COLOR-TO-SPIN RIBBON SCHENSTED ALGORITHMS
EXTENDED ABSTRACT

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ABSTRACT. A new Schensted bijection is given from colored permutations to pairs of ribbon tableaux, such that twice the total color of the colored permutation, is equal to the sum of the spins of the pair of tableaux.

1. INTRODUCTION

In [9] Stanton and White defined a k-rim hook Schensted, that is, a bijection between k-colored permutations and pairs of k-rim hook tableaux of the same shape. This bijection is transported to a k-fold product of ordinary Schensteds by Littlewood's bijection, which sends a partition with empty k-core, to the k-tuples of partitions given by its k-quotient. A formulation of this bijection using chains of partitions was given by Fomin and Stanton [3].

The goal of this paper is to define a different bijection from colored permutations to pairs of standard rim hook tableaux of the same shape, such that twice the total color of a colored permutation, equals the sum of the spins of the corresponding pair of rim hook tableaux (Theorem 4.2). This new bijection has the involution property, that is, taking a suitable inverse of the colored permutation has the effect of exchanging the two tableaux (Theorem 6.2). This color-to-spin ribbon Schensted is defined using chains of partitions, in the spirit of [3].

In the domino case ($k = 2$) there is yet another Schensted due to Barbasch and Vogan [1] and Garfinkle [4] that is different from the domino special cases of both of the above bijections. Garfinkle's recursive definition for the algorithm has been translated into the language of chains of partitions by van Leeuwen [10], who extended the algorithm to handle the case of nonempty 2-core. It is common knowledge that the algorithm of [1] which involves interleaved row and column insertions and jeu de taquin, and the domino insertion algorithm of [4], coincide; this is proven elegantly in [8] using Haiman's mixed and left-right insertion [5]. Surprisingly, it appears to be a new observation, that this domino Schensted also preserves spin [8]; this fact leads to a new formula for the q-analogue $c^{\lambda}_{\mu,\nu}(q)$ of the Littlewood-Richardson coefficients defined by Carré and Leclerc [2], who had already given a combinatorial description of the q-LR coefficients using Yamanouchi domino tableaux with spin. In [8] the new rule for the q-LR coefficients is used to derive an explicit formula in the case that μ and ν are rectangles; currently few other explicit formulas exist for these polynomials.

By counting k-ribbon tableaux (a "column-strict" version of k-rim hook tableaux) by weight and spin, Lascoux, Leclerc, and Thibon defined a q-analogue $c^{\lambda}_{\mu^1,\dots,\mu^k}(q)$ of the multiplicity of the Schur function s_λ in a product $s_{\mu^1}s_{\mu^2}\dots s_{\mu^k}$ of k Schur

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functions [6]. It has been shown by Leclerc and Thibon that these q-analogues are certain parabolic Kazhdan-Lusztig polynomials of affine type A [7].

All of the above rim hook Schensteds reduce to the ordinary Schensted bijection when $k = 1$.

Our insertion is defined here for the "standard" or k-rim hook case, but it can be "ribbonized", that is, it has a "column-strict" extension, which is not obtainable by an obvious standardization. Moreover it can be extended to work in the presence of a k-core, but for simplicity it is assumed here that this core is empty.

The authors hope that this color-to-spin ribbon Schensted will prove to be a useful tool in studying the q-LR coefficients.

2. Preliminaries

Here the English (matrix-style) indexing is used for partitions and tableaux. Let k be a fixed positive integer. A k-colored (partial) permutation is a biword whose bottom word uses letters in the alphabet $\{1, 2, \ldots, n\}$ with no repeats, and whose top word has letters in the alphabet $\{0, 1, 2, \ldots, k-1\}$. The color (resp. value) of a biletter is its upper (resp. lower) letter. The total color of a colored permutation is the sum of the colors of its biletters. A rim hook is a connected skew shape that contains at most one cell in each (northwest-to-southeast) diagonal. A k-rim hook is one that has exactly k cells. The head of a k-rim hook is the cell that lies on the northeastmost diagonal. From now on, colored permutation means k-colored permutation and rim hook means k-rim hook. The spin of a rim hook H is defined to be one less than the number of rows in which H contains a cell. A rim hook H is said to be μ-addable and λ-removable if there are partitions $\mu \subset \lambda$ such that $H = \lambda/\mu$. Let H_1 and H_2 be rim hooks. Say that H_1 is northeast of H_2 if the head of H_1 lies on a diagonal that is northeast of that of H_2. A k-rim hook tableau of shape λ/μ is a chain of partitions $\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda$ such that each skew shape $\lambda^{(i)}/\lambda^{(i-1)}$ is either a k-rim hook or is empty. A rim hook tableau can be depicted by placing the number i in the subshape $\lambda^{(i)}/\lambda^{(i-1)}$ if it is nonempty; in this case the tableau is said to contain i. The spin of a rim hook tableau is the sum of the spins of its rim hooks.

3. Rim hook bumping rules

Here the bumping rules for rim hooks are given. The following two properties of the rim hook lattice are interesting in their own right, and are fundamental to the definition of the color-to-spin ribbon Schensted.

Lemma 3.1. *Let μ be a partition and c a color. Then there is a μ-addable rim hook of spin c.*

Define firsthook(μ, c) to be the northeastmost μ-addable rim hook of spin c, which exists by Lemma 3.1.

Lemma 3.2. *Let μ be a shape and H a μ-addable rim hook. Then there is a $(\mu \cup H)$-addable rim hook that is strictly southwest of H and has the same spin.*

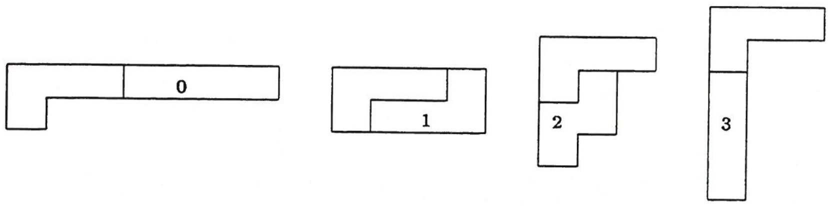
Define nexthook(μ, H) to be the northeastmost rim hook satisfying the conclusion of Lemma 3.2.

We now recall the operation of bumpout defined in [9]. Let H_1 and H_2 be k-rim hooks. Define the set of cells

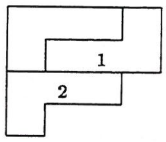
$$\text{bumpout}(H_1, H_2) = (H_2 \backslash H_1) \cup \{(i+1, j+1) \mid (i,j) \in H_1 \cap H_2\}.$$

If μ is a partition and H_1 and H_2 distinct and μ-addable, then it is not hard to verify that bumpout(H_1, H_2) is a $(\mu \cup H_1)$-addable k-rim hook.

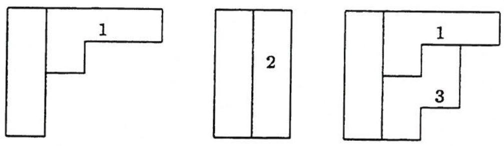
Example 3.3. Let $k = 4$ and $\mu = (3,1)$. For $c \in \{0,1,2,3\}$ the μ-addable hooks firsthook(μ, c) are given below.



With $\mu = (3,1)$ and H_1 indicated by the letter 1, the hook $H = \text{nexthook}(\mu, H_1)$ (indicated by the letter 2) is given below.



Finally, let $\mu = (1,1,1,1)$ and H_i μ-addable rim hooks indicated by the letter i for $1 \leq i \leq 2$. Then the hook $H = \text{bumpout}(H_1, H_2)$ is indicated by the letter 3 below.



This given, let us define an operation whose input is a partition μ and two (possibly empty) μ-addable rim hooks H_1 and H_2, and whose output is a (possibly empty) $(\mu \cup H_1)$-addable k-rim hook H that has the same spin as H_2. In analogy with Schensted's row insertion algorithm, one should imagine that the shape μ contains the rim hooks in some rim hook tableau whose values are smaller than that of H_1, and the rim hook H_1 is bumping the rim hook H_2 to a new position H. Define

$$H = \begin{cases} H_2 & \text{if } H_1 \cap H_2 = \emptyset \\ \text{bumpout}(H_1, H_2) & \text{if } H_1 \cap H_2 \neq \emptyset \text{ and } H_1 \neq H_2 \\ \text{nexthook}(\mu, H_1) & \text{if } H_1 = H_2. \end{cases} \tag{3.1}$$

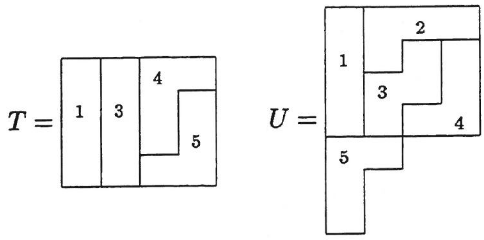
4. Inductive definition of insertion

Let T be a standard k-rim hook tableau of shape λ that does not contain i. The insertion $T \xleftarrow{cs} \binom{c}{i}$ into T of the rim hook of value i and color c, is defined as follows.

Suppose first that i is greater than any value in T. Then $T \xleftarrow{cs} \binom{c}{i}$ is obtained by adjoining to T, the rim hook firsthook(λ, c) of value i. This contains the base case $T = \emptyset$.

Otherwise assume i is less than the largest value n in T. Let T^- be obtained from T by removing the the rim hook H_2 containing the value n. Suppose μ is the shape of T^-. By induction on the size of T, the insertion $T^- \xleftarrow{cs} \binom{c}{i}$ has been defined; suppose it has shape ν, which by induction is such that ν/μ is a k-rim hook H_1. Then define $T \xleftarrow{cs} \binom{c}{i}$ by adjoining to $T^- \xleftarrow{cs} \binom{c}{i}$, the rim hook H of value n defined by (3.1) in terms of μ, H_1, and H_2.

Example 4.1. The insertion of $\binom{1}{2}$ into the rim hook tableau T results in the rim hook tableau U given below.



One first removes all hooks from T except those that are smaller than the value 2 being inserted. Hook 2 is adjoined at its initial position outside hook 1. Next hook 3 is adjoined; since its original position is now partially overlapped by smaller hooks, its new position is computed by a "bumpout". The hooks 4 and 5 move to their new positions under the "nexthook" case.

Consider a k-colored permutation

$$\binom{\underline{c}}{\underline{i}} = \begin{pmatrix} c_1 & c_2 & \cdots & c_L \\ i_1 & i_2 & \cdots & i_L \end{pmatrix}$$

such that the i_j are distinct elements in $\{1, 2, \ldots, n\}$ and $c_j \in \{0, 1, \ldots, k-1\}$. Let P_0 be the empty tableau and P_k the rim hook tableau given by $P_k = P_{k-1} \xleftarrow{cs} \binom{c_k}{i_k}$ for $1 \le k \le L$. Let $P = P_L$. Then we write

$$P = (\cdots((\emptyset \xleftarrow{cs} \binom{c_1}{i_1}) \xleftarrow{cs} \binom{c_2}{i_2}) \cdots \xleftarrow{cs} \binom{c_L}{i_L}).$$

Define the standard k-rim hook tableau Q by the sequence of partitions given by the shapes of the tableaux P_k; Q contains each of the values 1 through L exactly once.

Theorem 4.2. *The map $\binom{\underline{c}}{\underline{i}} \mapsto (P, Q)$ defines a bijection from the set of k-colored permutations of length L, to pairs of standard k-rim hook tableaux of the same shape, such that:*

(a) *The value m appears in i if and only if it appears in P (that is, the map is content-preserving).*
(b) *Q contains each of the values 1 through L exactly once.*
(c) *Twice the total color of the colored permutation $\binom{\underline{c}}{\underline{i}}$, is equal to the sum of the spins of the tableaux P and Q.*

5. Computation by array of partitions

Now it is shown how to compute the tableau pair (P, Q) using an $(n+1) \times (n+1)$-matrix of partitions whose rows and columns are indexed by the sets $\{0, 1, \ldots, n\}$ using only local rules.

Fill the zero-th row and column with empty partitions. Consider the computation of the partition ρ in the k-th row and ℓ-th column of the matrix. Assume that the partitions μ, λ, ν that lie one position to the northwest, north, and west of ρ have already been calculated.

Suppose first that $\ell = i_k$. In this case it follows by induction that $\mu = \lambda = \nu$. Define ρ by the shape given by adjoining to ν, the rim hook firsthook(ν, c_k).

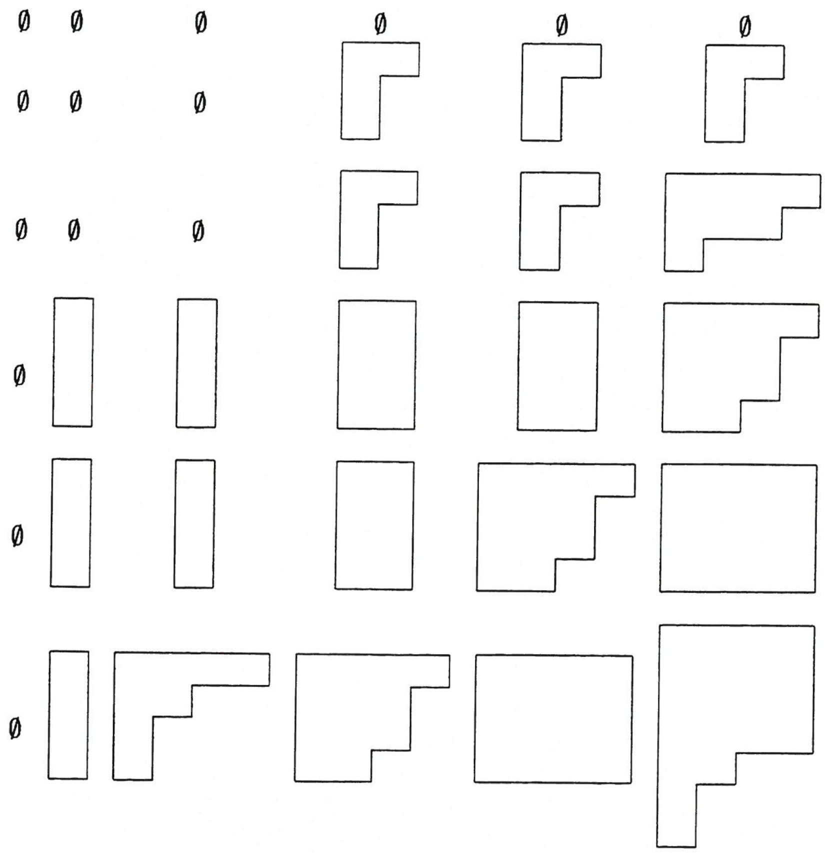
Otherwise suppose $\ell \neq i_k$. Let $H_1 = \nu/\mu$ and $H_2 = \lambda/\mu$. By induction each of H_1 and H_2 is either empty or a μ-addable k-rim hook. Then define ρ to be the shape given by adjoining to ν the hook H defined by the triple (μ, H_1, H_2) in (3.1). Then the rim hook tableau P is given by the chain of partitions in the n-th row of the matrix and Q by the n-th column.

The (k, ℓ)-th entry of the matrix is given by the shape of the tableau obtained by inserting the subword of the first k biletters in $\binom{\underline{c}}{\underline{i}}$ whose values are at most ℓ.

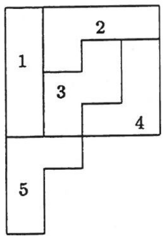
Example 5.1. Consider the colored permutation

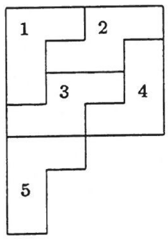
$$\begin{pmatrix} 2 & 1 & 3 & 2 & 1 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}.$$

Observe that its total color is $2+1+3+2+1=9$. Its matrix of partitions is given below.



Therefore the P and Q tableaux are given by

$P =$ 

$Q =$ 

Now the spins of P and Q are 10 and 8 respectively, so that $2 \cdot 9 = 10 + 8$ as stated in Theorem 4.2 (c).

6. Involution property

Define an indexed k-colored partial permutation to be a triword

$$\begin{pmatrix} \underline{c} \\ \underline{p} \\ \underline{i} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_L \\ p_1 & p_2 & \dots & p_L \\ i_1 & i_2 & \dots & i_L \end{pmatrix}$$

such that $\binom{\underline{c}}{\underline{i}}$ is a k-colored permutation and $1 \leq p_1 < p_2 < \cdots < p_L \leq n$. One can define a corresponding pair (P, Q) where P is as before but Q is the standard k-rim hook tableau that contains the letters p_1 through p_L exactly once instead of the letters 1 through L.

Define the inverse of an indexed colored partial permutation to be that given by viewing each of the columns as inseparable units, sorting the collection of columns in increasing order according to the i's and then exchanging the second and third rows.

Example 6.1. An indexing of the colored permutation of the running example and its inverse are given below.

$$\begin{pmatrix} 2 & 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

Theorem 6.2. *If an indexed colored permutation is sent to the tableau pair (P, Q) then its inverse is sent to (Q, P).*

The proof is given by transposing the matrix defined in Section 5 and observing that the local rules are preserved.

7. Comparison with other Schensteds

7.1. The Stanton-White Schensted. Consider the insertion of a biletter $\binom{c}{i}$ into a rim hook tableau T that does not contain i. The Stanton-White Schensted and the color-to-spin Schensted differ only in two cases.

Suppose a biletter of maximum value M is inserted into T. Let λ be the shape of T. In the Stanton-White Schensted, a rim hook labeled M is adjoined to T at the northeastmost λ-addable rim hook whose head lies on a diagonal whose index is congruent to c mod k, whereas in the color-to-spin Schensted, it is adjoined at the northeastmost λ-addable rim hook of spin c.

The other case is when $H_1 = H_2$ in the hook-bumping rule (3.1). In the Stanton-White Schensted, the rim hook H_2 is bumped to the northeastmost $(\mu \cup H_1)$-addable rim hook that is strictly southwest of H_1 and whose head lies on a diagonal whose index is congruent mod k to the diagonal of the head of H_1. In the color-to-spin Schensted, H_2 is bumped to the northeastmost $(\mu \cup H_1)$-addable rim hook that is strictly southwest of H_1 and has the same spin as H_2.

7.2. **Garfinkle's domino Schensted.** In the domino case, it was mentioned that the Garfinkle domino Schensted also has the color-to-spin property. The Garfinkle domino Schensted and the $k = 2$ case of the color-to-spin Schensted differ in the same two cases as above.

Suppose a domino of color c of maximum value is inserted. In the Garfinkle insertion, this domino is adjoined at the southwestmost position of spin 1 if $c = 1$ and at the northeastmost position of spin 0 if $c = 0$. In the domino case of our insertion, the domino is adjoined at the northeastmost position of spin c.

The only other case where these two domino insertions differ, is when $H_1 = H_2$ in the hook-bumping rule (3.1). In the Garfinkle insertion, the dominoes of spin 0 and 1 are bumped to the southwest and northeast respectively, while in our insertion all dominoes are bumped to the southwest. In both of these domino insertions, all bumped dominoes retain their spin.

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