-1}{d-1} \qquad (= S_A(m)),$$

satisfies the recurrence

$$m^2 h_1(m) + (-2m^2 - d^2 - 4m - d - 2)h_1(m+1) + (m+2)^2 h_1(m+2) = 0, \tag{3}$$

since the right side of (2) telescopes to 0.

One proceeds as before for the identity right hand side of Lemma 7; let

$$H_2(m,i) = \binom{d+1}{i}\binom{m-1}{i-1}\binom{d+1-i+m-1}{m},$$

then $H_2(m,i)$ satisfies the recurrence

$$m^2 H_2(m,i) + (-2m^2 - d^2 - 4m - d - 2)H_2(m+1,i) + (m+2)^2 H_2(m+2,i) = G_2(m,i+1) - G_2(m,i), \tag{4}$$

in which the left side is the same recurrence as in (1) above, but on the right side $G_2(m,i)$ is now

$$G_2(m,i) = \frac{m(-d+i-m-1)i(i-1)d}{(i-1-m)(i-2-m)(m+1)}.$$

Again, the recurrence (4) is summed over all integers i,

$$\sum_{i=-\infty}^{\infty} \left[m^2 H_2(m,i) + (-2m^2 - d^2 - 4m - d - 2)H_2(m+1,i) + (m+2)^2 H_2(m+2,i)\right] = \sum_{i=-\infty}^{\infty} \left[G_2(m,i+1) - G_2(m,i)\right], \tag{5}$$

which, in its turn, and letting

$$h_2(m) = \sum_{i=-\infty}^{\infty} \binom{d+1}{i} \binom{m-1}{i-1} \binom{d+1-i+m-1}{m} \qquad (= N_A(d, 2m)),$$

satisfies the recurrence

$$m^2 h_2(m) + (-2m^2 - d^2 - 4m - d - 2) h_2(m+1) + (m+2)^2 h_2(m+2) = 0, \tag{6}$$

since the right side of (5) telescopes to 0.

And, since the two sums in question, $h_1(m)$ and $h_2(m)$, satisfy exactly the same recurrence of the second order, together with the same two starting values, they must be identical for all values of m. ∎

Example 8 For A_2, $S_A(3) = N_A(2, 6) = 18$. The points are

$(3,-3,0),(-3,3,0)$	$(3,-1,-2),(-3,1,2)$	$(-1,3,-2),(1,-3,2)$
$(3,0,-3),(-3,0,3)$	$(3,-2,-1),(-3,2,1)$	$(-2,3,-1),(2,-3,1)$
$(0,3,-3),(0,-3,3)$	$(-1,-2,3),(1,2,-3)$	$(-2,-1,3),(2,1,-3)$.

For the sake of clarity, the first values of $N_A(d, 2m)$ are tabulated:

A_{d+1}	0	2	4	6	8	10
1	1	0	0	0	0	0
2	1	2	2	2	2	2
3	1	6	12	18	24	30
4	1	12	42	92	162	252
5	1	20	110	340	780	1500
8	1	56	812	5768	26474	91112
10	1	90	2070	22530	151560	731502
16	1	240	14520	400080	6447660	70006512
24	1	552	76452	4756952	169447302	3956576472
32	1	992	246512	27390112	1728939192	70807488864

All entries corresponding to a column for an odd norm are 0.

Combinatorial interpretation of these results is best captured by an alternative expression for $N_A(d, 2m)$:

$$N'_A(d, 2m) = \sum_{c=2}^{d+1} \binom{d+1}{c} \sum_{p=1}^{c-1} \binom{c}{p} \binom{m-1}{p-1} \binom{m-1}{(c-p)-1},$$

where c counts how many coordinates different from 0 there are (at least 2, one positive and one negative, and at most, all of them); the first binomial coefficient gives the ways to place them; p counts how many strictly positive coordinates there are (at least 1, at most $c-1$); the second binomial coefficient gives the ways to place these; and the third and fourth binomial coefficients respectively count the different combinations of strictly positive and negative coordinates that add up to half the norm.

Now $N'_A(d, 2m) =$

$$
\begin{aligned}
&= \sum_{c=2}^{d+1} \binom{d+1}{c} \sum_{p=1}^{c-1} \binom{c}{p} \binom{m-1}{p-1} \binom{m-1}{c-p-1} \\
&= \sum_{c} \binom{d+1}{c} \sum_{p} \binom{c}{p} \binom{m-1}{p-1} \binom{m-1}{c-p-1} \\
&= \sum_{p} \binom{m-1}{p-1} \sum_{c} \binom{d+1}{c} \binom{c}{p} \binom{m-1}{c-p-1} \\
&= \sum_{p} \binom{m-1}{p-1} \sum_{c} \binom{d+1}{p} \binom{d+1-p}{c-p} \binom{m-1}{c-p-1} \quad \text{by [8, (iv)]} \\
&= \sum_{p} \binom{m-1}{p-1} \binom{d+1}{p} \sum_{c} \binom{d+1-p}{c-p} \binom{m-1}{c-p-1} \\
&= \sum_{p} \binom{m-1}{p-1} \binom{d+1}{p} \binom{(d+1-p)+(m-1)}{(d+1-p)-(0)+(-1)} \quad \text{by [6, 5.23]} \\
&= \sum_{p} \binom{m-1}{p-1} \binom{d+1}{p} \binom{d-p+m}{d-p} \\
&= \sum_{p} \binom{d+1}{p} \binom{m-1}{p-1} \binom{d-p+m}{m} \\
&= N_A(d, 2m).
\end{aligned}
$$

4 Concluding remarks

No closed form has yet been found for any of the given combinatorial expression. To find them or to prove that they do not exist remains an open question.

Extension of this work to lattices A_d^*, D_d, D_d^*, D_d^+ (including Gosset lattice $E_8 = D_8^+$) is also being investigated and will be reported somewhere else [10].

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