Products of class-sums of the alternating group

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Abstract

Most of the structure constants in the class-algebra of the alternating group, $\mathcal{A}_n$, can be expressed in terms of corresponding structure constants in the class-algebra of the symmetric group, $\mathcal{S}_n$. This statement has to be modified for $\mathcal{A}_n$-structure constants that involve more than one conjugacy class stemming from a common $\mathcal{S}_n$ conjugacy class, when the latter consists of cycles of odd and distinct lengths. The $\mathcal{A}_n$-structure constants of the latter type depend on the $\mathcal{S}_n$ character that corresponds to the self-conjugate Young diagram whose principal hook lengths are equal to the set of odd and distinct cycle lengths mentioned above, evaluated over the third conjugacy class involved. The results provide a combinatorial interpretation of the $\mathcal{S}_n$ self-conjugate irreducible characters.

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1 Introduction

Let G be a finite group. By $\mathcal{C}(G)$ we denote the set of conjugacy classes of G. For a non empty subset A of G let $[A] = \sum_{x\in A} x \in \mathbf{C}G$, where $\mathbf{C}G$ is the corresponding group-algebra. If $A \subset G$ is a G-conjugacy class, then $[A]$ is the corresponding conjugacy class-sum. The conjugacy class-sums span the center of the group algebra, and play a major role in its representation theory [1, 2].

The product of any pair of class-sums $[A]$ and $[B]$ can be expressed as a linear combination of class-sums with non-negative integral coefficients $(A\cdot B)\big|_C$, which we call the class-algebra structure-constants. Thus,

$$[A]\cdot[B] = \sum_{C\in\mathcal{C}(G)} (A\cdot B)\big|_C [C]\,.$$

The coefficient $(A\cdot B)\big|_C$ can be expressed in terms of the ordinary irreducible characters in the form [1]

$$(A\cdot B)\big|_C = \frac{|A||B|}{|G|}\sum_{\Gamma\in\mathcal{I}(G)}\frac{1}{|\Gamma|}\chi_A^\Gamma\chi_B^\Gamma\overline{\chi_C^\Gamma}\,. \tag{1}$$

Here, $\mathcal{I}(G)$ is a complete set of inequivalent ordinary irreducible representations (IRREPS) of G, χ_C^Γ is the ordinary character corresponding to the IRREP Γ evaluated at C, and $|\Gamma|$ is the dimension of Γ. $|X|$ stands for the cardinality of the (finite) set X.

One circle of problems that have been addressed rather extensively involves the establishment of criteria for distinguishing between vanishing and non-vanishing structure constants, without actually evaluating the latter. In particular, class-sums $[A]$ such that $(A\cdot A)\big|_C \neq 0\ \forall C\in\mathcal{C}(G)$ have been looked for. In the alternating group this issue has been extensively investigated [3, 4].

In the present article we express the structure constants in the alternating group in terms of data that involve the symmetric group, as follows; Most structure constants are found to be immediately related to corresponding structure constants in the symmetric group (Theorem A). Structure constants in a certain well-defined subset differ from the corresponding $\mathcal{S}_n$ structure constants by a term that depends on a particular $\mathcal{S}_n$-irreducible character (Theorem

B). This result can easily be inverted to derive an expression for the $\mathcal{S}_n$ irreducible characters that correspond to self-conjugate $\mathcal{S}_n$-IRREPS in terms of the difference between two $\mathcal{A}_n$ structure constants (Corollary 1). The investigation presently reported is motivated by the recently formulated, albeit conjecturally, combinatorial procedures for the evaluation of both structure constants [5, 6] and (central) characters [7] in the symmetric group, but is independent of these procedures. However, using these procedures the present results allow the evaluation of all the structure constants in the class-algebras of the alternating groups.

2 Conjugacy classes and irreducible characters of the alternating group

The alternating group $\mathcal{A}_n$ consists of the even permutations of $\mathcal{S}_n$, i.e., permutations that consist of an even number of cycles of even length and any number of cycles of odd length. The orders of these groups satisfy

$$|\mathcal{A}_n| = \frac{|\mathcal{S}_n|}{2} = \frac{n!}{2}. \tag{2}$$

The set of the $\mathcal{S}_n$ conjugacy classes that consist of cycles of odd and distinct lengths will be denoted by $\mathcal{C}_o(\mathcal{S}_n)$. The set consisting of all the other $\mathcal{S}_n$ even conjugacy classes, i.e., classes which consist of permutations that contain (an even number of) cycles of even lengths, or at least one pair of cycles of equal odd lengths, will be denoted by $\mathcal{C}_e(\mathcal{S}_n)$.

Fact 1: *Each conjugacy class $C \in \mathcal{C}_o(\mathcal{S}_n)$ splits within $\mathcal{A}_n$ into a pair of conjugacy classes of equal cardinalities, that will be denoted by $C^{\pm}$. The conjugacy classes $C \in \mathcal{C}_e(\mathcal{S}_n)$ remain single conjugacy classes within $\mathcal{A}_n$.*

From Fact 1 it follows that $\forall C \in \mathcal{C}_o(\mathcal{S}_n) \Rightarrow C = C^+ \dot{\cup} C^-$ where $C^{\pm} \in \mathcal{C}_o(\mathcal{A}_n)$. Hence, $|\mathcal{C}_o(\mathcal{A}_n)| = 2|\mathcal{C}_o(\mathcal{S}_n)|$. On the other hand, $\mathcal{C}_e(\mathcal{S}_n) = \mathcal{C}_e(\mathcal{A}_n)$.

For the symmetric group Γ will denote both an IRREP and the corresponding Young diagram. The set of self-conjugate $\mathcal{S}_n$ IRREPS will be denoted by $\mathcal{I}_S(\mathcal{S}_n)$, and the set of non self-conjugate $\mathcal{S}_n$ IRREPS will be denoted by $\mathcal{I}_N(\mathcal{S}_n)$.

Fact 2: *The restriction of the $\mathcal{S}_n$-IRREPS $\Gamma \in \mathcal{I}_S(\mathcal{S}_n)$ to $\mathcal{A}_n$ is reducible, splitting into a pair of $\mathcal{A}_n$-IRREPS $\Gamma^{\pm}$, of equal dimensions. $\mathcal{S}_n$-IRREPS $\Gamma \in \mathcal{I}_N(\mathcal{S}_n)$ remain irreducible within $\mathcal{A}_n$; however, the restrictions to $\mathcal{A}_n$ of $\mathcal{S}_n$-IRREPS with conjugate Young diagrams are found to coincide.*

Denote the $\mathcal{S}_n$-IRREP which is conjugate to $\Gamma = \{\lambda_1, \lambda_2, \cdots\} \in \mathcal{I}_N(\mathcal{S}_n)$ by $\tilde{\Gamma} = \{\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots\}$, λ_i and $\tilde{\lambda}_i$ being the row lengths in the corresponding Young diagrams (i.e., $\tilde{\lambda}_i$ are the column lengths of Γ). We say that $\Gamma > \tilde{\Gamma}$ if $\exists k$ such that $\lambda_i = \tilde{\lambda}_i$ for $i = 1, 2, \cdots, k-1$ and $\lambda_k > \tilde{\lambda}_k$. We specify a set of inequivalent, non self-conjugate, $\mathcal{A}_n$-IRREPS by

$$\mathcal{I}_N(\mathcal{A}_n) = \{\Gamma \in \mathcal{I}_N(\mathcal{S}_n) \,\big|\, \Gamma > \tilde{\Gamma}\}.$$

From Fact 2 it follows immediately that $|\mathcal{I}_S(\mathcal{A}_n)| = 2|\mathcal{I}_S(\mathcal{S}_n)|$ and $|\mathcal{I}_N(\mathcal{A}_n)| = \frac{1}{2}|\mathcal{I}_N(\mathcal{S}_n)|$.
Finally,

$$\mathcal{I}(\mathcal{S}_n) = \mathcal{I}_S(\mathcal{S}_n)\dot{\cup}\mathcal{I}_N(\mathcal{S}_n) \tag{3}$$

and

$$\mathcal{I}(\mathcal{A}_n) = \mathcal{I}_S(\mathcal{A}_n)\dot{\cup}\mathcal{I}_N(\mathcal{A}_n). \tag{4}$$

Let $C \in \mathcal{C}_0(\mathcal{S}_n)$ have d (odd and distinct) cycles. Γ_C denotes the self-conjugate $\mathcal{S}_n$-IRREP whose principal hook lengths $\{h_{i,i}\ ;\ i = 1, 2, \cdots, d\}$ are equal to the cycle lengths of C. (The principal hook length $h_{i,i}$ in a Young diagram Γ is the number of boxes in the hook that consists of the box in the (diagonal) position (i,i) and the boxes to the right and below that box). $\{\Gamma_C^{\pm}\}$ consists of the pair of $\mathcal{A}_n$-IRREPS corresponding to Γ_C. For $A \in \mathcal{C}_e(\mathcal{S}_n)$ we define $\{\Gamma_A^{\pm}\} = \emptyset$. For $c \in \mathcal{A}_n$ C will denote the $\mathcal{S}_n$-conjugacy class that contains c.

Lemma 1 [8, 9]: *The $\mathcal{A}_n$ irreducible characters can be evaluated as follows:*

(i) for $c \in \mathcal{A}_n$ and $\Gamma \in \mathcal{I}_N(\mathcal{A}_n)$

$$\chi_c^{\Gamma}(\mathcal{A}_n) = \chi_C^{\Gamma}(\mathcal{S}_n) = \chi_C^{\tilde{\Gamma}}(\mathcal{S}_n).$$

(ii) for $c \in \mathcal{A}_n$ and $\Gamma \in \mathcal{I}_S(\mathcal{S}_n)\backslash\{\Gamma_C\}$

$$\chi_c^{\Gamma^+}(\mathcal{A}_n) = \chi_c^{\Gamma^-}(\mathcal{A}_n) = \frac{1}{2}\chi_C^{\Gamma}(\mathcal{S}_n).$$

(iii) for $C \in \mathcal{C}_0(\mathcal{S}_n)$ and $\Gamma = \Gamma_C$

$$\chi^{\Gamma+}_{C^+}(\mathcal{A}_n) = \chi^{\Gamma-}_{C^-}(\mathcal{A}_n) = \frac{1}{2}\chi^{\Gamma}_{C}(\mathcal{S}_n) + x_\Gamma$$

$$\chi^{\Gamma-}_{C^+}(\mathcal{A}_n) = \chi^{\Gamma+}_{C^-}(\mathcal{A}_n) = \frac{1}{2}\chi^{\Gamma}_{C}(\mathcal{S}_n) - x_\Gamma$$

where $x_\Gamma = \frac{1}{2}\sqrt{(-1)^m \prod_{i=1}^{d} h_{i,i}}$ and $m = \frac{1}{2}(n-d)$.

3 Structure constants in the class-algebra of the alternating group

It will be convenient to define the reduced structure constant

$$(a \cdot b)\Big|_C \equiv \frac{(A \cdot B)\Big|_C}{|A||B|}. \tag{5}$$

Thus, by eq. 1

$$(a \cdot b)\Big|_C = \frac{1}{|G|} \sum_{\Gamma \in \mathcal{I}(G)} \frac{1}{|\Gamma|} \chi^{\Gamma}_{A} \chi^{\Gamma}_{B} \overline{\chi^{\Gamma}_{C}}. \tag{6}$$

To simplify the notation, the $\mathcal{A}_n$ irreducible characters will be denoted by $\chi^{\Gamma}_{A}(\mathcal{A}_n)$ while the $\mathcal{S}_n$ irreducible characters will be denoted by χ^{Γ}_{A}, where Γ and A are understood to denote irreps and conjugacy classes of $\mathcal{A}_n$ and $\mathcal{S}_n$, respectively.

Theorem A: *In the alternating group algebra the reduced structure constants can be written in the form*

$$(a \cdot b)\Big|_C(\mathcal{A}_n) = (a \cdot b)\Big|_C(\mathcal{S}_n) + \mathcal{R}^{C}_{A,B}$$

where

$$\mathcal{R}^{C}_{A,B} = \frac{4}{n!} \sum_{\Gamma \in \Delta_{A,B,C}} \frac{1}{|\Gamma|} \left\{ \chi^{\Gamma}_{A}(\mathcal{A}_n)\chi^{\Gamma}_{B}(\mathcal{A}_n)\overline{\chi^{\Gamma}_{C}(\mathcal{A}_n)} - \frac{\chi^{\Gamma}_{A}}{2}\frac{\chi^{\Gamma}_{B}}{2}\frac{\overline{\chi^{\Gamma}_{C}}}{2} \right\}, \tag{7}$$

and

$$\Delta_{A,B,C} = \{\Gamma^{\pm}_{A}\} \cup \{\Gamma^{\pm}_{B}\} \cup \{\Gamma^{\pm}_{C}\}.$$

PROOF: Let A, B and C be $\mathcal{S}_n$-conjugacy classes consisting of even permutations (i.e., permutations that belong to $\mathcal{A}_n$). Using eq. 3 and Fact 2, eq. 6 yields

$$(a \cdot b)\Big|_C(\mathcal{S}_n) = \frac{1}{n!}\left\{2\sum_{\Gamma<\bar{\Gamma}}\frac{1}{|\Gamma|}\chi_A^\Gamma\chi_B^\Gamma\overline{\chi_C^\Gamma} + \sum_{\Gamma\in\mathcal{I}_S(\mathcal{S}_n)}\frac{1}{|\Gamma|}\chi_A^\Gamma\chi_B^\Gamma\overline{\chi_C^\Gamma}\right\}. \tag{8}$$

Similarly, using eqs. 2, 4, Lemma 1 and Fact 2, eq. 6 yields

$$(a \cdot b)\Big|_C(\mathcal{A}_n) = \frac{2}{n!}\left\{\sum_{\Gamma\in\mathcal{I}_N(\mathcal{A}_n)}\frac{1}{|\Gamma|}\chi_A^\Gamma\chi_B^\Gamma\overline{\chi_C^\Gamma} + \sum_{\Gamma\in\mathcal{I}_S(\mathcal{A}_n)}\frac{1}{|\Gamma|}\chi_A^\Gamma(\mathcal{A}_n)\chi_B^\Gamma(\mathcal{A}_n)\overline{\chi_C^\Gamma(\mathcal{A}_n)}\right\}. \tag{9}$$

The Theorem follows by using Lemma 1 to compare eq. 8 and eq. 9. ■

Theorem B: *The non vanishing residual terms are*

(i) $\mathcal{R}^B_{A^+,A^+} = -\mathcal{R}^B_{A^+,A^-} = \rho_A\chi_B^{\Gamma_A}(-1)^m$

(ii) $\mathcal{R}^{A^-}_{A^+,B} = -\mathcal{R}^{A^+}_{A^+,B} = -\rho_A\chi_B^{\Gamma_A}$

(iii) $\mathcal{R}^{A^+}_{A^+,A^+} = \rho_A(1+2(-1)^m)$

(iv) $\mathcal{R}^{A^+}_{A^+,A^-} = -\rho_A$

(v) $\mathcal{R}^{A^-}_{A^+,A^+} = \rho_A(1-2(-1)^m)$

and their conjugates (in which A^+ and A^- are interchanged). Here, $\rho_A = \frac{1}{|\Gamma_A||A|}$, $|\Gamma_A|$ is the dimension of Γ_A as an $\mathcal{S}_n$-IRREP, $\chi_B^{\Gamma_A}$ is an $\mathcal{S}_n$ character, and $B \neq A$ as $\mathcal{S}_n$ conjugacy classes.

PROOF: Using Lemma 1 we note that the residual term $\mathcal{R}^C_{A,B}$ vanishes unless at least two of the conjugacy classes involved correspond to the same member of $\mathcal{C}_o(\mathcal{S}_n)$; otherwise, at most one character differs from the corresponding $\mathcal{S}_n$ character in each term, so the corresponding contributions of Γ^+ and Γ^- add up to zero. The various cases in the Theorem follow by evaluating eq. 7, using Lemma 1 and the identity $\chi_A^{\Gamma_A} = (-1)^m$. ■

Let

$$n_C = \begin{cases} 0 & \text{if } C \in \mathcal{C}_e(\mathcal{A}_n) \\ 1 & \text{if } C \in \mathcal{C}_o(\mathcal{A}_n) \end{cases}$$

Remark: *The structure constant* $(A\cdot B)\big|_C(\mathcal{A}_n)$ *is obtained by multiplication of the reduced structure constant* $(a\cdot b)\big|_C(\mathcal{A}_n)$ *with the product of the cardinalities of the conjugacy classes* A *and* B *in* $\mathcal{A}_n$*, i.e.,*

$$|A|_{\mathcal{A}_n}|B|_{\mathcal{A}_n} = |A|_{\mathcal{S}_n}|B|_{\mathcal{S}_n}\cdot\left(\frac{1}{2}\right)^{n_A+n_B}.$$

PROOF: Use eq. 5 and Fact 1. ■

Finally,

Corollary 1: *For* $A\in\mathcal{C}_o(\mathcal{S}_n)$, $B\in\mathcal{C}(\mathcal{S}_n)$, $B\neq A$ *and* $m=\frac{1}{2}(n-d)$ *where* d *is the number of cycles in* A *(=number of principal hooks in* Γ_A*)*

$$\chi_B^{\Gamma_A} = (-1)^m\frac{2|\Gamma_A|_{\mathcal{S}_n}}{|A|_{\mathcal{S}_n}}\left((A^+\cdot A^+)\big|_B(\mathcal{A}_n)-(A^+\cdot A^-)\big|_B(\mathcal{A}_n)\right) \tag{10}$$

PROOF: Use the first case in Theorem B. ■

Comments:

(i) *Eq. 10 is easily checked to hold for* $B=A\in\mathcal{C}_o(\mathcal{S}_n)$ *if* $(A^+\cdot A^\pm)\big|_A$ *is interpreted as the average of* $(A^+\cdot A^\pm)\big|_{A^+}$ *and* $(A^+\cdot A^\pm)\big|_{A^-}$*; note that* $(A^+\cdot A^\pm)\big|_{B^+}=(A^+\cdot A^\pm)\big|_{B^-}$ *for* $\mathcal{C}_o(\mathcal{S}_n)\ni B\neq A$.

(ii) *If* B *is an odd conjugacy class of* $\mathcal{S}_n$ *eq. 10 holds trivially, since in this case* $\chi_B^{\Gamma_A}=0$.

In view of these Comments a more general version of Corollary 1 can be formulated as follows

Corollary 1': *For* $A\in\mathcal{C}_o(\mathcal{S}_n)$, $B\in\mathcal{C}(\mathcal{S}_n)$, *and* $m=\frac{1}{2}(n-d)$ *where* d *is the number of cycles in* A *(=number of principal hooks in* Γ_A*)*

$$\begin{aligned}\chi_B^{\Gamma_A} &= (-1)^m\frac{|\Gamma_A|_{\mathcal{S}_n}}{|A|_{\mathcal{S}_n}}\Big((A^+\cdot A^+)\big|_{B^+}(\mathcal{A}_n)+(A^+\cdot A^+)\big|_{B^-}(\mathcal{A}_n)\\ &\qquad -(A^+\cdot A^-)\big|_{B^+}(\mathcal{A}_n)-(A^+\cdot A^-)\big|_{B^-}(\mathcal{A}_n)\Big)\end{aligned}$$

where, if $B\notin\mathcal{C}_o(\mathcal{S}_n)$ *then it is understood that* $B^+=B^-=B$.

Corollary 1 provides a combinatorial interpretation for the $\mathcal{S}_n$ characters corresponding to self-conjugate IRREPS, evaluated over arbitrary conjugacy classes.

4 Some illustrative examples

$\mathcal{A}_3$:

The $\mathbf{C}\mathcal{S}_3$-conjugacy class-sum $[(3)] = (123) + (132)$ gives rise to the two $\mathbf{C}\mathcal{A}_3$ conjugacy class-sums $[(3)]^+ = (123)$ and $[(3)]^- = (132)$. Since

$$
\begin{aligned}
[(3)]^+ \cdot [(3)]^+ &= [(3)]^- , \\
[(3)]^+ \cdot [(3)]^- &= [(1)^3] ,
\end{aligned}
$$

corollary 1' yields $\chi^{[2,1]}_{(1)^3} = 2$ and $\chi^{[2,1]}_{(3)} = -1$.

$\mathcal{A}_4$:

The $\mathbf{C}\mathcal{A}_4$ conjugacy class-sums

$$
\begin{aligned}
[(3)(1)]^+ &= (123)(4) + (214)(3) + (341)(2) + (432)(1) , \\
&\text{e. g., } \big((12)(34)\big)(123)(4)\big((12)(34)\big) = (214)(3) , \\
[(3)(1)]^- &= (132)(4) + (241)(3) + (314)(2) + (234)(1) .
\end{aligned}
$$

yield

$$
\begin{aligned}
[(3)(1)]^+ \cdot [(3)(1)]^+ &= 4[(3)(1)]^- , \\
[(3)(1)]^+ \cdot [(3)(1)]^- &= 4[(1)^4] + 4[(2)^2] .
\end{aligned}
$$

Using corollary 1' it follows that $\chi^{[2,2]}_{(1)^4} = 2$, $\chi^{[2,2]}_{(2)^2} = 2$ and $\chi^{[2,2]}_{(3)(5)} = -1$.

$\mathcal{A}_5$:

Evaluating

$$[(5)]^+ \cdot [(5)]^+ = 5[(5)]^+ + [(5)]^- + 3[(3)(1)^2] + 12[(1)^5]$$

and

$$[(5)]^+ \cdot [(5)]^- = [(5)]^+ + [(5)]^- + 3[(3)(1)^2] + 4[(2)^2(1)]$$

we obtain $\chi^{[3,1,1]}_{(1)^5} = 6$, $\chi^{[3,1,1]}_{(3)(1)^2} = 0$, $\chi^{[3,1,1]}_{(2)^2(1)} = -2$ and $\chi^{[3,1,1]}_{(5)} = 1$.

$\mathcal{A}_6$:

Although already a bit cumbersome, the direct evaluation of the relevant conjugacy class sum products is still feasible, yielding

$$\begin{aligned}[(5)(1)]^+ \cdot [(5)(1)]^+ &= 72[(1)^6] + 9[(3)(1)^3] + 16[(2)^2(1)^2] + 9[(3)^2] + 16[(4)(2)] \\ &\quad +20[(5)(1)]^+ + 11[(5)(1)]^-\end{aligned}$$

and

$$\begin{aligned}[(5)(1)]^+ \cdot [(5)(1)]^- &= 18[(3)(1)^3] + 16[(2)^2(1)^2] + 18[(3)^2] + 16[(4)(2)] \\ &\quad +11[(5)(1)]^+ + 11[(5)(1)]^- .\end{aligned}$$

Hence, $\chi^{[3,2,1]}_{(1)^6} = 16$, $\chi^{[3,2,1]}_{(2)^2(1)^2} = 0$, $\chi^{[3,2,1]}_{(3)(1)^3} = -2$, $\chi^{[3,2,1]}_{(3)^2} = -2$, $\chi^{[3,2,1]}_{(4)(2)} = 0$ and $\chi^{[3,2,1]}_{(5)(1)} = 1$.

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