Decomposition of certain tensor products for some non-compact Lie groups

Frédéric TOUMAZET
Université de Marne-la-Vallée
Institut Gaspard Monge
5, boulevard Descartes 77 454 Marne-la-Vallée Cedex II - France
e-mail : `toumazet@weyl.univ-mlv.fr`

Abstract

The finite dimensional irreducible representations of the non-compact Lie groups $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ are non-unitary while the non-trivial unitary projective irreducible representations are all of infinite dimension. We intoduce the notion of generic tensor product of a non-unitary irreducible representation with a unitary irreducible representation . This product would be generic if it is fully resolvable into a finite set of unitary irreducible representations. The conditions for a product to be generic are given and methods and algorithms for the resolution of generic products developed. Relationships between the generic products for the two groups, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ are established. Conditions for the stabilisation of products are given.

Résumé

Les représentations irréductibles de dimension finie des groupes de Lie non compacts $Sp(2n,R)$ et $SO^*(2n)$ sont non unitaires alors que les représentations unitaires irréductibles de ces groupes sont des représentations projectives de dimension infinie. Nous appelons produit tensoriel générique le produit d'une représentation finie par une représentation de dimension infinie qui peut s'exprimer comme une somme finie de représentations irréductibles unitaires. Nous donnons des conditions de généricité pour ces produits ainsi qu'un algorithme pour les calculer. Nous donnons enfin des conditions de stabilisation de ces produits et des relations entre les produits génériques pour $Sp(2n,\mathbb{R})$ et $SO^*(2n)$.

1 Introduction

Multiplicities in tensor products of irreducible representations of compact Lie groups can be calculated by combinatorial algorithms such as the Littlewood-Richardson rule (corresponding to $U(n)$) or more recent ones derived from Kashiwara's crystal base [2, 3] or Littelmann's paths. [14].

The irreducible representations of the compact forms of the classical Lie groups are all finite dimensional and unitary. On the contrary, the interesting unitary representations of their non-compact real forms (such as $U(p,q)$, $Sp(2n,\mathbb{R})$, $SO(p,q)$, $SO^*(2n)$, ...) are all infinite dimensional, and in fact, only projective representations coming from linear representations of some covering groups (such as the metaplectic group $Mp(2n)$ for $Sp(2n,\mathbb{R})$).

Although no combinatorial rule is known for decomposing tensor products of such representation, explicit formulas which have been worked out in some particular cases [4, 11, 12, 9, 18, 19], suggest the existence of an underlying combinatorics somewhat similar to the one encountered in the compact case, and T. Roby [16] actually succeeded in constructing a Robinson-Schensted type correspondence accounting for the decomposition of particular tensor products.

Here, we present some partial results on another interesting case : tensor products of a (non-unitary) finite dimensional representation by a unitary one. While such products are actually difficult to understand in the general case, under some genericity condition their behaviour presents some similarities with the compact case.

2 Background and notations

2.1 The orthogonal and symplectic groups

First we recall the definition of the orthogonal and symplectic groups.

If V is a $2n$-dimensional linear vector space and g and h are symmetric and antisymmetric bilinear forms, then

$$SO(2n) := \{A \in GL(V) = GL(2n) | A^T gA = g, \det A = 1\} ,$$

$$Sp(2n) := \{A \in GL(V) = GL(2n) | A^T hA = g, \det A = 1\} .$$

Now, briefly recall a few results about the representations for these groups. We first examine the tensor representations of these groups by remarking that (see [7, 13]) the tensor powers of the defining representation V of $GL(2n)$ can be decomposed in this manner

$$V^{\otimes m} = \bigoplus_{\substack{\mu \vdash 2m \\ l(\mu) \leq 2m}} V_\mu^{\oplus f^\mu}$$

where the summation is over partitions μ of m and the multiplicity of the irreducible representation (irrep) V_μ is f^μ, the dimension of the irrep (μ) of the symmetric group S_m. Now it is possible to decompose these representations of $GL(2n)$ under restriction to the groups $SO(2n)$ and $Sp(2n)$. In terms of characters if we write $\{\mu\}$, $[\mu]$ and $\langle\mu\rangle$ respectively for the standard, orthogonal and symplectic Schur functions, we have the following relations

$$\begin{array}{lll}
\{\mu\} & = [\mu/D] & = [\mu] + [\mu/2] + [\mu/4] + \cdots \\
\{\mu\} & = \langle\mu/B\rangle & = \langle\mu\rangle + \langle\mu/11\rangle + \langle\mu/22\rangle + \langle\mu/1111\rangle + \cdots \\
[\mu] & = \{\mu/C\} & = \{\mu\} - \{\mu/2\} + \{\mu/31\} + \cdots \\
\langle\mu\rangle & = \{\mu/A\} & = \{\mu\} - \{\mu/11\} + \{\mu/211\} + \cdots
\end{array}$$

where A, B, C and D are infinite series of S-functions [1, 6, 7], that we define in this manner

$$B = \prod_{i<j} \frac{1}{1 - x_i x_j} = \sum_\beta s_\beta(X) = \sum_\beta \{\beta\}$$

where β runs over partition with even columns. and D is

$$D = \prod_{i \leq j} \frac{1}{1 - x_i x_j} = \sum_\delta s_\delta(X) = \sum_\delta \{\delta\}$$

where δ runs over partition with even rows. The series C and A are defines by

$$C = \frac{1}{D} = \sum_{\nu}(-1)^{\frac{|\nu|}{2}} s_\nu(X) = \sum_{\nu}(-1)^{\frac{|\nu|}{2}}\{\nu\}$$

$$A = \frac{1}{B} = \sum_{\alpha}(-1)^{\frac{|\alpha|}{2}} s_\alpha(X) = \sum_{\alpha}(-1)^{\frac{|\alpha|}{2}}\{\alpha\}$$

where in the Frobenius notation

$$\nu = \begin{pmatrix} a_1+1 & \dots & a_k+1 \\ a_1 & \dots & a_k \end{pmatrix}$$

and

$$\alpha = \begin{pmatrix} a_1 & \dots & a_k \\ a_1+1 & \dots & a_k+1 \end{pmatrix}$$

and the notation $\{\mu/\lambda\}$ corresponds to the skew Schur function $s_{\mu/\lambda}$.

At this stage, it is now possible to give a caracterisation of the irreducible tensor representations of $SO(2n)$ and $Sp(2n)$. They are labelled by partitions (λ) and if we denote the lenght of λ by $l(\lambda)$ we obtain

- for $SO(2n)$: $[\lambda]$ for $l(\lambda) < n$ and $[\lambda]_+, [\lambda]_-$ for $l(\lambda) = n$
- for $Sp(2n)$: $\langle\lambda\rangle$ for $l(\lambda) \leq n$

The existence of the terms $[\lambda]_\pm$ for $l(\lambda) = n$ is linked with the existence of self-associate irreps of $O(2n)$ [1]. In addition we can recall that the group $SO(2n)$ has irreducible spinor representations that we write $[\Delta;\lambda]_+$, $[\Delta;\lambda]_-$.

Now we consider the Lie algebras C_n and D_n associated with $Sp(2n)$ and $SO(2n)$. For the algebra D_n, we can identify the two finite-dimensionnal spin irreps as the representations with lowest weights

$$\Delta_+ = \omega_{n-1}, \\ \Delta_- = \omega_n.$$

For C_n we have no spin representations but the role of the basic spin representation is in some sense played by the Weil or metaplectic representation. In fact C_n can be realised in terms of differential operators on $\mathbb{C}[x_1, x_2, \dots, x_n]$:

$$e_i = x_i\partial_{i+1},\ f_i = x_{i+1}\partial_i,\ h_i = x_i\partial_i - x_{i+1}\partial_{i+1}$$

for $i = 1, 2, \dots, n-1$ which generate A_{n-1} and with

$$e_n = \frac{1}{2}x_n^2,\ f_n = -\frac{1}{2}\partial_n^2,\ h_n = \frac{1}{2} + x_n\partial_n\ .$$

If we write $\tilde{V}_\pm$ the module generated by the action of C_n on $v_+ = 1$ and $v_- = x_n$, this have lowest weight

$$\tilde{\Delta}_+ = \tfrac{1}{2}\omega_n\ , \\ \tilde{\Delta}_- = -\omega_{n-1} + \tfrac{3}{2}\omega_n\ ,$$

and they are also irreducible.

In this paper we will write this representations as $\tilde{\Delta}_+ = \langle \frac{1}{2}(0) \rangle$ and $\tilde{\Delta}_- = \langle \frac{1}{2}(1) \rangle$. The tensor powers of $\tilde{\Delta}$ generate a set of projective unitary infinite-dimensional irreducible representations of $Sp(2n, \mathbb{R})$ that we will denote by $\langle \frac{k}{2}(\lambda) \rangle$.

One can construct analogous metaplectic representations for the group $SO^*(2n)$ and we will write it $[k(\lambda)]$ [11, 9].

2.2 Finite dimensional irreducible representations of $Sp(2n, \mathbb{R})$ and $SO^*(2n)$

The finite dimensional non-unitary irreducible representations of the non-compact Lie groups $Sp(2n, \mathbb{R})$ and $SO^*(2n)$ are in one-to-one correspondence with those of the tensor unitary irreducible representations of the compact Lie groups $Sp(2n)$ and $O(2n)$ and may be similarly labelled by partitions of integers into at most n non-zero parts. Likewise, the branching rules for the finite non-unitary irreducible representations under the reductions $Sp(2n, \mathbb{R}) \to U(n)$ and $SO^*(2n) \to U(n)$ are identical to those of the corresponding unitary irreducible representations of compact Lie groups and hence we have [6] for $Sp(2n, \mathbb{R}) \to U(n)$

$$\langle \lambda \rangle \to \sum_{\zeta} \{\bar{\zeta}; \lambda / D\zeta\} \tag{1}$$

and for $SO^*(2n) \to U(n)$

$$[\lambda] \to \sum_{\zeta} \{\bar{\zeta}; \lambda / B\zeta\} \tag{2}$$

where the B and D are two infinite series of S-functions [1, 8]. Thus under $Sp(8, R) \to U(4)$ we have

$$\begin{aligned} \langle 31 \rangle \to \ & \{\overline{31}; 0\} + \{\bar{3}; 1\} + \{\overline{21}; 1\} + \{\bar{2}; 2\} + \{\bar{2}; 1^2\} + \{\bar{2}; 0\} + \{\overline{1^2}; 2\} \\ & + \{\overline{1^2}; 0\} + \{\bar{1}; 3\} + \{\bar{1}; 21\} + 2\{\bar{1}; 1\} + \{31\} + \{2\} + \{1^2\} \end{aligned} \tag{3}$$

while for $SO^*(8) \to U(4)$ we have

$$\begin{aligned} [31] \to \ & \{\overline{31}; 0\} + \{\bar{3}; 1\} + \{\overline{21}; 1\} + \{\bar{2}; 2\} + \{\bar{2}; 1^2\} + \{\bar{2}; 0\} + \{\overline{1^2}; 2\} \\ & + \{\bar{1}; 3\} + \{\bar{1}; 21\} + \{\bar{1}; 1\} + \{31\} + \{2\} \, . \end{aligned} \tag{4}$$

For any irreducible representation $\{\overline{\mu}; \lambda\}$ of $U(n)$, with a mixed tensor basis, there always exists an associated irreducible representation $\{\rho\}$, with a covariant basis, such that [6]:

$$\{\overline{\mu}; \lambda\} = \varepsilon^{\omega} \{\rho\} \tag{5}$$

for some integer ω. Specifically one has for $\{\overline{\mu}; \lambda\}$ with (μ) a partition with p non-zero parts and (λ) with q non-zero parts in $U(n)$

$$\begin{aligned} \rho_1 &= \lambda_1 + \mu_1 \\ \rho_2 &= \lambda_2 + \mu_1 \\ \cdots &= \cdots \\ \rho_q &= \lambda_q + \mu_1 \\ \rho_{q+1} &= \mu_1 - \mu_{n-q} \\ \rho_{q+2} &= \mu_1 - \mu_{n-q-1} \\ \cdots &= \cdots \\ \rho_n &= 0 \end{aligned} \tag{6}$$

with

$$\varepsilon^{\omega} = \varepsilon^{-\mu_1} \ . \tag{7}$$

Thus for the group $U(4)$ we obtain the result of (4) in a covariant basis as

$$\begin{aligned}[31] \to \ & \varepsilon^0(\{2\} + \{31\}) + \varepsilon^{-1}(\{21^2\} + \{321\} + \{41^2\} + \{31\}) \\ & \varepsilon^{-2}(\{2^3\} + \{3^22\} + \{42^2\} + \{321\}) + \varepsilon^{-3}(43^2\} + \{3^22\}) \ .\end{aligned} \tag{8}$$

In the case of partitions into n non-zero parts we have the $U(n)$ equivalence

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \equiv \varepsilon^{-\lambda_n}\{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, 0\} \ . \tag{9}$$

For subsequent usage we note that the irreducible representations $\langle\lambda\rangle$ and $[\lambda]$ may be mapped into one another and that [6, 10]

$$\begin{aligned}\langle\lambda\rangle &\to \ [\lambda/W] \\ [\lambda] &\to \ \langle\lambda/V\rangle\end{aligned} \tag{10}$$

where W is the infinite $S-$function series [1]

$$W = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\{r, s\} \quad \text{with } r - s \text{ even}, \ . \tag{11}$$

and $V = W^{-1}$.

If $(\lambda)_{sc} = (a, a-1, \ldots, 1)$ then $(\lambda)_{sc}$ is said to be a *staircase partition* with the important property [10] that

$$(\lambda/W)_{sc} = (\lambda/V)_{sc} = (\lambda)_{sc} \ . \tag{12}$$

Thus the decompositions obtained for the irreducible representations $\langle 21\rangle$ and $[21]$ have the same $U(n)$ content. In the case of $n = 4$ we obtain the $U(4)$ content

$$\begin{aligned}&\{\overline{21}; 0\} + \{\bar{2}; 1\} + \{\overline{1^2}; 1\} + \{\bar{1}; 2\} + \{\bar{1}; 1^2\} + \{\bar{1}; 0\} + \{21\} + \{1\} \\ &= \varepsilon^0(\{1\} + \{21\}) + \varepsilon^{-1}(\{1^3\} + \{2^21\} + \{31^2\} + \{21\}) + \varepsilon^{-2}(32^2\} + \{2^21\}) \ .\end{aligned} \tag{13}$$

Products of $U(n)$ irreducible representations may be evaluated by noting that [1]

$$\{\bar{\mu}; \lambda\} \times \{\bar{\rho}; \nu\} = \sum_{\sigma,\tau}\{\overline{(\mu/\sigma)\cdot(\rho/\tau)}; (\lambda/\tau)\cdot(\nu/\sigma)\} \ . \tag{14}$$

Two useful specialisations are

$$\{\bar{\mu}; \lambda\} \times \{\nu\} = \sum_{\sigma}\{\overline{\mu/\sigma}; \lambda\cdot(\nu/\sigma)\} \tag{15}$$

and

$$\{\bar{\mu}\} \times \{\nu\} = \sum_{\zeta}\{\overline{\mu/\zeta}; \nu/\zeta\} \ . \tag{16}$$

2.3 The metaplectic representation of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$

The standard discrete series unitary irreducible representations of $Sp(2n,\mathbb{R})$ may be labelled as $\langle\frac{1}{2}k(\lambda)\rangle$ where

$$\lambda_1' + \lambda_2' \leq k \quad and \quad \lambda_1' \leq n \tag{17}$$

while those of $SO^*(2n)$ may be labelled as [11, 9] $[k(\lambda)]$ where

$$\lambda_1' \leq min(k,n) \ . \tag{18}$$

Under restriction to the $U(n)$ subgroup we have [17, 11] for $Sp(2n,\mathbb{R}) \to U(n)$

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^{[k]} \cdot D_N\}_N \tag{19}$$

with $N = min(n,k)$. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the final subscript N. $\{\lambda_s\}_N^k$ is a signed sequence [17, 11] of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules [1, 5] of $O(k)$.

Equation (19) greatly simplifies if

$$\lambda_1' \leq n \leq k - \lambda_2' \tag{20}$$

to just

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda\} \cdot D_N\}_N \ . \tag{21}$$

Irreducible representations satisfying (20) will be said to be *highly standard* [11] and the products are all evaluated in $U(n)$. In such cases the signed sequence consists of the single leading term.

Likewise, under restriction to the $U(n)$ subgroup we have [9] for $SO^*(2n) \to U(n)$

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda_s\}_N^{\langle 2k\rangle} \cdot B_N\}_N \tag{22}$$

where $N = min(2k,n)$ and the signed sequences are evaluated in $Sp(2k)$. An irreducible representation of $SO^*(2n)$ will be highly standard if

$$2k + 2 - \lambda_1' > N = min(n,2k) \tag{23}$$

and (23) simplifies to

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda\} \cdot B_N\}_N \ . \tag{24}$$

Highly standard irreducible representations have the important, and useful, distinction that (21) and (24) can be inverted by forming the product, in $U(n)$, of the inverse $S-$function series ($C = D^{-1}$) and ($A = B^{-1}$) respectively. For irreducible representations that are standard but not highly standard the occurrence of signed sequences prevents such a simple inversion.

Two irreducible representations of $Sp(2n,\mathbb{R})$ (or $SO^*(2n)$) will be said to be *equivalent* if under $Sp(2n,\mathbb{R}) \to U(n)$ (or under $SO^*(2n) \to U(n)$) they yield the same $U(n)$ characters [11]. These equivalences arise in the particular case of highly standard irreducible representations. Thus in $Sp(6,R)$ we have the equivalences

$$\langle 6(1)\rangle \equiv \langle 5(211)\rangle \equiv \langle 4(322)\rangle \equiv \langle 3(433)\rangle \ . \tag{25}$$

All these irreducible representations satisfy the constraints of (17) whereas the irreducible representation $\langle 2(544)\rangle$ is necessarily null.

3 Generic products

Here we consider the tensor product of a finite dimensional non-unitary irreducible representation with an infinite dimensional unitary irreducible representation. In general the problem of determining the content of an arbitrary tensor product of a finite dimensional non-unitary irreducible representation with an infinite dimensional unitary irreducible representation is complicated by the diversity of representations of non-compact groups. In practical applications there is a class of products that are of great importance, namely the class of generic products. A product is said to be *generic* if it is fully resolvable into just a finite series of standard unitary irreducible representations of the relevant non-compact Lie group. A product will be said to be *highly generic* if it is fully resolvable into highly standard unitary irreducible representations . In most of this paper we shall be considering the highly generic case.

We now take up the problem of determining the highly generic conditions, first for $SO^*(2n)$. Consider the $SO^*(2n)$ product $[\mu] \times [k(\lambda)]$ where $[k(\lambda)]$ is a highly standard irreducible representation and $[\mu]$ is a finite non-unitary irreducible representation . Under $SO^*(2n) \to U(n)$the irreducible representation $[\mu]$ certainly yields a lowest weight $U(n)$ irreducible representation $\{\bar{\mu}\}$ such than in a covariant basis

$$\{\bar{\mu}\} = \varepsilon^{-\mu_1} \cdot \{\mu_1^{n-\ell_\mu}, \nu\} = \varepsilon^{-\mu_1} \cdot \{\mu_c\} \tag{26}$$

where ℓ_ν is the number of $\mu_i \neq \mu_1$. If the product is highly generic then certainly

$$[\mu] \times [k(\lambda)] \supset \varepsilon^{k-\mu_1} \cdot (\mu_c \cdot \lambda \cdot B)_n = [k - \mu_1(\mu_c \cdot \lambda)] \ . \tag{27}$$

The product will be assuredly highly generic if every term on the right-hand-side of (27) is highly standard.

Let us write

$$[k - \mu_1(\mu_c \cdot \lambda)] = \sum_\rho [k - \mu_1(\rho)] \ . \tag{28}$$

We seek the smallest value of k that ensures that every $[k - \mu_1(\rho)]$ is highly standard. If $\rho_1' = n$ then $[k - \mu_1(\rho)] \equiv [k - \mu_1 + \rho_n(\rho - \rho_n)]$ which is of higher weight than terms with $\rho_1' < n$. The worst case scenario is when $\rho_1' = n - 1$. In that case (23) leads to

$$k \geq n + \mu_1 - 1 \ . \tag{29}$$

In the general case the minimum value of k depends upon the longest term in $(\mu_c \cdot \lambda)$. Setting

$$\ell = \min(n - 1, n - \ell_\mu + \ell_\nu + \ell_\lambda) \tag{30}$$

gives, for $2k \geq n$ the condition for highly standardness as

$$k \geq n + \mu_1 + \frac{\ell - n - 1}{2} \ . \tag{31}$$

If $n > 2k$ then necessarily $\rho_1' \leq 1$ and $[k(\rho)] = [k(n)]$ is highly standard for all k and n.

Now consider the case of $Sp(2n, R)$. The condition for highly standardness becomes, from (21),

$$k + 1 - \rho_2' > \min(n, k) \tag{32}$$

Two cases arise.

Firstly, if $k \geq n$ then

$$k \geq n - \ell \tag{33}$$

where ℓ is as defined in (30).

Secondly, if $n < k$ then necessarily $\rho'_2 = 0$ and $\langle k(\rho)\rangle = \langle k(1^n)\rangle$ is highly standard for all k and n.

We continue with an example. Consider the $SO^*(6)$ fip $[1] \times [3(21)]$. Under $SO^*(6) \to U(3)$

$$[1] \to \{\bar{1};0\} + \{0;1\} = \varepsilon^{-1}\{1^2\} + \varepsilon^0\{1\} \tag{34}$$

and since $[3(21)]$ is highly standard we have from (24)

$$[3(21)] \to (\varepsilon^3 \cdot \{21\} \cdot B_3)_3 \ . \tag{35}$$

Thus at the $U(3)$ level we have the product of (4.9) with (4.10) leading to

$$\begin{aligned} [1] \times [3(21)] \to \ & ((\{\bar{1};0\} + \{\bar{0};1\}) \cdot (\varepsilon^3 \cdot \{21\} \cdot B))_3 \\ &= ((\varepsilon^{-1}\{1^2\} + \varepsilon^0\{1\}) \cdot (\varepsilon^2 \cdot \{21\}) \cdot B) \\ &= ((\varepsilon^2(\{32\} + \{31^2\} + \{2^21\}) + \varepsilon^3(\{2^2\} + \{31\} + \{21^2\}) \cdot B)_3 \\ &= ((\varepsilon^4\{1\} + \varepsilon^3(\{2^2\} + \{31\} + \{2\} + \{1^2\}) + \varepsilon^2\{32\}) \cdot B)_3 \ . \end{aligned} \tag{36}$$

The result of (36) may be inverted using the A−series which is the inverse of the B−series to give the final result

$$[1] \times [3(21)] \to [4(1)] + [3(2^2)] + [3(31)] + [3(2)] + [3(1^2)] + [2(32)] \ .$$

Note that the irreducible representations in the result are all highly standard and hence the product is highly generic. That the result is highly generic could have been predicted from (29).

Algorithm I. To evaluate the generic product $[\mu] \times [k(\lambda)]$.

1. *Perform the decomposition of the irreducible representation $[\mu]$ under $SO^*(2n) \to U(n)$ using (2) and convert the $U(n)$ irreducible representations into their covariant form using (6) and (7).*

2. *Perform the decomposition of the irreducible representation $[k(\lambda)]$ under $SO^*(2n) \to U(n)$ using (24).*

3. *Form the tensor product of the above two sets of $U(n)$ irreducible representations to give a list of irreducible representations each of the form $\varepsilon^p\{\rho\}$.*

4. *Use (9) to reduce every partition of n non-zero parts to give an equivalent $U(n)$ irreducible representation involving a partition into fewer than n non-zero parts.*

5. *Replace every $\varepsilon^p\{\rho\}$ by $[p(\rho)]$ to give the final $SO^*(2n)$ content of the fip.*

The corresponding algorithm for evaluating $Sp(2n, \mathbb{R})$ fips follows in the same manner.

Algorithm II. To evaluate the generic product $\langle\mu\rangle \times \langle\frac{1}{2}k(\lambda)\rangle$.

1. *Perform the decomposition of the irreducible representation $\langle\mu\rangle$ under $Sp(2n, \mathbb{R}) \to U(n)$ using (1) and convert the $U(n)$ irreducible representations into their covariant form using (6) and (7).*

2. *Perform the decomposition of the irreducible representation $\langle\frac{1}{2}k(\lambda)\rangle$ under $Sp(2n, \mathbb{R}) \to U(n)$ using (22).*

3. *Form the tensor product of the above two sets of $U(n)$ irreducible representations to give a list of irreducible representations each of the form $\varepsilon^p\{\rho\}$.*

4. *Use (9) to reduce every partition of n non-zero parts to give an equivalent $U(n)$ irreducible representation involving a partition into fewer than n non-zero parts.*

5. *Replace every $\varepsilon^p\{\rho\}$ by $\langle p(\rho)\rangle$ to give the final $Sp(2n, \mathbb{R})$ content of the fip.*

The above two algorithms are tedious in application but can form the basis for more direct methods of evaluating fips. In the case of highly generic products considerable simplification is possible. It follows from (2), (15) and (24) that a highly generic fip for $SO^*(2n)$ may be resolved by evaluating the expression

$$[\mu] \times [k(\lambda)] \to \sum_{\zeta,\sigma} [k(\{\overline{\zeta/\sigma}; (\mu/B\zeta) \cdot (\lambda/\sigma)\})] \tag{37}$$

with the $\cdot$ indicating a product in $U(n)$.

Likewise for $Sp(2n, \mathbb{R})$ we have

$$\langle\mu\rangle \times \langle\tfrac{1}{2}k(\lambda)\rangle \to \textstyle\sum_{\zeta,\sigma} \langle\frac{1}{2}k(\{\overline{\zeta/\sigma}; (\mu/D\zeta) \cdot (\lambda/\sigma)\})\rangle \; . \tag{38}$$

At this stage we remark that a product can be generic but not highly generic. Consider first the group $SO^*(8)$ for which $[3(1)]$ is highly standard. If we computed the product using (33) we would obtain

$$[1] \times [3(1)] = [3(1^2)] + [3(2)] + [3(0)] + [2(21^2)]$$

but $[2(21^2)] \to 0$. In this case algorithm I fails as the lowest weight $U(4)$ irreducible representation is $\varepsilon^2\{21^2\}$ and cannot be inverted to give a standard $SO^*(8)$ irreducible representation . In this case the product is clearly not a generic product.

Consider now $Sp(8, \mathbb{R})$ and the generic products $\langle 1\rangle \times \langle 3(1)\rangle$ and $\langle 1\rangle \times \langle 3(2)\rangle$. Using algorithm II we obtain

$$\begin{aligned} \langle 1\rangle \times \langle 3(1)\rangle &= 2\langle 3(1^2)\rangle + \langle 3(2)\rangle + \langle 3(0)\rangle + \langle 2(21^2)\rangle \\ \langle 1\rangle \times \langle 3(2)\rangle &= 2\langle 3(21)\rangle + \langle 3(3)\rangle + \langle 3(1)\rangle + \langle 2(31^2)\rangle \end{aligned} \tag{39}$$

where $\langle 2(21^2)\rangle$ and $\langle 2(31^2)\rangle$ are standard but not highly standard. In this case while in each case algorithm II holds (30) does not. These products are generic but not highly generic.

4 Stabilisation of Products

The first algorithm makes it possible to calculate fips for the group $SO^*(2n)$. The results exhibit certain stabilisation properties. Two particular types of stabilisation can occur.

Firstly, we note that if the product is highly generic we can obtain an infinite set of products from that product by increasing k by an integer or half-integer x. This follows by noting that in such a case the value of ε^k is incremented while the Schur function products are unchanged. Thus we can write

$$C^{[p(\rho)]}_{[\mu],[k(\lambda)]} = C^{[p+x(\rho)]}_{[\mu],[k+x(\lambda)]} \,. \tag{40}$$

Secondly, we can consider a form of stabilisation, as where k remains constant, as exemplified in (35), and the product is at least generic. Let us introduce the notation $(\lambda + h) = (\lambda_1 + h, \lambda_2, \dots)$ where h is an integer. Consider the Schur function product

$$\{\lambda\} \cdot \{\mu\} = \sum C^{\rho}_{\lambda\mu}\{\rho\}$$

where the $C^{\rho}_{\lambda\mu}$ are the usual Littlewood-Richardson coefficients. Increasing the first part of λ by h corresponds to increasing the difference $\lambda_1 - \lambda_2$ to h. But rewriting the new product in terms of Yamanouchi words is equivalent to putting h more 1's in the first row and this is only possible if $\mu_1 \geq h$. Otherwise we must add some 2's, 3's etc.. to the other rows. This has the consequence that

$$C^{[p(\rho)]}_{[\mu],[k(\lambda)]} = C^{[p(\rho+h)]}_{[\mu],[k(\lambda+h)]} \tag{41}$$

with the condition that $\lambda_1 - \lambda_2 \geq \rho_{1max}$, where ρ_{1max} corresponds to the maximal first part in the list of the covariant representations obtained using (1). For this reason the condition (41) is available with the condition that $\lambda_1 - \lambda_2 \geq \mu_1 + \mu_2$.

We can illustrate these results by four examples obtained for $Sp(6, R)$.

$\langle 21\rangle \times \langle 5(31)\rangle =$

$$\begin{array}{lllll}
\langle 7(1)\rangle & + \langle 6(4)\rangle & + 2\langle 6(31)\rangle & + \langle 6(2^2)\rangle & + 3\langle 6(2)\rangle \\
+ 2\langle 6(1^2)\rangle & + \langle 6(0)\rangle & + \langle 5(52)\rangle & + \langle 5(5)\rangle & + \langle 5(43)\rangle \\
+ 4\langle 5(41)\rangle & + 4\langle 5(32)\rangle & + 3\langle 5(3)\rangle & + 4\langle 5(21)\rangle & + \langle 5(1)\rangle \\
+ \langle 4(62)\rangle & + 2\langle 4(53)\rangle & + 2\langle 4(51)\rangle & + \langle 4(4^2)\rangle & + 4\langle 4(42)\rangle \\
+ \langle 4(4)\rangle & + 2\langle 4(3^2)\rangle & + 2\langle 4(31)\rangle & + \langle 4(2^2)\rangle & + \langle 3(63)\rangle \\
+ \langle 3(54)\rangle & + \langle 3(52)\rangle & + \langle 3(43)\rangle , & &
\end{array} \tag{42}$$

$\langle 21\rangle \times \langle 6(31)\rangle =$

$$\begin{array}{lllll}
\langle 8(1)\rangle & + \langle 7(4)\rangle & + 2\langle 7(31)\rangle & + \langle 7(2^2)\rangle & + 3\langle 7(2)\rangle \\
+ 2\langle 7(1^2)\rangle & + \langle 7(0)\rangle & + \langle 6(52)\rangle & + \langle 6(5)\rangle & + \langle 6(43)\rangle \\
+ 4\langle 6(41)\rangle & + 4\langle 6(32)\rangle & + 3\langle 6(3)\rangle & + 4\langle 6(21)\rangle & + \langle 6(1)\rangle \\
+ \langle 5(62)\rangle & + 2\langle 5(53)\rangle & + 2\langle 5(51)\rangle & + \langle 5(4^2)\rangle & + 4\langle 5(42)\rangle \\
+ \langle 5(4)\rangle & + 2\langle 5(3^2)\rangle & + 2\langle 5(31)\rangle & + \langle 5(2^2)\rangle & + \langle 4(63)\rangle \\
+ \langle 4(54)\rangle & + \langle 4(52)\rangle & + \langle 4(43)\rangle , & &
\end{array} \tag{43}$$

$\langle 21\rangle \times \langle 5(41)\rangle =$

$$\begin{array}{lllll} \langle 7(2)\rangle & + \langle 6(5)\rangle & + 2\langle 6(41)\rangle & + \langle 6(32)\rangle & + 3\langle 6(3)\rangle \\ + 2\langle 6(21)\rangle & + \langle 6(1)\rangle & + \langle 5(62)\rangle & + \langle 5(6)\rangle & + \langle 5(53)\rangle \\ + 4\langle 5(51)\rangle & + 4\langle 5(42)\rangle & + 3\langle 5(4)\rangle & + \langle 5(3^2)\rangle & + 4\langle 5(31)\rangle \\ + \langle 5(2^2)\rangle & + \langle 5(2)\rangle & + \langle 4(72)\rangle & + 2\langle 4(63)\rangle & + 2\langle 4(61)\rangle \\ + \langle 4(54)\rangle & + 4\langle 4(52)\rangle & + \langle 4(5)\rangle & + 2\langle 4(43)\rangle & + 2\langle 4(41)\rangle \\ + \langle 4(32)\rangle & + \langle 3(73)\rangle & + \langle 3(64)\rangle & + \langle 3(62)\rangle & + \langle 3(53)\rangle, \end{array} \tag{44}$$

$\langle 21\rangle \times \langle 5(51)\rangle =$

$$\begin{array}{lllll} \langle 7(3)\rangle & + \langle 6(6)\rangle & + 2\langle 6(51)\rangle & + \langle 6(42)\rangle & + 3\langle 6(4)\rangle \\ + 2\langle 6(31)\rangle & + \langle 6(2)\rangle & + \langle 5(72)\rangle & + \langle 5(7)\rangle & + \langle 5(63)\rangle \\ + 4\langle 5(61)\rangle & + 4\langle 5(52)\rangle & + 3\langle 5(5)\rangle & + \langle 5(43)\rangle & + 4\langle 5(41)\rangle \\ + \langle 5(32)\rangle & + \langle 5(3)\rangle & + \langle 4(82)\rangle & + 2\langle 4(73)\rangle & + 2\langle 4(71)\rangle \\ + \langle 4(64)\rangle & + 4\langle 4(62)\rangle & + \langle 4(6)\rangle & + 2\langle 4(53)\rangle & + 2\langle 4(51)\rangle \\ + \langle 4(42)\rangle & + \langle 3(83)\rangle & + \langle 3(74)\rangle & + \langle 3(72)\rangle & + \langle 3(63)\rangle\ . \end{array} \tag{45}$$

The examples illustrate the different stability properties. Examples (42) and (43) are an illustration of the property (40). Examples (42) and (44) show a situation where the rule (41) does not hold while (44) and (45) respect that particular condition.

5 Relationships between $Sp(2n, \mathbb{R})$ and $SO^*(2n)$ products

Recalling (10) and that $WB = D$ and $VD = B$ we can establish the fip equivalences for the multiplicities

$$\begin{aligned} C^{[p(\rho)]}_{[\mu/W],[k(\lambda)]} &= C^{\langle p(\rho)\rangle}_{\langle\mu\rangle,\langle k(\lambda)\rangle}\ , \\ C^{\langle p(\rho)\rangle}_{\langle\mu/V\rangle,\langle k(\lambda)\rangle} &= C^{[p(\rho)]}_{[\mu],[k(\lambda)]}\ . \end{aligned} \tag{46}$$

In the special case of (μ) being a staircase partition we recall (12).

6 Concluding remarks

The aim of this paper has been to obtain a series of results for the practical evaluation of certain generic products of finite and infinite representations of $Sp(2n, \mathbb{R})$ and $SO^*(2n)$. In the case of highly generic products there is a considerable simplification since by avoiding irreducible representations associated with signed sequences it becomes unnecessary to consider branching rules for the infinite dimensional irreducible representations, effectively reducing the entire problem to applications of the Littlewood-Richardson rule with certain constraints.

The results exhibit certain stabilisation properties which make it possible to deduce the form of an infinite set of generic products from a particular minimal product. The conditions for stabilisation have been specified.

References

[1] G. R. E. Black, R. C. King and B. G. Wybourne *Kronecker products for compact semisimple Lie groups*, J. Phys. A : Math. Gen. **16** (1983), 1555-1589

[2] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63**, No.2 (1991), 465-516.

[3] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras* J. Algebra **165**, No.2 (1994), 295-345.

[4] M. Kashiwara et M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1-48

[5] R. C. King, *Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups*, J. Math. Phys. **12** (1971), 1588-1598.

[6] R. C. King, *Branching rules for classical Lie groups using tensor and spinor methods*, J. Phys. A : Math. Gen. **8** (1975), 429-449

[7] R. C. King, *Some analogies between the representation theory of* $SO(2n)$ *and* $Sp(2n,\mathbb{R})$, Poc. Fifth Intnl School on Theoritical Physics, (1998) World Scientific (In Press).

[8] R. C. King, Luan Dehuai and B. G. Wybourne, *Symmetrised powers of rotation group representations*, J. Phys. A: Math. Gen. **14** (1981), 2509-2538

[9] R. C. King, F. Toumazet and B. G. Wybourne , *Products and symmetrized powers of irreducible representations of* $SO^*(2n)$, J. Phys. A: Math. Gen. **31** (1998), 6691-6705.

[10] R. C. King and B. G. Wybourne *Some noteworthy spin plethysms.* J. Phys. A : Math. Gen. **15** (1982), 1137-1141.

[11] R. C. King and B. G. Wybourne *Holomorphic discrete series and harmonic series unitary irreducible representations of non-compact Lie groups :* $Sp(2n,\mathbb{R})$, $U(p,q)$ *and* $SO^*(2n)$, J. Phys. A : Math. Gen. **18** (1985), 3113-3139

[12] R. C. King and B. G. Wybourne *Products and symmetrized powers of irreducible representations of* $Sp(2n,\mathbb{R})$ *and their associates*, J. Phys. A : Math. Gen. **31** (1998), 6669-6689.

[13] K. Koike, I. Terada *Young-Diagrammatic Methods for the Representation Theory of the Classical Groups of type* B_n, C_n, D_n, Journal of Algebra **107** No 2 (1987), 466-511.

[14] P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras.*, Invent. Math. bf 116 No.1-3 (1994), 329-346.

[15] I. G. Macdonald *Symmetric Functions and Hall Polynomials, second edition*, Oxford science publications, Oxford, England, 1995.

[16] T. Roby, *A Schensted insertion for an infinite dimensional representation*, Proc. FPSAC 97, 496-507.

[17] D. J. Rowe, B. G. Wybourne et P. H. Butler, *Unitary representations, branching rules and matrix elements for the non-compact symplectic groups*, J. Phys. A: Math. Gen. **18** (1985), 939-953.

[18] J.-Y. Thibon, F. Toumazet et B. G. Wybourne, *Products and plethysms for the fundamental harmonic series representations of* $U(p,q)$, J. Phys. A: Math. Gen. **30** (1997), 4851-4856.

[19] J.-Y. Thibon, F. Toumazet et B. G. Wybourne, *Symmetrized squares and cubes of the fundamental unirreps of* $Sp(2n,\mathbb{R})$, J. Phys. A: Math. Gen. **31** (1998), 1073-1086 .