INTERSECTIONS OF SCHUBERT CELLS AND GROUPS GENERATED BY SYMPLECTIC TRANSVECTIONS

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ABSTRACT. We prove that the number of connected components in the intersection of two open Schubert cells in relative position w in the space of complete real n-dimensional flags equals $3 \cdot 2^{n-1}$ for a generic permutation $w \in S_n$. Our construction combines the machinery of pseudo-line arrangements with the theory of groups generated by symplectic transvections.

The point of departure for this talk is the following result obtained in [SSV2, SSV3]. Let N_n^0 denote the semi-algebraic set of all unipotent upper-triangular $n \times n$ matrices x with real entries such that for every $k = 1, \dots, n-1$, the minor of x with rows $1, \dots, k$ and columns $n-k+1, \dots, n$ is non-zero. Then the number $\#_n$ of connected components of N_n^0 is as follows: $\#_2 = 2$, $\#_3 = 6$, $\#_4 = 20$, $\#_5 = 52$, and $\#_n = 3 \cdot 2^{n-1}$ for $n \geqslant 6$.

Let Fl_n be the space of complete flags in $\mathbb{R}^n$. Given any $f \in Fl_n$, the Schubert cell decomposition assigns to each $w \in S_n$ a Schubert cell C_w^f whose dimension equals the number of inversions in w. For any $g \in \mathrm{Fl}_n$ we put $C_{w_1,w_2}^{f,g} = C_{w_1}^f \cap C_{w_2}^g$. Recall that $C_{w_1,w_2}^{f,g}$ depends only on the relative position of f and g, that is, C_{w_1,w_2}^{f,g_1} and C_{w_1,w_2}^{f,g_2} are isomorphic, provided g_1, g_2 belong to the same C_w^f. For this reason, we write C_{w_1,w_2}^w instead of $C_{w_1,w_2}^{f,g}$, $g \in C_w^f$. The geometry and combinatorics of pairwise intersections C_{w_1,w_2}^w for arbitrary w_1, w_2, w was studied in [SSV1].

Let $C^w = C_{w_0,w_0}^w$, where $w_0 = n\, n-1 \dots 1$ is the longest permutation in S_n. Intersections C^w appeared in the literature in various contexts, and were studied (in various degrees of generality) in [BFZ,BZ,R1,R2]. It is easy to

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see that $N_n^0 = C^{w_0}$, so the result cited above gives the number of connected components in C^{w_0}. The main result of the present talk is the following generalization of the above result.

Theorem 1. *For a generic $w \in S_n$ the number of connected components in C^w equals $3 \cdot 2^{n-1}$.*

Our basic construction relies on the machinery of pseudoline arrangements associated with reduced expressions in the symmetric group developed in [BFZ]. Let $w \in S_n$ be an arbitrary permutation, and let $\omega = s_{i_1}s_{i_2}\dots s_{i_m}$ be an arbitrary reduced expression for w. We denote by $\sigma = \sigma^\omega$ the sequence $1\,2\dots n-1\,i_1\dots i_m$, and by σ_i the ith element of σ. A pair (σ_i, σ_j), $i < j$, is called a chamber (of level k) if $\sigma_i = \sigma_j = k$ and $\sigma_p \neq k$ for $i < p < j$. A chamber (σ_i, σ_j) is called bounded if $i > n-1$ and unbounded otherwise.

We assign to σ^ω an undirected graph G^ω in the following way. The vertices of G^ω are all the chambers of σ^ω; we denote the vertex set of G^ω by Γ^ω. Let $\gamma_1 = (\sigma_{i_1}, \sigma_{j_1})$ be a chamber of level k_1, $\gamma_2 = (\sigma_{i_2}, \sigma_{j_2})$ be a chamber of level k_2; assume without loss of generality that $i_1 < i_2$. The vertices γ_1 and γ_2 are joined by an edge in G^ω if and only if one of the following two conditions holds:

(i) $k_1 = k_2$ and $\sigma_i \neq k_1$ for $j_1 < i < i_2$;

(ii) $|k_1 - k_2| = 1$ and $i_2 < j_1 < j_2$.

Denote by $V^\omega = \mathbb{F}_2^{\Gamma^\omega}$ the vector space over $\mathbb{F}_2$ with the basis e_γ, $\gamma \in \Gamma^\omega$. The graph G^ω induces an alternating bilinear form Ω^ω on V^ω, namely,

$$\Omega^\omega = \sum_{(\alpha,\beta)\in G^\omega} e_\alpha^* \wedge e_\beta^*.$$

For any $\gamma \in \Gamma^\omega$ we define a linear transformation $\tau_\gamma \colon V^\omega \to V^\omega$ by the following rule:

$$\tau_\gamma v = v - \Omega^\omega(v, e_\gamma)e_\gamma \qquad \text{for any } v \in V^\omega;$$

τ_γ is called the symplectic transvection at γ with respect to Ω^ω. Let $\Gamma_0^\omega \subset \Gamma^\omega$ be the set of bounded chambers of σ^ω. We denote by $\mathfrak{G}^\omega$ the group of linear transformations $V^\omega \to V^\omega$ generated by $\{\tau_\gamma, \gamma \in \Gamma_0^\omega\}$.

The following result is a generalization of the main theorem of [SSV2].

Theorem 2. *For any $w \in S_n$ and any reduced expression ω for w, the number of connected components of C^w equals the number of orbits of $\mathfrak{G}^\omega$ in V^ω.*

Our construction of $\mathfrak{G}^\omega$ is a special case of a more general construction, which uses an arbitrary finite set Γ, an arbitrary undirected graph G without loops and multiple edges on the vertex set Γ, and an arbitrary subset $\Gamma_0 \subseteq \Gamma$. The orbits of $\mathfrak{G}$ in $V = \mathbb{F}_2^\Gamma$ in case $\Gamma_0 = \Gamma$ were studied in [Ja] under certain assumptions of general position; the number of orbits in this case is equal to 3. Here we prove that under similar assumptions the number of $\mathfrak{G}$-orbits in V equals $3 \cdot 2^r$, where $r = |\Gamma| - |\Gamma_0|$.

Let us define the linear operator $L\colon V \to V^*$ by $\Omega(x, y) = (x, Ly)$, where $(\cdot, \cdot)$ is the standard pairing between V and V^*; evidently, $L^* = L$. Let $V_0 = \mathbb{F}_2^{\Gamma_0}$ be the space spanned by e_γ, $\gamma \in \Gamma_0$; by ι we denote the injection $V_0 \hookrightarrow V$, and by ι^* its dual $V^* \to V_0^*$. Combining L with ι and ι^* we get the maps $\Lambda = L \circ \iota\colon V_0 \to V^*$ and $L_0 = \iota^* \circ \Lambda\colon V_0 \to V_0^*$; evidently, $L_0^* = i^* \circ L^* \circ i = i^* \circ L \circ i = L_0$.

The following observation turns out to be crucial for our construction. Let Ω_0 denote the restriction of the form Ω to V_0.

Theorem 3. *Let Γ, Γ_0 and G satisfy condition*

$$(* +) \qquad \Lambda^*\colon V \to V_0^* \text{ is a surjection,}$$

then the $\mathfrak{G}$-action on V_0 by symplectic transvections with respect to Ω_0 is dual to the $\mathfrak{G}$-action on V_0^* induced by Λ^*.

We say that a subset $\Delta \subseteq \Gamma_0$ is a transversal if $|\Gamma_0| - |\Delta| = \dim \ker L_0$ and the restriction of Ω to $\mathbb{F}_2^{\Delta}$ is nondegenerate. A transversal is called nonspecial if the subgraph of G induced by Δ is connected and contains six vertices that span a subgraph isomorphic to the Dynkin diagram E_6.

Theorem 4. *Let Γ, Γ_0 and G satisfy $(*)$ and let there exist two nonspecial transversals $\Delta_1, \Delta_2 \subseteq \Gamma_0$ such that*

$$(**) \qquad \Delta_1 \cup \Delta_2 = \Gamma_0,$$

$$(***) \qquad (\gamma_1, \gamma_2) \in G \text{ for some pair } \gamma_1, \gamma_2 \in \Delta_1 \cap \Delta_2.$$

Then the number of orbits of $\mathfrak{G}$ on V equals $3 \cdot 2^r$, where $r = |\Gamma| - |\Gamma_0|$.

To apply Theorem 4 we have to verify its assumptions for the triple $(\Gamma^\omega, \Gamma_0^\omega, G^\omega)$. First of all, we have the following proposition.

Proposition 5. *For any $w \in S_n$ and any reduced expression ω for w, the linear operator $(\Lambda^\omega)^*$ satisfies $(*)$.*

Let ω be a reduced expression for some $w \in S_n$. If the subgraph G_0^ω of G^ω induced by Γ_0^ω is not connected, then w can be decomposed into the product of two permutations on the index sets $[1, \dots, i]$ and $[i, \dots, n]$ with $1 < i < n$; moreover, $G_0^{\omega'}$ remains not connected for any other reduced expression ω' of w. Therefore, the fraction of permutations for which G_0^ω is connected tends to 1 with $n \to \infty$, and hence this property is generic. On the other hand, the existence of a reduced expression ω such that σ^ω contains three consequent indices $i-1$, i, $i+1$ at least k times is also generic for any fixed k.

Proposition 6. *Let $w \in S_n$ possess a reduced expression ω such that G_0^ω is connected and σ^ω contains each of the indices $i-1$, i, $i+1$ at least 5 times for some i, $1 < i < n-2$. Then there exist two nonspecial transversals $\Delta_1, \Delta_2 \subseteq \Gamma_0^\omega$ satisfying $(**)$ and $(***)$.*

To get Theorem 1 from Theorem 4 it remains to observe that $r^\omega = |\Gamma^\omega| - |\Gamma_0^\omega|$ is just the number of unbounded chambers, $r^\omega = n - 1$.

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