A Strahler bijection between Dyck paths and planar trees

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Abstract

The Strahler number of binary trees has been introduced by hydrogeologists and rediscovered in computer science in relation with some optimization problems. Explicit expressions have been given for the Strahler distribution, i.e. binary trees enumerated by number of vertices and Strahler number. Two other Strahler distributions have been discovered with the logarithmic height of Dyck paths and the pruning number of forests of planar trees in relation with molecular biology. Each of these three classes are enumerated by the Catalan numbers, but only two bijections preserving the Strahler parameters have been explicited: by Françon between binary trees and Dyck paths, by Zeilberger between binary trees and forests of planar trees. We present here the missing bijection between forests of planar trees and Dyck paths sending the pruning number onto the logarithmic height. A new functional equation for the Strahler generating function is deduced. Some orthogonal polynomials appear, they are one parameter Tchebycheff polynomials.

Résumé

Le nombre de Strahler d'un arbre binaire a été introduit en hydrogéologie et redécouvert en informatique en relation avec certains problèmes d'optimisation. Des expressions explicites ont été données pour la distribution du paramètre nombre de Strahler, c'est-à-dire pour le nombre d'arbres binaires énumérés selon le nombre de sommets et leur nombre de Strahler. Depuis, deux autres distributions de Strahler ont été découvertes: la hauteur logarithmique des chemins de Dyck et l'ordre des forêts d'arbres planaires en relation avec la biologie moleculaire. Chacune de ces trois classes d'objets est énumérée par les nombres de Catalan, mais seulement deux bijections conservant les paramètres des distributions de Strahler ont été explicitées: l'une de Françon entre les arbres binaires et les mots de Dyck, l'autre de Zeilberger entre les arbres binaires et les forêts d'arbres planaires. Nous donnons une bijection entre les forêts d'arbres planaires et les mots de Dyck envoyant le paramètre ordre sur le paramètre hauteur logarithmique. Une nouvelle équation fonctionnelle est déduite pour la série génératrice associée la distribution de Strahler. Certains polynômes orthogonaux apparaissent, ce sont des extensions à un paramètre des polynômes de Tchebycheff.

1 Strahler number of a binary tree

We use the following classical notations for binary trees. A binary tree is a triple $B = (L, r, R)$ or is reduced to an external vertex denoted by "$\square$". Here L and R are binary trees (left and right subtree respectively) and r denotes the root of B. In Figure 1, internal vertices are denoted by "$\bigcirc$" . The number of binary trees with n internal vertices (and $(n+1)$ external vertices) is the Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}.$$

Definition 1. *The Strahler number $St(B)$ of the binary tree B is defined inductively by the relation*

$$St(\square) = 1,$$

$$St(L,r,R) = \begin{cases} \text{if } St(L) = St(R), \text{then } 1 + St(R), \\ \text{else } \max\{St(L), St(R)\}. \end{cases} \tag{1.0}$$

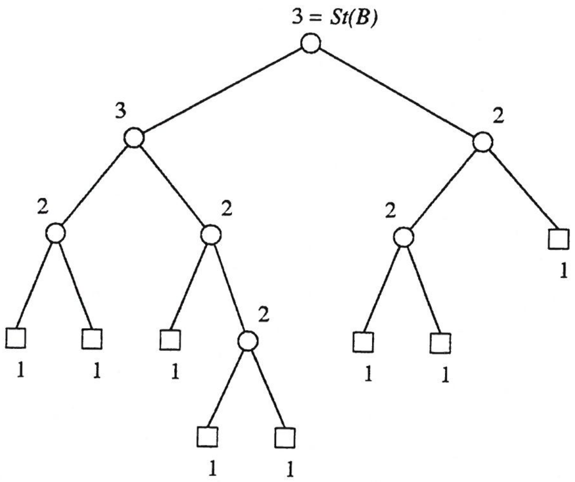


Figure 1 : The Strahler number of a binary tree

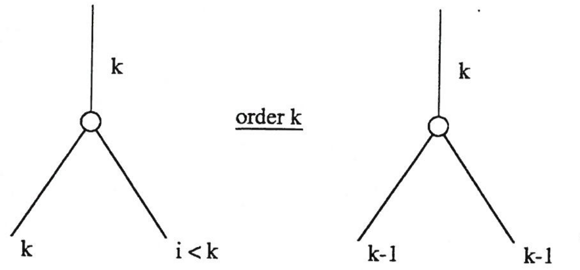


Figure 2 : The Strahler labelling of a binary tree

In other words, the Strahler number of a binary tree is obtained by the following process: label the external vertices by 1 and each internal vertex by the recursive rule displayed in Figure 2. The Strahler number is the label of the root.

The Strahler number of binary trees appeared in computer science in relation with the coding and computation of arithmetic expressions using only binary operations. Such an expression is encoded by a binary tree. Each internal vertex corresponds to a binary operation. Each external vertex corresponds to a variable (see Figure 3).

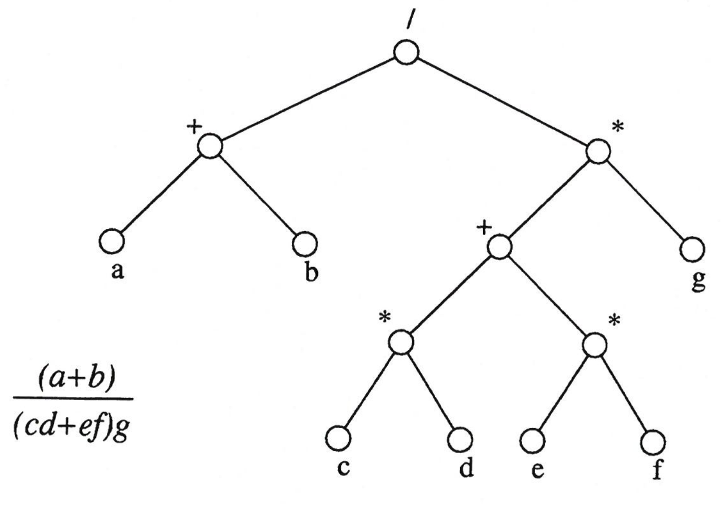


Figure 3 : The binary tree associated to an arithmetic expression

The minimum number of registers needed to compute an arithmetic expression is exactly the Strahler number of the underlying binary tree. Computer scientists have shown remarkable properties for the asymptotic behaviour of the mean of the parameter Strahler number over all binary trees with n internal vertices, see Flajolet, Raoult, Vuillemin [8], Kemp [11]. For that, they established the generating function for Strahler numbers.

Let $S_{n,k}$ (resp. $S_{n,\leq k}$) be the number of binary trees B with n internal vertices and Strahler number

$St(B) = k$ (resp. $St(B) \leq k$). We denote the corresponding generating functions by

$$\begin{aligned} S_k(t) &= \sum_{n \geq 0} S_{n,k} t^n, \\ S_{\leq k}(t) &= \sum_{n \geq 0} S_{n,\leq k} t^n. \end{aligned} \tag{1.1}$$

Let $U_n(t)$ be the n^{th} Tchebycheff polynomial of second kind, that is the polynomial defined by the relation

$$\sin(n+1)\theta = (\sin\theta)\, U_n(\cos\theta). \tag{1.2}$$

We denote by $F_n(t)$ the polynomial

$$F_n(t) = U_n(t/2). \tag{1.3}$$

These polynomials form a sequence of monic (i.e. the coefficient of t^n is 1) orthogonal polynomials. The moments are the Catalan numbers $\{C_n\}_{n \geq 0}$. These polynomials satisfy the following 3-terms linear recurrence relation

$$F_{n+1}(t) = tF_n(t) - F_{n-1}(t) \quad ; \quad F_0 = 1, \quad F_1 = t. \tag{1.4}$$

The reciprocal polynomials $F_n^*(t) = t^n F_n(1/t)$ are even polynomials and we define $R_n(t)$ by the following relation

$$R_n(t^2) = t^n F_n(1/t). \tag{1.5}$$

Then, Flajolet, Raoult and Vuillemin [8] and Kemp [11] gave the following explicit expression for the Strahler generating function:

$$S_{\leq k}(t) = \frac{R_{2^k-2}(t)}{R_{2^k-1}(t)} \tag{1.6a}$$

$$S_k(t) = \frac{t^{2^{k-1}-1}}{R_{2^k-1}(t)}. \tag{1.6b}$$

We will call *Strahler distribution* the distribution given by the relation (1.6b) for a parameter defined on a set of objects enumerated by the Catalan numbers. Here are the first values of the generating functions $S_k(t)$:

$$S_1(t) = 1 \quad ; \quad S_2(t) = \frac{t}{1-2t} \quad ; \quad S_3(t) = \frac{t^3}{1 - 6t + 10t^2 - 4t^3}. \tag{1.7}$$

In fact, the Strahler number of a binary tree was introduced much earlier in hydrogeology in the morphological study of rivers networks. Horton introduced in [10] a process classifying the rivers of a fluvial network by "order". This rule was simplified by Strahler [16] and corresponds to the labeling of a binary tree defined in Figure 2: a river starting at a source point has order 1, two rivers of order k joining together give rise to a river of $k+1$, while a river of order i joining a river of order $k > i$ gives rise to a river of order k (see Figure 4 for the Garonne rivers network in the South-Ouest of France). We have supposed that the river network has no island, delta and that no more than two rivers join at the same point. This labeling of river by order is the starting point for many works in hydrogeology for quantitative study of morphology of rivers network.

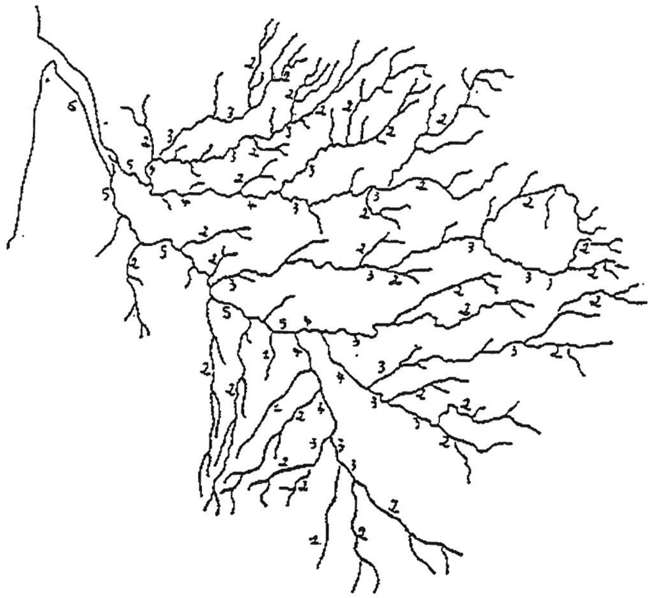


Figure 4: The Garonne river network

More recently, the Horton-Stahler analysis for ramified patterns has been refined with the introduction of the "ramification matrix" of a binary tree, and applied in computer graphics in order to give synthetic images of trees and landscapes, see Viennot, Eyrolles, Janey, Arques [23] and Figure 5. Other applications have also been given in the analysis of fractal ramified patterns in physics by Vannimenus and Viennot [18]. Strahler numbers also appeared in some constructions related to the derived series of the free Lie algebra, see Reutenauer [15]. A survey paper about Horton-Strahler analysis in various sciences is Viennot [22].

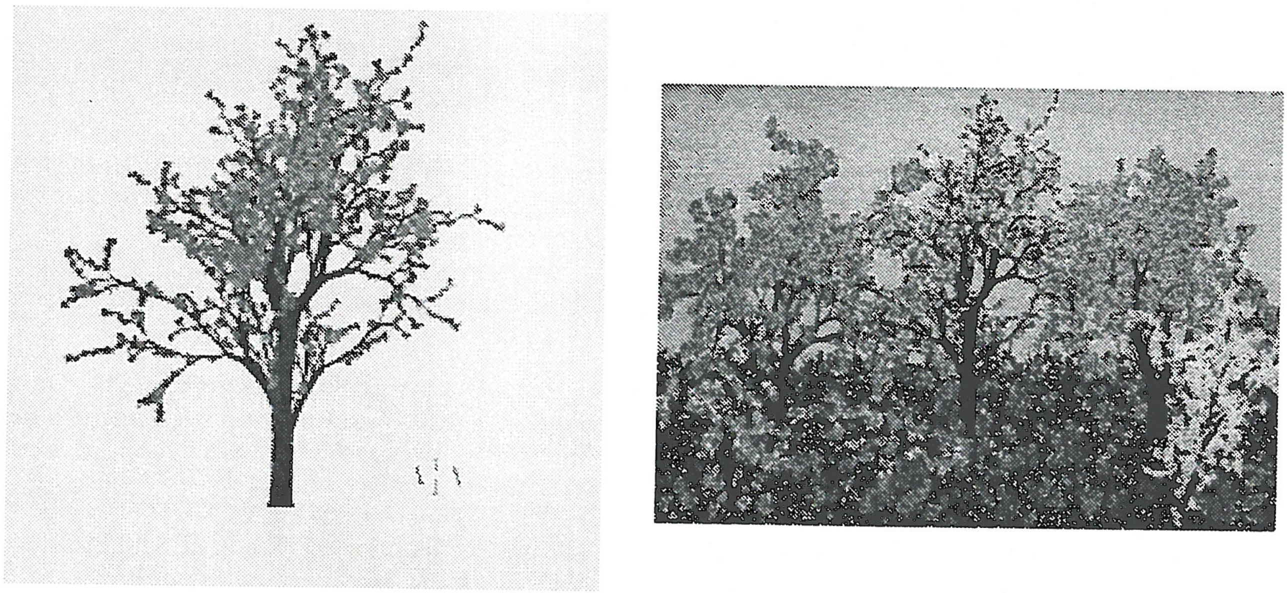


Figure 5: Synthetic images of trees based on Horton-Strahler analysis (from [23])

2 Logarithmic height of Dyck paths

Françon [9] gave a bijective proof of identity (1.6a) by relating the Strahler number with another parameter on the so called Dyck paths.

Recall that a *Dyck path* is a sequence of points $(s_0, \dots, s_{2n})$ of $\mathbb{N} \times \mathbb{N}$ such that $s_0 = (0,0)$, $s_{2n} = (2n, 0)$ and each elementary step is North-East (NE) or South-East (SE). An example is displayed in Figure 6. A very classical bijection between Dyck paths of length $2n$ and binary trees with n internal vertices, using the *prefix order* of binary tree, is well known.

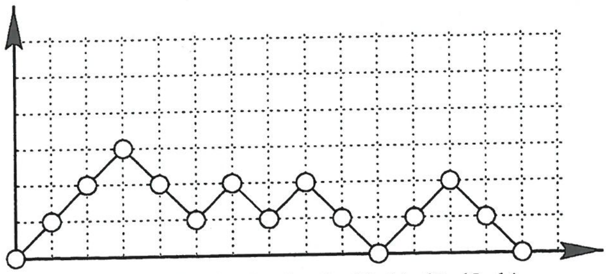


Figure 6 : A Dyck path

The *height* $H(\omega)$ of a Dyck path is the maximal height (or level) of its vertices, i.e.

$$H(\omega) = \max_{0 \le i \le 2n} \{y_i \mid s_i = (x_i, y_i)\}. \tag{2.1}$$

It is well known that the generating function for Dyck paths ω with bounded height $H(\omega) \le p$ is given by the following relation (see Kreweras [12], Berstel [2]).

$$\sum_{\omega : H(\omega) \le p} t^{||\omega||/2} = \frac{R_p(t)}{R_{p+1}(t)}, \tag{2.2}$$

where $R_p(t)$ is the "modified" Tchebycheff polynomial defined by (1.5), the summation is over Dyck paths and $||\omega||$ denotes the length of the path.

Identity (1.6a) says that the generating function $S_{\le k}(t)$ for binary trees having Strahler number $\le k$ is the same as the generating function for Dyck paths bounded by height $2^k - 2$. In other words, the parameter Strahler has the same distribution than the following parameter defined on Dyck paths:

$$LH(\omega) = \lfloor \log_2(1 + H(\omega)) \rfloor. \tag{2.3}$$

We will call the parameter $LH(\omega)$ the *logarithmic height* of the path ω.

Françon [9] gave a bijection between binary trees and Dyck paths sending the paramenter "Strahler number" onto the parameter "logarithmic height". This bijection is defined recursively and in fact proves that the double generating function $S(t, x)$ defined by the following relation

$$S(t, x) = \sum_{k \ge 1} S_k(t) x^{k-1} = \sum_{n,k} S_{n,k} x^{k-1} t^n, \tag{2.4}$$

satisfies the functional equation

$$S(t, x) = 1 + \frac{xt}{(1-2t)} S\left(\left(\frac{t}{1-2t}\right)^2, x\right). \tag{2.5}$$

Such an equation is a linear Read-Bajraktarević equation, as considered by Bergeron, Labelle, Leroux [1], pages 230 and 235.

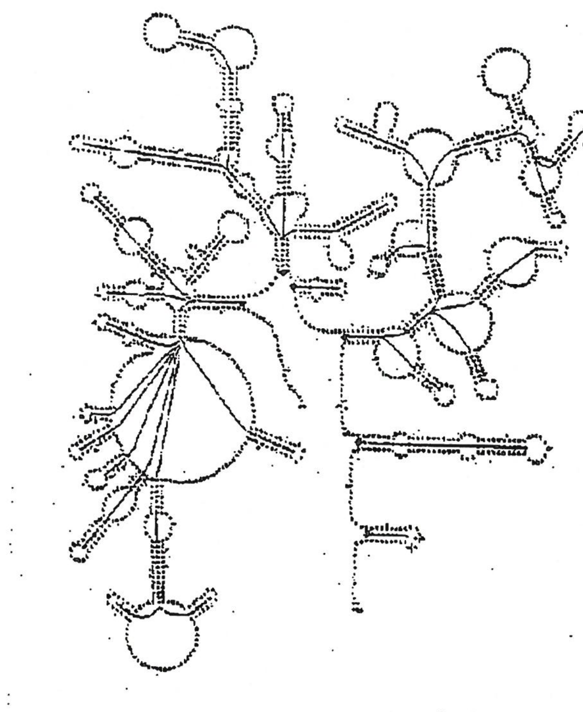


Figure 7 : Secondary structure of 16S-RNA of E. Coli (from [17]) and its associated forest of planar trees

3 Order or pruning number of planar trees

Surprisingly, Strahler distribution reappeared in the context of molecular biology. We consider molecules of *single-stranded nucleic acids*, as for example RNAs. The *primary structure* is the sequence of bases linked by phosphodiester bonds. There are four possible bases denoted by A (Adenine), U (Uracyl), G (Guanine) and C (Cytosine). Such bases can be linked together by hydrogen bonds (A with U, and G with C). The primary structure is thus folded into a planar graph called the *secondary structure*, as shown in Figure 7.

The notion of *order* or *complexity* of a secondary structure has been introduced, in relation with the computation of the energy of the secondary structure, see Mitiko Gô [14], de Gennes [5], Waterman [24]. Secondary structure of RNAs are branching structures. The important underlying structure is a forest of planar trees, as shown on Figure 7. The order (or complexity) of the molecule is the order of the forest, as defined below.

Recall that a *planar rooted tree* T (or *planar tree* for short) is : if T has only one vertex, then T is reduced to that point; else $T = (r; T_1, \ldots, T_p)$, where r is a vertex called the root and $T_1, \ldots, T_p$ is an ordered sequence of planar trees. A *forest* of planar (rooted) trees is an ordered sequence of planar trees, see Figure 8. The number of forests of planar trees with n vertices is the Catalan number (of course, this is also the number of planar trees with $n+1$ vertices). Very classical bijections are known between forests of planar trees, Dyck paths and binary trees making a "golden triangle" of bijections which commute. Let us now define the *order* of a forest.

A *filament* of a forest F is a maximal sequence $(s_1, \ldots, s_q)$ of vertices such that, for $i = 1, \ldots, q-1$, s_i has only one son and this son is s_{i+1}, and moreover s_q is a *leaf* of the forest (or *end point*, i.e it has no son). The filaments are two by two disjoint. We introduce the operator δ "deletion of filaments". The forest $\delta(F)$ is obtained from F by deleting all the vertices of all filaments of F. The *order* of the forest is the minimum integer k such that $\delta^k(F) = \emptyset$ (see Figure 8). Zeilberger [25] called this parameter the *pruning number* of the forest.

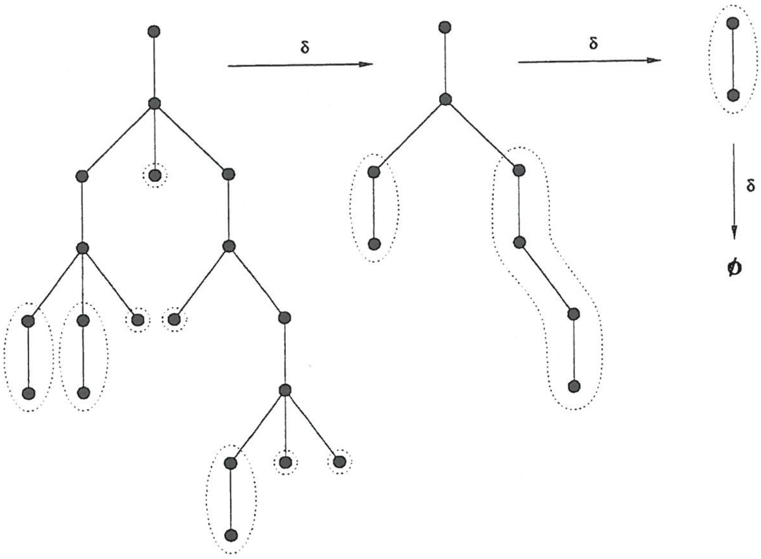


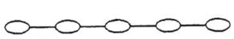
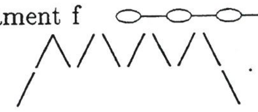
Figure 8 : Filament, operator δ and pruning number of a forest of plane trees

Waterman [24] raised the problem of finding the generating function for all possible secondary structures of order k. This was solved by Vauchaussade de Chaumont and Viennot [19], [20] by applying the so-called DSV-methodology due to M. P. Schützenberger (using algebraic langages, see a survey in Delest [4]) and analytic calculus. A preliminary problem was first to find the generating function for the pruning number of forest. The surprise was that this parameter has indeed the Strahler distribution. At the UQAM meeting in 1985, I offered a price of ten bottles of Bordeaux wine "domaine des Mattes" 1982 for a bijection between binary trees and forest of planar trees sending the Strahler number onto the pruning number, (see Labelle, Leroux [13], p 386). Zeilberger [25] obtained a bijection and consequently received the ten bottles of wine which were given to him at the 3rd FPSAC in Bordeaux. Later and independently, E. Bender and R.Canfield also constructed a bijection. As with Françon's bijection quoted in section 2, Zeilberger's bijection is also highly recursive.

A natural problem is thus to complete the "golden triangle" of Strahler bijections between the three class: binary trees, Dyck paths and forest of planar trees. The remaining problem is to construct a direct bijection between forest of planar trees and Dyck paths, sending the parameter pruning number onto the parameter logarithmic height.

4 A Strahler bijection between forest of planar trees and Dyck paths

As with Françon's and Zeilberger's bijection, our bijection is defined recursively. We suppose that we have constructed a bijection ϕ_k between forests of planar trees with n vertices having pruning number k, and Dyck paths of length $2n$ having logarithmic height k. We then construct the bijection ϕ_{k+1} for the same objects but with respective parameters $k+1$.

First consider the case $k = 1$. The filament f  of length n is associated with the following Dyck path $\phi_1(f)$ of length $2n$:  .

A forest F of order 1 is a sequence of filaments $F_1, \ldots, F_p$. The associated Dyck path $\phi_1(F)$ is obtained by concatenating the sequence of Dyck paths $\phi_1(F_1), \ldots, \phi_1(F_p)$.

The transition from ϕ_k to ϕ_{k+1} is based on the following basic remark. Let G be a forest of order $k+1$ and let $F = \delta(G)$ be the forest of order k obtained by deleting the filaments. Suppose that F has n vertices.

Conversely, we can derive G from F by adding some filaments. At the end of each leaf of F, we must add at least two filaments. Figure 9 is symbolic. We have colored in red the two leftmost filaments. All the other filaments added are colored in green. These filaments are added to G by package of "fans of filaments" (sequences of filaments hanging from the same vertex) at each of possible positions marked by a green asterisk in Figure 9. The number of such positions is $2n+1$. Such a fan of green filaments may be empty, but the red filaments have length at least 1. Remark that for the positions on the orange line (Figure 9), each filament of the fan creates a new root for the subtrees of the forest.

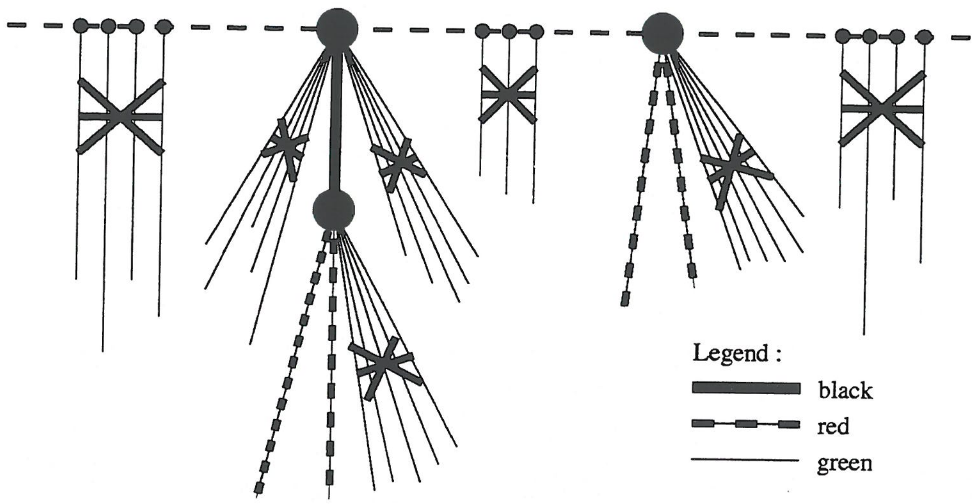


Figure 9 : From a forest of order k to a forest of order $k+1$

Figure 10 gives a precise example of a passage from a forest F of order k to a forest G of order $k+1$ (here $k=1$). The forest F is colored black. The first edge of the red filaments is colored red, then all the other edges are colored blue. The edges of the green filaments are colored in green. The forest G is obtained from F by adding the red, blue and green edges. Recall that the two red edges for each leaf are compulsory, the blue and green edges are not.

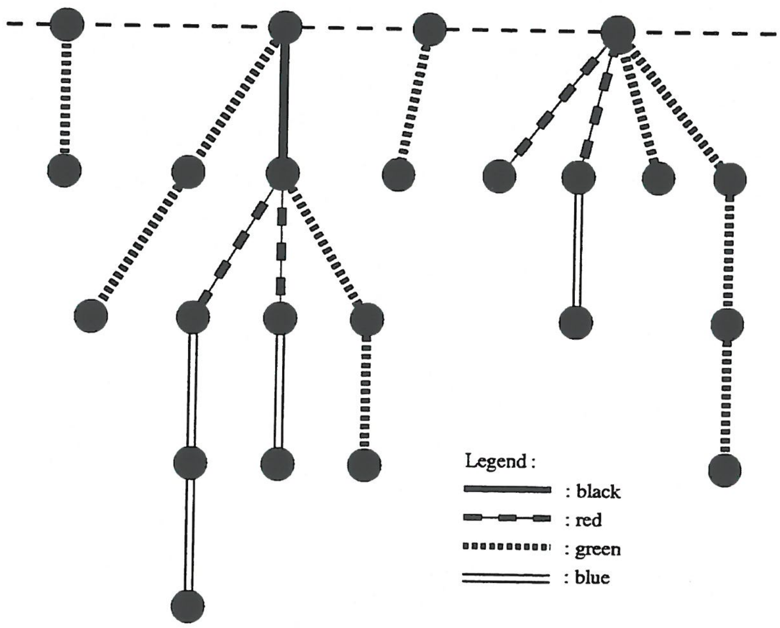


Figure 10 : From a forest of order 1 to a forest of order 2

If ω is the path of length $2n$ related to F by ϕ_k , we construct the path $\eta = \phi_{k+1}(G)$ using some "rewriting rules" inside the elementary steps of ω according to the three possible colors of the edges added to F in order to get G.

Define a *strip* as to be the portion of $\mathbb{N} \times \mathbb{N}$ contained between two consecutive lines with equation $y = p$ and $y = p + 1$. We color alternatively the strips into two colors yellow or white, according to the parity of p, starting with color yellow for the strip between the line of equation $y = 0$ and $y = 1$. The border of each strip is displayed in green, see Figure 11a. Assume by induction that the number of leaves in the forest F is exactly the number of SE steps in the yellow strips (colored in red in Figure 11). The construction is as follows.

Step 1. The yellow strips are "dilated" vertically by a factor 3. The elementary steps located inside the white strips (colored in black in the Figures) are left invariant. Each elementary step in the yellow strip (colored in red in Figure 11a) is replaced according to the rewriting rules displayed in Figure 11b. We get a path $\alpha(\omega)$ as shown in Figure 11c. In this last figure, the green lines, borders of the yellow strips, have been preserved in the dilatation. We have colored in green each of the $(2n + 1)$ intersection points of the paths ω and $\alpha(\omega)$ with the green lines in Figures 11a and 11c. Intuitively, in Figure 11c, the red SE steps correspond to the leaves of the forest obtained from F by adding the red edges. The blue NE and SE steps correspond to the positions of possible blue filaments. The green points correspond to the $(2n + 1)$ possible positions for adding fan of green filaments.

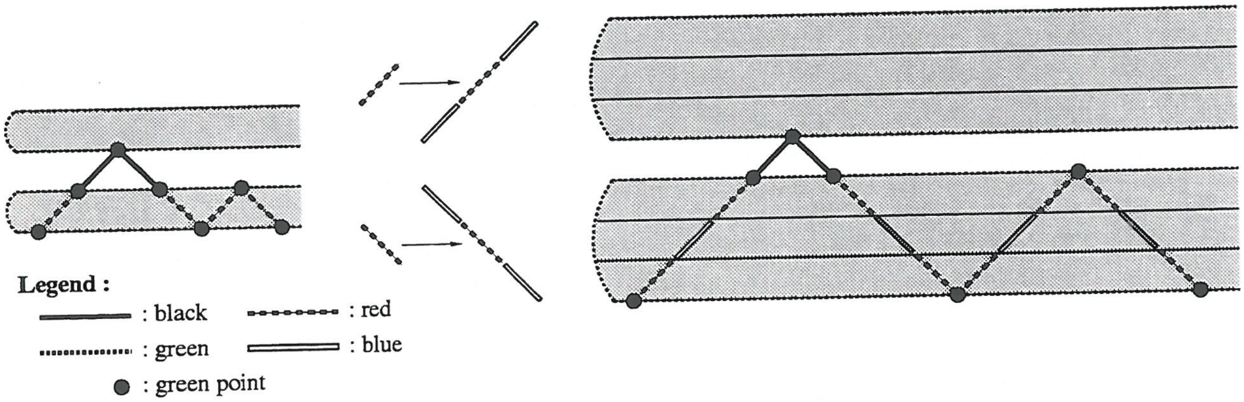


Figure 11 : (a) yellow strips, green border ; (b) rewriting rules ; (c) Step 1

Step 2. Each blue step / or \ is replaced by /\/\.../ or \/\.../\ according to the length of the corresponding filaments, see the example in Figure 12, according to the forest of Figure 10.

Step3. At the place of each green vertices of ω is inserted a sequence (in green in Figure 12) of the following form. If the vertex corresponds to the i^{th} fan F_i of green filaments of G, let ω_i be the path $\phi_1(F_i)$.

a) if the vertex is on a lower border of a yellow strip then we insert ω_i,

b) if the vertex is an upper border of a yellow strip, then we insert the path ω_i^* obtained from ω_i by replacing the SE steps with NE steps, and conversely.

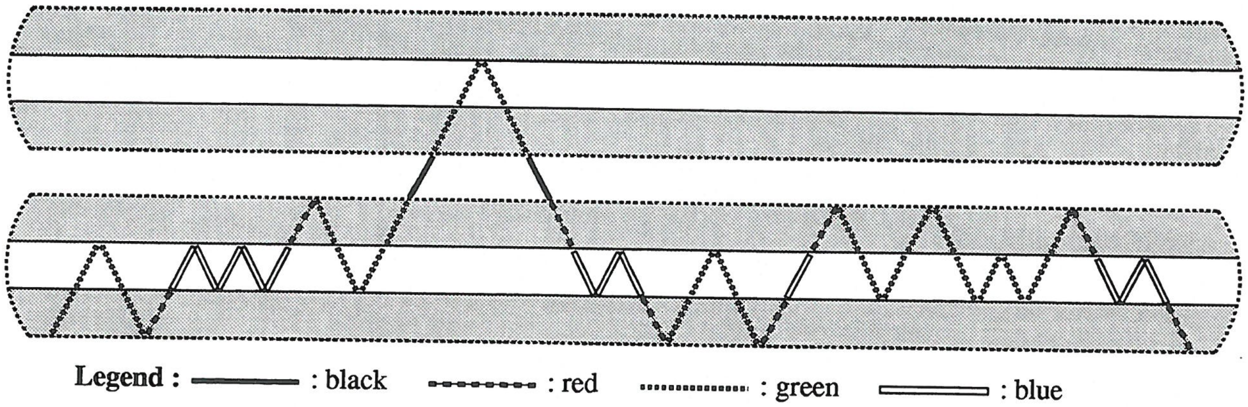


Figure 12: Steps 2 and 3

Steps 1,2, and 3 yield the path $\eta = \phi_{k+1}(G)$. An example is displayed in Figure 12, from the forest of Figure 10. The recurrence hypothesis about the number of leaves and SE steps in the yellow strips is satisfied and it is easily seen that $LH(\eta) = k+1$. Finally we can check that the map ϕ_{k+1} is a bijection.

5 A new functional equation for Strahler numbers

In fact, the recursive construction presented in the previous section gives rise to a functional equation, different from the one obtained by Françon [9]. We need here to introduce a triple generating function $S(t,x,y)$ counting forests of planar trees by their number of vertices (variable t), by their pruning number (variable x) and by their number of leaves (variable y).

Proposition 1 *The generating function $S(t,x,y)$ satisfies the following functional equation*

$$S(t,x,y) = 1 + \frac{xyt}{1-(1+y)t} + \frac{x(1-t)}{(1-(1+y)t)} S\left[\left(\frac{(1-t)}{1-(1+y)t}\right)^2, x, \frac{t^2y^2}{(1-t)^2}\right]. \tag{5.1}$$

An idea of the proof is given with the formula (5.1) displayed with colors in Figure 13. Each block of terms of the same color corresponds to some construction in the passage from the forest F to the forest G, as displayed in Figure 10. For example the red term "ty" and the blue term "$1/(1-t)$" correspond to the blue and red edges of the forest F. The green term is the generating function for fans of filaments, that is

$$\frac{(1-t)}{(1-(1+y)t)} = \frac{1}{1-\dfrac{ty}{1-t}}. \tag{5.2}$$

The second term of the right handside corresponds to a forest of order 1. The proposition follows from standard techniques in enumerative combinatorics, using for example the theory of species, Bergeron, Labelle, Leroux [1], or Flajolet's model of decomposable structures [7].

$$\mathbf{S}(t,x,\underbrace{y}_{\textbf{red}}) = \underbrace{1}_{\textbf{red}} + x\frac{yt}{1-(1+y)t} + x\underbrace{\frac{(1-t)}{(1-(1+y)t)}}_{\textbf{green}}\,\mathbf{S}\left[\underbrace{\left(\frac{(1-t)}{1-(1+y)t}\right)^2}_{\textbf{green}}, x, \underbrace{t^2y^2}_{\textbf{red}}\underbrace{\frac{1}{(1-t)^2}}_{\textbf{blue}}\right]$$

Figure 13: Colors for the proof of (5.1)

From section 4, we can also deduce that the same functional equation holds for the generating function $C(t, x, y)$ of Dyck paths counted by their length (variable t), by their logarithmic height (variable x) and by their number of steps going from an odd level to an even level (variable y). If we write

$$C(t,x,y) = \sum_{k\geq 1} C_k(t,y)x^k\,, \tag{5.3}$$

we can deduce from standard techniques (Flajolet [6] or Viennot [21]) the following identity

$$C_k(t,y) = \frac{t^{2^k-1}y^{2^{k-1}}}{R_{2^{k+1}-1}(t,y)}\,, \tag{5.4}$$

where the $R(t, y)$ are "y-analogues" of the modified Tchebycheff polynomials of the second kind. They are defined as in section 1 by replacing the recurrence (1.4) by the recurrence

$$F_{n+1}(t,y) = tF_n(t,y) - \delta(y)F_{n-1}(t,y) \qquad \text{with} \quad F_0 = 1, \quad F_1 = ty\,, \tag{5.5}$$

and $\delta(y)$ is 1 or y according to whether n is odd or even. These polynomials are a kind of "sieve polynomials", as considered in Chihara [3].

Combining Françon's bijection with our bijection, we get a bijective proof of the equality of the three following double distributions:

- binary trees by Strahler number and number of left "internal" edges (i.e. edges connecting two internal vertices),
- Dyck paths by logarithmic height and number of SE steps starting from an odd level,
- forests of planar trees by pruning number and number of leaves.

Recall that each of the three second parameters has individually the so-called β-distribution (Kreweras [12]), given by the Narayana numbers $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. In fact the generating polynomial (in the variable y) of these numbers are nothing but the moments of the orthogonal polynomials $F_n(t, y)$ introduced by (5.5). Finally we can show that a slight modification of Zeilberger's bijection preserves the corresponding double distribution.

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