A CHARACTERIZATION OF $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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ABSTRACT. Posets containing no subposet isomorphic to the disjoint sums of chains $\mathbf{3}+\mathbf{1}$ and/or $\mathbf{2}+\mathbf{2}$ are known to have many special properties [4], [5], [8], [9]. However, while posets free of $\mathbf{2}+\mathbf{2}$ and posets free of both $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$ may be characterized as interval orders, no such characterization is known for posets free of only $\mathbf{3}+\mathbf{1}$. We give here a characterization of $(\mathbf{3}+\mathbf{1})$-free posets in terms of their anti-adjacency matrices. Using results about totally positive matrices, we show that this characterization leads to a simple proof that the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros.

RÉSUMÉ. Les ensembles partiellement ordonnés qui ne contiennent pas un sous-ensemble partiellement ordonné isomorphique à $\mathbf{3}+\mathbf{1}$ et/ou $\mathbf{2}+\mathbf{2}$ possèdent propriétés interessantes [4], [5], [8], [9]. Cependant, alors que les ensembles sans $\mathbf{2}+\mathbf{2}$, et les ensembles sans $\mathbf{2}+\mathbf{2}$ et $\mathbf{3}+\mathbf{1}$ possèdent une caracterisation par ordres d'intervalle, aucune caracterisation analogue pour les ensembles seulement sans $\mathbf{3}+\mathbf{1}$ n'est connue. Nous présentons une caracterisation basée sur leurs matrices anti-adjacentes. En utilisant les resultats sur les matrices totallement positives, nous montrons que cette caracterisation produit une preuve simple que le polynôme des chaînes d'un ensemble partiellement ordonné sans $\mathbf{3}+\mathbf{1}$ ne possède que des zéros réelles.

1. INTRODUCTION

Fishburn [3] introduced the term *interval order* to refer to posets having no induced subposet isomorphic to the disjoint sum $\mathbf{2}+\mathbf{2}$ of two two-element chains. Such a poset P may be thought of as a set of closed intervals of the form $[a_i, b_i]$, ordered by defining $[a_i, b_i] <_P [a_j, b_j]$ whenever $b_i < a_j$. If, in addition, P has no induced subposet isomorphic to the disjoint sum $\mathbf{3}+\mathbf{1}$ of a three-element chain and a single element, then P may be represented as a set of constant size intervals. The converses of both of these statments are also true. That is, the interval representations are in fact characterizations of the posets. An analogous characterization for posets free only of $\mathbf{3}+\mathbf{1}$ remains an open problem.

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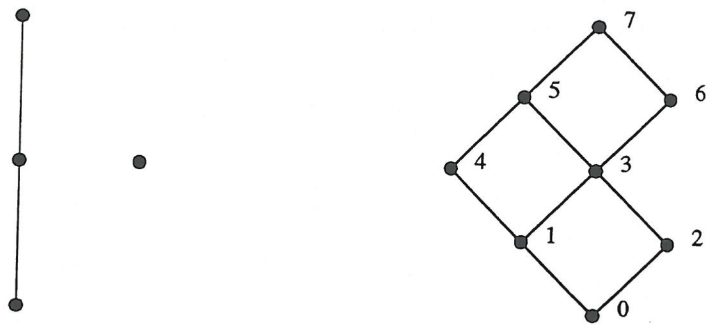

FIGURE 2.1

Characterization of $(\mathbf{3}+\mathbf{1})$-free posets is an interesting problem, because several results and conjectures about posets require avoidance only of $\mathbf{3}+\mathbf{1}$. For instance, Stanley's generalization of the chromatic polynomial [7] is known to be s-positive for the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets [5], and is conjectured to be e-positive for these graphs as well [7, 9, 10]. Further, the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros. (See [9] and Corollary 4.1).

The main theorem of this paper characterizes a $(\mathbf{3}+\mathbf{1})$-free poset in terms of its anti-adjacency matrix (Section 3). From this characterization, reality of the zeros of the chain polynomial follows as an easy corollary (Section 4). We finish with some related open questions.

2. PRELIMINARIES

We denote by $\mathbf{a}+\mathbf{b}$ the poset which is the disjoint sum of an a-element chain and a b-element chain. We say that a poset is $(\mathbf{a}+\mathbf{b})$-*free* if it contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. In Figure 2.1, the poset on the left is $\mathbf{3}+\mathbf{1}$. The poset on the right is not $(\mathbf{3}+\mathbf{1})$-free because the subposet induced by elements $\{2,3,4,6\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$.

Let P be a poset on n elements with order relation $<_P$. We say that a bijectve function $\phi : P \rightarrow \{1, \ldots n\}$ is a *natural labelling* of P if

(2.1) $$x <_P y \Rightarrow \phi(x) < \phi(y),$$

where $<$ is the usual relation on integers.

Given any labelling of P, we define its *anti-adjacency matrix* $A = [a_{ij}]$ by

(2.2) $$a_{ij} = \begin{cases} 0 & i <_P j \\ 1 & \text{otherwise.} \end{cases}$$

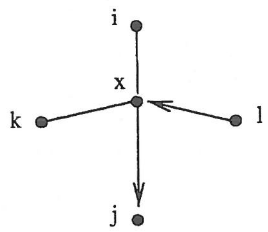

FIGURE 2.2

Let P be a poset with anti-adjacency matrix A, and consider the matrix $B = A^2$, which has the following combinatorial interpretation. If we let G be the directed graph whose *adjacency* matrix is A, then $B = A^2$ counts the number of paths of length two between each pair of vertices in G. That is, b_{ik} is the number of paths of length two in G from i to k. Note that edges exist in G between each pair of vertices: (i, j) is a directed edge if $i >_P j$, and an undirected edge if i and j are incomparable or identical.

Using the graph G, we can show that certain properties of a poset P may be read directly from its squared anti-adjacency matrix B.

Observation 2.1. *Suppose that i, j, k are three distinct elements of P forming a chain: $i <_P j <_P k$. Then either $b_{ik} = 0$, or P is not $(\mathbf{3}+\mathbf{1})$-free.*

Proof. If there is no path of length two in G from i to k, then $b_{ik} = 0$. If there is such a path, it must be of the form (i, x, k) where x is incomparable to i and k. Thus, the poset induced by $\{i, j, k, x\}$ is isomorphic to $\mathbf{3} + \mathbf{1}$. □

Observation 2.2. *Let i, j, k, and l be elements of P, not necessarily distinct.*

1. *If $b_{ik} > b_{il}$, then there is an element $x <_P l$, such that $x \not<_P k$, and $x \not>_P i$.*
2. *If $b_{ik} > b_{jk}$, then there is an element $j <_P y$, such that $y \not>_P i$, and $y \not<_P k$.*

Proof. (1) If $b_{ik} > b_{il}$, then there are more paths of length two in G from i to k than from i to l. It follows that there is a vertex x such that (i, x, k) is a path in G and (i, x, l) is not. In particular, the edge (l, x) is directed, and the edges (i, x) and (x, k) are not directed backward. Let us call such a vertex x a *(k,l)-advantage* for i, imagining that it "helps" i get to k, but not to l. (2) is similar. □

Note that in Figure 2.2, the vertex x is a (k, l)-advantage for i

Lemma 2.3. *If $b_{ik} - b_{il} > b_{jk} - b_{jl}$, then either i has a (k, l)-advantage that j doesn't have, or j has an (l, k)-advantage that i doesn't have. In particular, either there is an element x such that $j <_P x <_P l$ or there is an element y such that $i <_P y <_P k$.*

Proof. Noting that $b_{ik} - b_{il}$ is the number of (k, l)-advantages for i minus the number of (l, k)-advantages for i, one proves the first statement of the lemma immediately. Then, recalling that there is an edge between each pair of vertices in G, we see that an element x is a (k, l)-advantage for i and not for j if and only if there are directed edges (l, x) and (x, j) in G. □

Note that in Figure 2.2, x is not a (k, l)-advantage for j, although it is a (k, l)-advantage for i.

Observation 2.4. *For any matrix B, denote by* $\text{row}(i)$ *and* $\text{column}(i)$ *the ith row and column of B. The following two conditions on a poset P are equivalent.*

1. There is some natural labelling of P such that the squared anti-adjacency matrix B weakly increases to the southwest, i.e. $b_{i+1,j} \geq b_{ij}$ and $b_{i,j-1} \geq b_{ij}$.

2. In every natural labelling of P, the rows and columns of B corresponding to any pair of indices i and j satisfy one of the following pairs of vector inequalities.

$$\text{row}(i) \geq \text{row}(j) \text{ and } \text{column}(i) \leq \text{column}(j) \tag{2.3}$$

$$\text{row}(i) \leq \text{row}(j) \text{ and } \text{column}(i) \geq \text{column}(j) \tag{2.4}$$

Proof. The first statement simply says that we may sort the columns of B in weakly decreasing order while simultaneously sorting the rows in weakly increasing order. Clearly this is possible if and only if the conditions in the second statement are true. □

In examining the anti-adjacency matrix of a poset, we will use some facts about infinite Toeplitz matrices and totally positive matrices (see [1, 2]).

Definition 2.1. Given a sequence $(c_n)_{n \geq 0}$, we define the infinite *Toeplitz matrix* C by

$$C = (c_{i-j}) = \begin{bmatrix} c_0 & 0 & 0 & \cdots & \\ c_1 & c_0 & 0 & \cdots & \\ c_2 & c_1 & c_0 & \cdots & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{bmatrix}, \tag{2.5}$$

where $i, j \geq 0$ and $c_n = 0$ if $n < 0$.

A real matrix, finite or infinite, is called *totally positive* (or sometimes *totally nonnegative*) if each $k \times k$ minor is nonnegative. An important property of a finite totally positive matrix is that it has only nonnegative real eigenvalues (see [2, Corollary 6.6]).

Example 2.2. In particular, we will consider the infinite Toeplitz matrix corresponding to the positive integers,

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{bmatrix}, \tag{2.6}$$

and submatrices of this matrix. In this case, it is not hard to show that C is totally positive.

3. Main Result

Using the three observations and lemma from the previous section, we will characterize $(\mathbf{3}+\mathbf{1})$-free posets by relating their anti-adjacency matrices to the infinite Toeplitz matrix of Example 2.2.

Theorem 3.1. *A poset P is $(\mathbf{3}+\mathbf{1})$-free if and only if there is a natural labelling of P such that its squared anti-adjacency matrix is a submatrix of the infinite Toeplitz matrix C,*

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{bmatrix}, \tag{3.1}$$

with row and column repetition allowed.

Precisely, *row (column) repetition* is the replacement of any row (column) of C by any number of copies of itself, without reordering any other rows or columns. We allow the repetition of any number of rows and columns.

First we prove that the anti-adjacency matrices of $(\mathbf{3}+\mathbf{1})$-free posets have the two defining properties of submatrices of C, allowing row and column repetition.

Proposition 3.2. *If P is a $(\mathbf{3}+\mathbf{1})$-free poset then there is a natural labelling of P such that the entries of its squared anti-adjacency matrix B weakly increase toward the southwest.*

Proof. Assume that for each natural labelling of P, B does not weakly increase toward the southwest. Then, one of the pairs of vector inequalities in Observation 2.4 fails to hold. We consider two cases for a given natural labelling.

Case 1: Suppose that all columns and rows of B are comparable as vectors. We then may assume that for some i and j we have the following incorrect comparison, where the vectors in question are not identically equal.

$$\text{row}(i) \geq \text{row}(j) \text{ and column}(i) \geq \text{column}(j). \tag{3.2}$$

Then for some elements k, l, of P we have

$$b_{ki} > b_{kj} \text{ and } b_{il} > b_{jl}. \tag{3.3}$$

Since $b_{ki} > b_{kj}$, there must be an element $x \neq j$ such that $x <_P j$, $x \not<_P i$, and $x \not>_P k$. Similarly, since $b_{il} > b_{jl}$, there must be an element $y \neq j$ such that $j <_P y$, $y \not>_P i$, and $y \not<_P l$. In particular, either x is incomparable to i, or $x >_P i$, and either y is incomparable to i, or $y <_P i$. By the transitivity of P, both x and y must be incomparable to i, and by Observation 2.1, the subposet induced by $\{i, j, x, y\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$.

Case 2: Suppose that some pair of columns (or rows) is not comparable as a pair of vectors. In this case, we may assume without loss of generality that there is a 2×2 submatrix

$$\begin{bmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{bmatrix} \tag{3.4}$$

with $b_{ik} > b_{il}$ and $b_{jk} < b_{jl}$. Therefore, i has a (k,l)-advantage x, and j has an (l,k)-advantage y. In particular, we cannot have $k >_P x$, $x >_P i$, $l >_P y$, or $y >_P j$. Since

$$b_{ik} - b_{il} > 0 > b_{jk} - b_{jl}, \tag{3.5}$$

we may apply Lemma 2.3. Suppose first that x is not a (k,l)-advantage for j. Then, $j <_P x <_P l$. By transitivity, both j and l must be incomparable to y. Thus, the subposet induced by $\{l, x, j, y\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$. Supposing that y is not an (l,k)-advantage for i, we see by the same argument that the subposet induced by $\{k, y, i, x\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$. □

Proposition 3.3. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset, naturally labelled so that its squared anti-adjacency matrix B weakly increases toward the southwest. Let $i < j$ and $k < l$. Then the 2×2 submatrix*

$$\begin{bmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{bmatrix} \tag{3.6}$$

satisfies one of the following two conditions:

(i) $b_{ik} - b_{il} = b_{jk} - b_{jl}$

(ii) $b_{il} = 0$ *and* $b_{jk} - b_{jl} > b_{ik}$

Proof. Suppose that condition (i) is not satisfied.

Case 1: $(b_{ik} - b_{il} > b_{jk} - b_{jl})$. Applying Lemma 2.3 to this inequality, we suppose first that i has a (k, l)-advantage x that j doesn't have. Thus, $j <_P x <_P l$. By Observation 2.1, we have that $b_{jl} = 0$, implying that $b_{il} = 0$, since B weakly increases to the southwest. Thus, $b_{ik} > b_{jk}$, a contradiction. We conclude that j has an (l, k)-advantage y that i doesn't have. Thus, $i <_P y <_P k$. Applying Observation 2.1, we have that $b_{ik} = 0$, implying that $b_{il} = 0$. Thus, $0 > b_{jk} - b_{jl}$, again a contradiction.
Case 2: $(b_{ik} - b_{il} < b_{jk} - b_{jl})$. As in the previous case, we apply Lemma 2.3 to this inequality. First let us suppose that i has an (l, k)-advantage x that j doesn't have. Then, $j <_P x <_P k$. By Observation 2.1, $b_{jk} = 0$, implying that all four numbers are zero, a contradiction. We conclude that that j has a (k, l)-advantage y that i doesn't have. This implies that $i <_P y <_P l$, and that $b_{il} = 0$. Thus condition (ii) is satisfied. □

Since the properties stated in Proposition 3.2 and Proposition 3.3 characterize submatrices of the infinite Toeplitz matrix C in Example 2.2, allowing row and column repetition, we have proven one direction of the theorem. Now we show that the only posets whose anti-adjacency matrices have the desired form are those which are $(\mathbf{3} + \mathbf{1})$-free.

Proposition 3.4. *Let P be a naturally lablelled poset containing $\mathbf{3} + \mathbf{1}$ as an induced subposet, and let B be its squared anti-adjacency matrix. Then B is not a submatrix of the infinite Toeplitz matrix C of Example 2.2.*

Proof. Consider the poset $Q = \mathbf{3} + \mathbf{1}$. Let us label the elements $1 <_Q 2 <_Q 3$ and 4. The squared anti-adjacency matrix of Q is

$$B = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 3 & 2 & 4 \\ 4 & 3 & 2 & 4 \end{bmatrix}. \tag{3.7}$$

Note that the submatrix

$$\begin{bmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \tag{3.8}$$

weakly increases to the southwest, but does not satisfy

$$b_{11} - b_{13} = b_{31} - b_{33}. \tag{3.9}$$

Let us attempt to adjoin elements to Q to satisfy equation (3.9). To increase the difference $b_{11} - b_{13}$ without increasing the difference $b_{31} - b_{33}$, we need a new element x to be a $(1, 3)$-advantage for 1 and *not* a $(1, 3)$-advantage for 3. This is clearly impossible. Similarly, we cannot decrease the second difference without decreasing the first. Therefore there are no elements we can adjoin to Q so that B will be a submatrix of the infinite Toeplitz matrix C. $\square$

Combinining the last three propositions, we now have the desired equivalence.

4. Chain Polynomials and Open Questions

Definition 4.1. We define the *chain polynomial* of a finite poset P by

$$f_P(x) = \sum_{i=0}^{r} c_i x^i, \tag{4.1}$$

where c_i is the number of i-element chains in P, and r is the maximum cardinality of a chain in P. We define $c_0 = 1$.

If A is the anti-adjacency matrix of P then the chain polynomial is given by

$$f_P(x) = \det(I + xA). \tag{4.2}$$

(See [8].) From this formula we see that $f_P(x)$ has only real zeros if and only if A has only real eigenvalues. The following result now follows easily from Theorem 3.1.

Corollary 4.1. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the chain polynomial $f_P(x)$ has only real zeros.*

Proof. Since $B = A^2$ is a submatrix of a totally positive matrix, it is totally positive, and therefore has only nonnegative real eigenvalues (see Section 2). It follows that A has only real eigenvalues, and that $f_P(x)$ has only real zeros. $\square$

The converse of Corollary 4.1 is not true, for there are many posets containing $\mathbf{3}+\mathbf{1}$ as an induced subposet, whose chain polynomials have only real zeros. An important open problem is to determine which posets have this property. In particular, we have the following conjecture, due to Stanley and Neggers.

Conjecture 4.2. *Let $J(Q)$ be a finite distributive lattice. Then the chain polynomial $f_{J(Q)}(x)$ has only real zeros.*

By a result of Simion [6], the conjecture holds for the special case of products of chains. The question of whether the conjecture holds for the larger class of *modular* lattices is open as well. It would be interesting to apply formula (4.2) to either of the open questions or to the special case.

5. Acknowledgments

I would like to thank Richard Stanley for inspiration and for many helpful suggestions.

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