A DUALITY OF A TWISTED GROUP ALGEBRA OF THE HYPEROCTAHEDRAL GROUP AND THE QUEER LIE SUPERALGEBRA

MANABU YAMAGUCHI

Department of Mathematics, Aoyama Gakuin University
Chitosedai 6-16-1, Setagaya-ku, Tokyo, 157-8572 Japan

ABSTRACT. We establish a duality relation between one of the twisted group algebras of the hyperoctahedral group H_k and a Lie superalgebra $\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$ for any integers $k \geq 4$ and n_0, $n_1 \geq 1$, where $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ denote the "queer" Lie superalgebras. Note that this twisted group algebra $\mathcal{B}'_k$ belongs to a different cocycle from the one $\mathcal{B}_k$ used by A. N. Sergeev in [8] and by the present author in [11].

We will use the supertensor product $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$ of the 2^k-dimensional Clifford algebra $\mathcal{C}_k$ and $\mathcal{B}'_k$, as an intermediary for establishing our duality. We show that the algebra $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$ and $\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$ act on the k-fold tensor product $W = V^{\otimes k}$ of the natural representation V of $\mathfrak{q}(n_0+n_1)$ as mutual supercentralizers of each other (Theorem 4.1). Moreover, we show that $\mathcal{B}'_k$ and $\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$ act on a subspace W' of W as mutual supercentralizers of each other (Theorem 4.2). This duality relation gives a formula for the character values of simple $\mathcal{B}'_k$-modules. This formula is different from a formula (Theorem E) obtained by J. R. Stembridge (cf. [10, Lem 7.5]).

§1. INTRODUCTION

We establish a duality relation (Theorem 4.2) between one of the twisted group algebras of the hyperoctahedral group H_k and a Lie superalgebra $\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$ for any integers $k \geq 4$ and n_0, $n_1 \geq 1$. Here $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ denote the "queer" Lie superalgebras as called by some authors. The twisted group algebra $\mathcal{B}'_k$ in focus in this paper belongs to a different cocycle from the one $\mathcal{B}_k$ used by A. N. Sergeev in his work [8] on a duality with $\mathfrak{q}(n)$ and by the present author in a previous work [11]. This $\mathcal{B}'_k$ contains the twisted group algebra $\mathcal{A}_k$ of the symmetric group $\mathfrak{S}_k$ in a straightforward manner (cf. (2.1)), and has a structure similar to the semidirect product of $\mathcal{A}_k$ and $\mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^k]$.

We will use the algebra $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$, where $\mathcal{C}_k$ is the 2^k-dimensional Clifford algebra (cf. (3.2)) and $\dot{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product (the supertensor product) (cf. [1], [2], [11, §1]). We define a representation of $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$ in the k-fold tensor product $W = V^{\otimes k}$ of $V = \mathbb{C}^{n_0+n_1}\oplus\mathbb{C}^{n_0+n_1}$, the space of the natural representation of the Lie superalgebra $\mathfrak{q}(n_0+n_1)$. This representation of $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$ depends on n_0 and n_1, not just n_0+n_1. Note that $\mathcal{B}_k$ can be regarded as a subalgebra of $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$, since $\mathcal{B}_k$ is isomorphic to $\mathcal{C}_k\dot{\otimes}\mathcal{A}_k$ by our previous result (cf. [11, Th. 3.2]). Under this embedding, our representation of $\mathcal{C}_k\dot{\otimes}\mathcal{B}'_k$ restricts to the representation of

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

$\mathcal{B}_k$ in W defined by Sergeev (cf. Theorem C). We show that the centralizer of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ in $\mathrm{End}(W)$ is generated by the action of the Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ (Theorem 4.1). Moreover we show that $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ act on a subspace W' of W "as mutual centralizers of each other" (Theorem 4.2). This duality relation gives a formula for the character values of simple $\mathcal{B}'_k$-modules (Corollary 4.4). In this formula, the character values are described as the entries of the transition matrix between two bases of $\Omega_x \otimes \Omega_y$ (see the definition of these rings Ω_x and Ω_y in Appendix, **A**). These bases are different from those used by J. R. Stembridge in [10, Lem 7.5] (cf. Theorem E). Note that $\mathcal{A}_k$ and $\mathfrak{q}(n)$ act on the same space W' "as mutual centralizers of each other" (cf. Theorem D).

In Appendix, we include short explanations of some known reults, which we use in the previous sections.

In this paper, all vector spaces, and associative algebras, and representations are assumed to be finite dimensional over $\mathbb{C}$ unless specified otherwise. The precise statements of the results skeched in the introduction use the formulation of $\mathbb{Z}_2$-graded representations of $\mathbb{Z}_2$-graded algebras (superalgebras) (cf. Appendix, **B**) as was used in [1] and [2].

§2. Simple modules for a twisted group algebra $\mathcal{B}'_k$

For any $k \geq 1$, let $\mathcal{B}'_k$ denote the associative algebra generated by τ' and the γ_i, $1 \leq i \leq k-1$, with relations

(2.1)
$$\begin{aligned}&\tau'^2 = \gamma_i^2 = 1 \quad (1 \leq i \leq k-1), \quad (\gamma_i\gamma_{i+1})^3 = 1 \quad (1 \leq i \leq k-2),\\ &(\gamma_i\gamma_j)^2 = -1 \quad (|i-j| \geq 2), \quad (\tau'\gamma_i)^2 = 1 \quad (2 \leq i \leq k-1),\\ &(\tau'\gamma_1)^4 = 1.\end{aligned}$$

If $k \geq 4$, then $\mathcal{B}'_k$ is isomorphic to a twisted group algebra of the hyperoctahedral group H_k with a nontrivial 2-cocycle (cf. [10, Prop. 1.1]). We regard $\mathcal{B}'_k$ as a superalgebra by giving the generator τ' (resp. the generator γ_i, $1 \leq i \leq k-1$) degree 0 (resp. degree 1). Note that this grading of $\mathcal{B}'_k$ is different from that of $\mathcal{B}_k$ in (3.1) or in [11]. Let $\mathcal{A}_k$ denote the subsuperalgebra of $\mathcal{B}'_k$ generated by γ_i, $1 \leq i \leq k-1$. If $k \geq 4$, then $\mathcal{A}_k$ is isomorphic to a twisted group algebra of the symmetric group $\mathfrak{S}_k$ with a nontrivial 2-cocycle, with the $\mathbb{Z}_2$-grading as in [2] and [11]. The simple $\mathcal{A}_k$-modules are parametrized by DP_k, the distinct partitions (the strict partitions) of k (cf. [2], [7], [9]). For $\lambda \in DP_k$, let V_λ denote a simple $\mathcal{A}_k$-module indexed by λ.

The simple $\hat{\mathcal{B}}_k$-modules are parametrized by $(DP^2)_k$, where $(DP^2)_k$ denotes the set of all $(\lambda,\mu) \in DP^2$ such that $|\lambda| + |\mu| = k$ ($DP = \coprod_{k\geq 0} DP_k$). For $(\lambda,\mu) \in (DP^2)_k$, we construct a $\hat{\mathcal{B}}_k$-module $V_{\lambda,\mu}$ indexed by (λ,μ) as follows. Define a surjective homomorphism of superalgebras $\pi_k \colon \mathcal{B}'_k \to \mathcal{A}_k$ (resp. $\pi'_k \colon \mathcal{B}'_k \to \mathcal{A}_k$) by $\pi_k(\tau') = 1$, $\pi_k|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$ (resp.$\pi'_k(\tau') = -1$, $\pi'_k|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$). The simple $\mathcal{A}_{k'}$ (resp. $\mathcal{A}_{k-k'}$)-module V_λ (resp. V_μ) can be lifted to a $\mathcal{B}'_{k'}$ (resp. $\mathcal{B}'_{k-k'}$)-module via $\pi_{k'}$ (resp. $\pi'_{k-k'}$), where $k' = |\lambda|$. This (simple) $\mathcal{B}'_{k'}$ (resp. $\mathcal{B}'_{k-k'}$)-module is denoted by $V_{\lambda,\phi}$ (resp. $V_{\phi,\mu}$). Let $V_{\lambda,\mu}$ denote the $\mathcal{B}'_k$-module induced from the

$\mathcal{B}'_{k'} \otimes \mathcal{B}'_{k-k'}$-module $V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu}$, namely

$$V_{\lambda,\mu} = \mathcal{B}'_k \otimes_{\mathcal{B}'_{k'} \otimes \mathcal{B}'_{k-k'}} (V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu})$$

(see the definition of the operation $\dot{\circ}$ in Appendix, (B.1)), where $\mathcal{B}'_{k'}$ (resp. $\mathcal{B}'_{k-k'}$) is embedded into $\mathcal{B}'_k$ as a subsuperalgebra generated by τ'_1 and the γ_i, $1 \le i \le k'-1$ (resp. $\tau'_{k'+1}$ and the γ_j, $k'+1 \le j \le k-1$) (τ'_i denotes the element $\gamma_{i-1}\gamma_{i-2}\cdots\gamma_1\tau'\gamma_1\cdots\gamma_{i-2}\gamma_{i-1}$ of $\mathcal{B}'_k$).

Theorem 2.1. (cf. [10], Th. 7.1) $\{V_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ *is a complete set of the isomorphism classes of simple* $\hat{\mathcal{B}}_k$*-modules.*

The proof is analogous to the little group method, and is omitted. It can also be shown that this parametrization coincides with that by Stembridge in [10, Th. 7.1] modulo the usual difference between $\mathbb{Z}_2$-graded and non-graded modules.

§3. The algebras $\mathcal{B}_k$ and $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$

For any $k \ge 1$, let $\mathcal{B}_k$ denote the associative algebra generated by τ and the σ_i, $1 \le i \le k-1$, with relations

(3.1)
$$\begin{aligned}
&\tau^2 = \sigma_i^2 = 1 \quad (1 \le i \le k-1), \quad (\sigma_i\sigma_{i+1})^3 = 1 \quad (1 \le i \le k-2),\\
&(\sigma_i\sigma_j)^2 = 1 \quad (|i-j| \ge 2), \quad (\tau\sigma_i)^2 = 1 \quad (2 \le i \le k-1),\\
&(\tau\sigma_1)^4 = -1.
\end{aligned}$$

We regard $\mathcal{B}_k$ as a superalgebra by giving the generator τ' (resp. the generator σ_i, $1 \le i \le k-1$) degree 1 (resp. degree 0). The subgroup of $(\mathcal{B}_k)^\times$ generated by σ_i, $1 \le i \le k-1$, is isomorphic to the symmetric group of degree k and it is denoted by $\mathfrak{S}_k$.

Let $\mathcal{C}_k$ denote the 2^k-dimensional Clifford algebra, namely $\mathcal{C}_k$ is generated by $\xi_1, \dots, \xi_k$ with relations

$$\xi_i^2 = 1, \quad \xi_i\xi_j = -\xi_j\xi_i \quad (i \ne j) \ . \tag{3.2}$$

We regard $\mathcal{C}_k$ as a superalgebra by giving the generator ξ_i, $1 \le i \le k$, degree 1. $\mathcal{C}_k$ is a simple superalgebra. Let X_k be a unique simple $\mathcal{C}_k$-module.

The superalgebra $\mathcal{B}_k$ is isomorphic to the supertensor product of the superalgebras $\mathcal{C}_k$ and $\mathcal{A}_k$ (see the definition of the supertensor product in [1], [2], [11, §1]). Define a linear map $\vartheta\colon \mathcal{B}_k \to \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ by

(3.3)
$$\begin{aligned}
&\vartheta(\tau_i) \mapsto \xi_i \otimes 1 \qquad (1 \le i \le k),\\
&\vartheta(\sigma_j) \mapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}) \otimes \gamma_j \quad (1 \le j \le k-1)
\end{aligned}$$

where $\tau_i = \sigma_{i-1}\cdots\sigma_1\tau\sigma_1\ldots\sigma_{i-1}$. Then ϑ is an isomorphism of superalgebras (cf. [11, Th. 3.2]). For $\lambda \in DP_k$, define a $\mathcal{B}_k$-module W_λ by $W_\lambda = X_k \dot{\circ} V_\lambda$. By

Corollary B.2, $\{W_\lambda \mid \lambda \in DP_k\}$ is a complete set of the isomorphism classes of simple $\mathcal{B}_k$-modules.

Let $\hat{\mathcal{B}}_k$ denote the supertensor product of the algebras $\mathcal{C}_k$ and $\mathcal{B}'_k$, namely $\hat{\mathcal{B}}_k = \mathcal{C}_k \hat{\otimes} \mathcal{B}'_k$. Since $\mathcal{B}_k \cong \mathcal{C}_k \hat{\otimes} \mathcal{A}_k$, $\mathcal{B}_k$ can be regarded as a subsuperalgebra of $\hat{\mathcal{B}}_k$. For $(\lambda, \mu) \in (DP^2)_k$, put $W_{\lambda,\mu} = X_k \dot{\circ} V_{\lambda,\mu}$. By Corollary B.2, $\{W_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ is a complete set of the isomorphism classes of simple $\hat{\mathcal{B}}_k$-modules. Note that $W_{\lambda,\mu}$ is isomorphic to the $\hat{\mathcal{B}}_k$-module induced from the $\hat{\mathcal{B}}_{k'} \hat{\otimes} \hat{\mathcal{B}}_{k-k'}$-module $W_{\lambda,\phi} \dot{\circ} W_{\phi,\mu}$, where $k' = |\lambda|$, namely

$$W_{\lambda,\mu} \cong_{\hat{\mathcal{B}}_k} \hat{\mathcal{B}}_k \otimes_{\hat{\mathcal{B}}_{k'} \hat{\otimes} \hat{\mathcal{B}}_{k-k'}} (W_{\lambda,\phi} \dot{\circ} W_{\phi,\mu}). \tag{3.4}$$

§4. A duality of $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$

Let $\mathfrak{q}(n)$ denote the queer Lie superalgebra, namely $\mathfrak{q}(n)$ is the Lie subsuperalgebra of $\mathfrak{gl}(n,n)$ (denoted by $l(n,n)$ in [5]) consisting of the matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Let $\mathcal{U}_n = \mathcal{U}(\mathfrak{q}(n))$ denote the universal enveloping algebra of $\mathfrak{q}(n)$, which can be regarded as a superalgebra.

Let W denote the k-fold supertensor product of the $2n$-dimensional natural representation $V = \mathbb{C}^n \oplus \mathbb{C}^n$ of $\mathfrak{q}(n)$, namely $W = V^{\otimes k}$. We define a representation $\Theta\colon \mathcal{U}_n \to \mathrm{End}(W)$ by

$$\Theta(X)(v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} (-1)^{\overline{X}\cdot(\overline{v_1}+\cdots+\overline{v_{j-1}})} v_1 \otimes \cdots \otimes \underbrace{X v_j}_{j} \otimes \cdots \otimes v_k$$

for all homogeneous elements $X \in \mathfrak{q}(n)$ and $v_i \in V$ $(1 \le i \le k)$. Note that $\mathcal{U}_n$ is an infinite dimensional superalgebra. However, for a fixed number k, $\mathcal{U}_n$ acts on W through its finite dimensional image in $\mathrm{End}(W)$. Therefore we can use the results in Appendix, $\mathbf{E}$ on finite dimensional superalgebras and their finite dimensional modules.

Let n_0 and n_1 be two positive integers such that $n_0 + n_1 = n$. The Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ can be embedded into $\mathfrak{q}(n)$ via

$$\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1) \ni \left(\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right) \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ A & 0 & B & 0 \\ 0 & C & 0 & D \end{pmatrix} \in \mathfrak{q}(n). \tag{4.1}$$

The universal enveloping algebra of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ is isomorphic to $\mathcal{U}_{n_0} \hat{\otimes} \mathcal{U}_{n_1}$ which can be embedded into $\mathcal{U}_n$ as a subalgebra generated by the elements of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$.

Now we define a representation $\Psi\colon \hat{\mathcal{B}}_k \to \mathrm{End}(W)$ of $\hat{\mathcal{B}}_k$, which depends on n_0

and n_1, by

(4.2)

$$\Psi(\xi_i \otimes 1)(v_1 \otimes \cdots \otimes v_k) = (-1)^{\overline{v_1}+\cdots+\overline{v_{i-1}}} v_1 \otimes \cdots \otimes Pv_i \otimes \cdots \otimes v_k \quad (1 \le i \le k),$$
$$\Psi(1 \otimes \tau')(v_1 \otimes \cdots \otimes v_k) = (Qv_1) \otimes v_2 \otimes \cdots \otimes v_k,$$
$$\Psi(1 \otimes \gamma_j)(v_1 \otimes \cdots \otimes v_k)$$
$$= \frac{(-1)^{\overline{v_1}+\cdots+\overline{v_{j-1}}}}{\sqrt{2}} v_1 \otimes \cdots \otimes (Pv_j \otimes v_{j+1} - (-1)^{\overline{v_j}} v_j \otimes Pv_{j+1}) \otimes \cdots \otimes v_k$$
$$(1 \le j \le k-1)$$

for all homogeneous elements $v_j \in V$, $1 \le j \le k$, where P and Q denote the matrices $\begin{pmatrix} 0 & -\sqrt{-1}I_n \\ \sqrt{-1}I_n & 0 \end{pmatrix}$ and $\begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & -I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_0} & 0 \\ 0 & 0 & 0 & -I_{n_1} \end{pmatrix}$ respectively. Note that, by the isomorphism $\vartheta\colon \mathcal{B}_k \cong \mathcal{C}_k \otimes \mathcal{A}_k (\subset \hat{\mathcal{B}}_k)$, W can be regarded as a $\mathcal{B}_k$-module and this $\mathcal{B}_k$-module was investigated by Sergeev in [8] (cf. Theorem C).

Let W' be a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of W. Since $(\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1))_0 \cong \mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ as a Lie algebra, and V is a sum of two copies of the natural representation $\mathbb{C}^n = \mathbb{C}^{n_0} \oplus \mathbb{C}^{n_1}$ of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$, this embeds $W'|_{(\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1))_0}$ into a sum of tensor powers of the natural representation, so that this representation of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ can be integrated to a polynomial representation $\theta_{W'}$ of $GL(n_0, \mathbb{C}) \times GL(n_1, \mathbb{C})$. Let $\mathrm{Ch}[W']$ denote the character of $\theta_{W'}$, namely

$$\mathrm{Ch}[W'](x_1, \ldots, x_{n_0}, y_1, \cdots, y_{n_1}) = \mathrm{tr}\, \theta_{W'}(\mathrm{diag}(x_1, \ldots, x_{n_0}), \mathrm{diag}(y_1, \ldots, y_{n_1})).$$

The following theorem determines the supercentralizer of $\Psi(\hat{\mathcal{B}}_k)$ in $\mathrm{End}(W)$ and describes the characters of simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules appearing in W.

Theorem 4.1. (1) *The two superalgebras $\Psi(\hat{\mathcal{B}}_k)$ and $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$ act on W as the mutual supercentralizers of each other:*

$$\text{(4.4)} \qquad \mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W) = \Psi(\hat{\mathcal{B}}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}).$$

(2) *The simple $\hat{\mathcal{B}}_k$-module $W_{\lambda,\mu}$ $((\lambda, \mu) \in (DP^2)_k)$ occurs in W if and only if $l(\lambda) \le n_0$ and $l(\mu) \le n_1$. Moreover W is decomposed as a multiplicity-free sum of simple $\hat{\mathcal{B}}_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules as follows:*

$$\text{(4.5)} \qquad W \cong_{\hat{\mathcal{B}}_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{\substack{(\lambda,\mu) \in (DP^2)_k \\ l(\lambda) \le n_0, l(\mu) \le n_1}} W_{\lambda,\mu} \dot{\circ} (U_\lambda \dot{\circ} U_\mu)$$

where U_λ (resp. U_μ) denotes the simple $\mathcal{U}_{n_0}$ (resp. $\mathcal{U}_{n_1}$)-module corresponding to the simple $\mathcal{B}_{|\lambda|}$ (resp. $\mathcal{B}_{|\mu|}$)-module W_λ (resp. W_μ) in Sergeev's duality (cf. Theorem C).

(3) *Put* $U_{\lambda,\mu} = U_\lambda \dot\circ U_\mu$. *Then the character values of* $\mathrm{Ch}[U_{\lambda,\mu}]$ *are given as follows:*

(4.6)

$$\mathrm{Ch}[U_{\lambda,\mu}](x_1,\dots,x_{n_0},y_1,\dots,y_{n_1}) = 2^{\frac{d(\lambda,\mu)-l(\lambda)-l(\mu)}{2}} Q_\lambda(x_1,\dots,x_{n_0})Q_\mu(y_1,\dots,y_{n_1})$$

where $d\colon (DP^2)_k \to \mathbb{Z}_2$ *denotes a map defined by* $d(\lambda,\mu) = 0$ (*resp.* $d(\lambda,\mu) = 1$) *if* $l(\lambda) + l(\mu)$ *is even* (*resp.* $l(\lambda) + l(\mu)$ *is odd*) *and* Q_λ *and* Q_ν *denote Schur's* Q*-functions* (*cf. Appendix,* **A**).

Proof. First we will show the second equality of (4.4). Then the first equality also follows from the double supercentralizer theorem (abbreviated as DSCT) for semisimple superalgebras (cf. [11, Th. 2.1]).

By direct calculations, it can be checked that $\Theta(X\otimes Y)$ commutes with $\Psi(\hat{\mathcal{B}}_k)$ for any $X \in \mathfrak{q}(n_0)$, $Y \in \mathfrak{q}(n_1)$. Hence we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \supset \Theta(\mathcal{U}_{n_0}\otimes \mathcal{U}_{n_1})$. Moreover, it can be shown that any element f of $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$ belongs to the subsuperalgebra of $\mathrm{End}(W) = \mathrm{End}(V)^{\otimes k}$ generated by the elements of the form $\sum_{j=1}^k 1\otimes\cdots\otimes 1\otimes \underbrace{X}_{j}\otimes 1\otimes\cdots\otimes 1$ with $X \in \mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$. Hence we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset \Theta(\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})$.

Since V can be regarded the space of the natural representation of $\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1)$, it follows, by Theorem C, that any simple $\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1}$-module occurring in $W = V^{\otimes k}$ is of the form $U_\lambda\dot\circ U_\mu$, $(\lambda,\mu)\in(DP^2)_k$, and that $U_\lambda\dot\circ U_\mu$ occurs in W if and only if $l(\lambda)\le n_0$ and $l(\mu)\le n_1$. By (4.4) and DSCT, W can be decomposed as a multiplicity-free sum of non-isomorphic simple $\hat{\mathcal{B}}_k \dot\otimes (\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})$-modules. Again by Theorem C, it can be shown that the simple $\hat{\mathcal{B}}_k$-module, which is paired with $U_\lambda\dot\circ U_\mu$ in W, contains $W_{\lambda,\phi}\otimes W_{\phi,\mu}$ as a $\hat{\mathcal{B}}_{k'}\dot\otimes\hat{\mathcal{B}}_{k-k'}$-submodule. By (3.4), it follows that this simple $\hat{\mathcal{B}}_k$-module is isomorphic to $W_{\lambda,\mu}$. Therefore the result (2) follows.

The result (3) immediately follows from Theorem C, (3) and the fact that

$$\begin{aligned}&\mathrm{Ch}[U\dot\circ U'](x_1,\dots,x_{n_0},y_1,\dots,y_{n_1})\\ &=\begin{cases}\mathrm{Ch}[U](x_1,\dots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\dots,y_{n_1}) & \text{if } U \text{ or } U' \text{ is of type } M,\\ \frac{1}{2}\,\mathrm{Ch}[U](x_1,\dots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\dots,y_{n_1}) & \text{if } U,\ U' \text{ are of type } Q\end{cases}\end{aligned}$$

(see the definition of the "type" of simple modules of superalgebras in the argument before Theorem B.1). □

We can rewrite (4.5) using the isomorphism $W_{\lambda,\mu}\cong X_k\dot\circ V_{\lambda,\mu}$ as $\hat{\mathcal{B}}_k$-modules. We have

$$W \cong \bigoplus_{(\lambda,\mu)\in(DP^2)_k} (X_k\dot\circ V_{\lambda,\mu})\dot\circ U_{\lambda,\mu}.$$

The Clifford algebra $\mathcal{C}_k$ contains the commuting involutions $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i}$, $1\le i\le r = \lfloor\frac{k}{2}\rfloor$, of degree 0. For each $\varepsilon = (\varepsilon_1,\dots,\varepsilon_r)\in\mathbb{Z}_2^r$, put $W^\varepsilon = \{w\in W \mid \Psi(\zeta_i\otimes 1)(w) = (-1)^{\varepsilon_i}w\ (1\le^{\forall} i\le r)\}$. Then we have $W = \bigoplus_{\varepsilon\in\mathbb{Z}_2^r} W^\varepsilon$. Since $\zeta_i\otimes 1$ commutes with $1\dot\otimes\mathcal{B}'_k$ for each $1\le i\le r$, W^ε is a $\mathcal{B}'_k\dot\otimes(\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})$-module.

Theorem 4.2. *For each $\varepsilon \in \mathbb{Z}_2^r$, the submodule W^ε is decomposed as a multiplicity-free sum of simple $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules as follows:*

$$W^\varepsilon \cong_{\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu} \circ U_{\lambda,\mu}. \tag{4.7}$$

In the above decomposition, the simple $\mathcal{B}'_k$-modules are paired with the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules in a bijective manner.

If k is even, then we have

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W^\varepsilon) = \Psi(\mathcal{B}'_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) = \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}). \tag{4.8}$$

If k is odd, then we have

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Psi(\mathcal{B}'_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}). \tag{4.9}$$

Proof. For a simple $\hat{\mathcal{B}}_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module $U = (X_k \circ V_{\lambda,\mu}) \circ U_{\lambda,\mu}$ of W, put $U^\varepsilon = \{v \in U \mid \Psi(\zeta_i \otimes 1)v = (-1)^{\varepsilon_i}\xi \quad (1 \leq^\forall i \leq r)\}$ for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$. Then U^ε is a $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-submodule of $U|_{\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \cong (V_{\lambda,\mu} \circ U_{\lambda,\mu})^{\oplus 2^r}$. Since $\dim U^\varepsilon = 2^{-r} \dim U = \dim V_{\lambda,\mu} \circ U_{\lambda,\mu}$, we have $U^\varepsilon \cong V_{\lambda,\mu} \circ U_{\lambda,\mu}$. Therefore the result (4.7) follows.

Assume that k is even. We will show the second equality in (4.8). Then the first equality follows from DSCT. Since W^ε is a $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module, we have $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} \subset \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$. By DSCT, (4.7) and (4.5), we have

$$\dim \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) = \dim \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$$

since both equal $\sum_{(\lambda,\mu)\in(DP^2)_k^+} (\dim U_{\lambda,\mu})^2 + \sum_{(\lambda,\mu)\in(DP^2)_k^-} \frac{1}{2}(\dim U_{\lambda,\mu})^2$. By Theorem 4.1, (1), we have $\dim \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. Moreover, it can be shown that a linear map $\mathfrak{p}_\varepsilon \colon \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}) \to \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon}$ defined by $\mathfrak{p}_\varepsilon(f) = f|_{W^\varepsilon}$ for any $f \in \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$ is bijective. Hence we have $\dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \dim \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$. It follows that $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$, as required.

Assume that k is odd. Then the supercentralizer $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ contains an invertible element $\Psi(\xi_k) \in \Psi(\mathcal{C}_k)$. The subsuperalgebra of $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ generated by $\Psi(\xi_k)$ is isomorphic to $\mathcal{C}_1$. By the argument similar to the proof of (4.8), the result (4.9) follows from DSCT (cf. [11, Cor. 2.1]). □

Let us mention a relation between the branching rule of the $\mathfrak{q}(n)$-modules to $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ and that of the $\hat{\mathcal{B}}_k$-modules to $\mathcal{B}_k$ (or that of the $\mathcal{B}'_k$-modules to $\mathcal{A}_k$).

If an A-module V restricts to an B-module, we write $V \downarrow^A_B$ for this B-module, for a superalgebra A and a subsuperalgebra B of A. Moreover, we write $[V : U]_A$ (or simply write $[V : U]$) for the multiplicity of a simple A-module U in an A-module V.

Corollary 4.3. *Put*

$$m^\lambda_{\mu,\nu} = [U_\lambda \downarrow^{\mathcal{U}_n}_{\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1}} : U_{\mu,\nu}],$$
$$m'^\lambda_{\mu,\nu} = [W_{\mu,\nu} \downarrow^{\hat{\mathcal{B}}_k}_{\mathcal{B}_k} : W_\lambda] \quad \left(\text{resp.} [V_{\mu,\nu} \downarrow^{\mathcal{B}'_k}_{\mathcal{A}_k} : V_\lambda]\right).$$

Then we have

$$(4.10) \qquad m'^\lambda_{\mu,\nu} = \begin{cases} \frac{1}{2} m^\lambda_{\mu,\nu} & \text{if } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}) \text{ is of type } M \\ & \quad \text{and } W_\lambda \text{ (resp. } V_\lambda) \text{ is of type } Q, \\ 2m^\lambda_{\mu,\nu} & \text{if } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}) \text{ is of type } Q \\ & \quad \text{and } W_\lambda \text{ (resp. } V_\lambda) \text{ is of type } M, \\ m^\lambda_{\mu,\nu} & \text{otherwise.} \end{cases}$$

Proof. Using (4.5) and (C.2) we consider the multiplicities of the simple $\mathcal{B}_k \otimes (\mathcal{U}_{n_0} \otimes \mathcal{U}_{n_1})$-module $W_\lambda \circ U_{\mu,\nu}$ in $W \downarrow^{\mathcal{B}_k\otimes\mathcal{U}_n}_{\mathcal{B}_k\otimes(\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})}$ and $W \downarrow^{\hat{\mathcal{B}}_k\otimes(\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})}_{\mathcal{B}_k\otimes(\mathcal{U}_{n_0}\otimes\mathcal{U}_{n_1})}$ respectively. Then, in the case of $[W_{\mu,\nu} \downarrow^{\hat{\mathcal{B}}_k}_{\mathcal{B}_k} : W_\lambda]$, the result (4.10) follows.

In the case of $[V_{\mu,\nu} \downarrow^{\mathcal{B}'_k}_{\mathcal{A}_k} : V_\lambda]$, the result (4.10) similarly follows by using (4.7) and (D.1). □

Let H'_k be the subgroup of $(\mathcal{B}'_k)^\times$ generated by $-1, \tau', \gamma_1, \ldots, \gamma_{k-1}$. Then H'_k is a double cover (a central extension with a $\mathbb{Z}_2$ kernel) of H_k. For a pair (κ, ν) of partitions κ and ν, let $w^{\kappa,\nu}$ denote the element of H'_k defined by

$$w^{\kappa,\nu} = w_1 w_2 \cdots w_l w'_1 w'_2 \cdots w'_{l'} \quad (l = l(\kappa), l' = l(\nu)),$$
$$w_i = \gamma_{a+1}\gamma_{a+2}\cdots\gamma_{a+\kappa_i-1} \quad (a = \kappa_1 + \cdots + \kappa_{i-1}),$$
$$w'_i = \gamma_{b+1}\gamma_{b+2}\cdots\gamma_{b+\nu_i-1}\tau'_{b+\nu_i} \quad (b = |\kappa| + \nu_1 + \cdots + \nu_{i-1}).$$

Note that the image of $w^{\kappa,\nu}$ in H_k is a representative of the conjugacy class of H_k indexed by (κ, ν). Let $\mathrm{Ch}[V_{\lambda,\mu}]$ denote the character of $V_{\lambda,\mu}$, namely $\mathrm{Ch}[V_{\lambda,\mu}](w) = \mathrm{tr}(w_{V_{\lambda,\mu}})$ for $w \in \mathcal{B}'_k$ where $w_{V_{\lambda,\mu}}$ denotes the action of $w \in \mathcal{B}'_k$ on $V_{\lambda,\mu}$. Then $\mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) = 0$ unless $(\kappa, \nu) \in (OP^2)_k$, where $(OP^2)_k$ denotes the set of all $(\kappa, \nu) \in OP^2$ such that $|\kappa| + |\nu| = k$ (OP denotes the odd partitions). We describe a formula for the character values of simple $\mathcal{B}'_k$-modules. Define a map $\varepsilon\colon (DP^2)_k \to \mathbb{Z}_2$ by $\varepsilon(\lambda, \mu) = 1$ (resp. $\varepsilon(\lambda, \mu) = 0$) if $(\lambda, \mu) \in (DP^2)^+_k$ (resp. $(\lambda, \mu) \in (DP^2)^-_k$).

Corollary 4.4. *We have*

(4.11)
$$2^{\frac{l(\kappa)+l(\nu)}{2}} p_\kappa(x, y) p_\nu(x, -y) = \sum_{(\lambda,\mu)\in(DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x) Q_\mu(y)$$

for all $(\kappa, \nu) \in (OP^2)_k$*, where* $p_\kappa(x, y)$ *(resp.* $p_\nu(x, -y)$*) denotes the power sum symmetric function* $p_\kappa(x_1, x_2, \ldots, y_1, y_2, \ldots)$ *(resp.* $p_\nu(x_1, x_2, \ldots, -y_1, -y_2, \ldots)$*) (cf. Appendix,* **A***).*

Proof. By what we noted before Theorem 4.1, any $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \otimes \mathcal{U}_{n_1})$-submodule W' of W can be regarded as a $\mathcal{B}'_k$-module with a commuting polynomial representation

$\theta_{W'}$ of $GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$. Here we extend our notation in Theorem 4.1 to let $\mathrm{Ch}[W'](x \otimes g)$ denote the trace $\mathrm{tr}(x_{W'} \circ \theta_{W'}(g))$ for $x \in \mathcal{B}'_k$ and $g \in GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$, where $x_{W'}$ denotes the action of $x \in \mathcal{B}'_k$ on W'.

For any $\varepsilon, \varepsilon' \in \mathbb{Z}_2^r$, we have $W^\varepsilon \cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} W^{\varepsilon'}$. Hence, for $(\kappa,\nu) \in (OP^2)_k$ and $E = \mathrm{diag}(x_1,\dots,x_{n_0},y_1,\dots,y_{n_1}) \in GL(n,\mathbb{C})$, we have

$$\mathrm{Ch}[W^\varepsilon](w^{\kappa,\nu} \otimes E) = 2^{-r}\,\mathrm{Ch}[W]\left((1 \otimes w^{\kappa,\nu}) \otimes E\right) \tag{4.12}$$

where $1 \otimes w^{\kappa,\nu} \in \mathcal{C}_k \otimes \mathcal{B}'_k = \hat{\mathcal{B}}_k$. Put $k' = |\kappa|$ and $l = l(\kappa)$. Then $k - k' = |\nu|$. Moreover put $W' = V^{\otimes k'}$ and $W'' = V^{\otimes k-k'}$. We have $w^{\kappa,\nu} = w^{\kappa,\phi} w^{\phi,\nu}$, where $w^{\kappa,\phi} \in \mathcal{B}'_{k'}$, $w^{\phi,\nu} \in \mathcal{B}'_{k-k'}$. Define a representations of $\hat{\mathcal{B}}_{k'}$ on W' (resp. a representation of $\hat{\mathcal{B}}_{k-k'}$ on W'') by the same manner as the representation Ψ of $\hat{\mathcal{B}}_k$ in W. Then we have

(4.13)

$$\mathrm{Ch}[W]\left((1 \otimes w^{\kappa,\nu}) \otimes E\right) = \mathrm{Ch}[W']\left((1 \otimes w^{\kappa,\phi}) \otimes E\right) \mathrm{Ch}[W'']\left((1 \otimes w^{\phi,\nu}) \otimes E\right).$$

The element $1 \otimes w^{\kappa,\phi}$ of $\hat{\mathcal{B}}_{k'}$ is a product of $k'-l$ elements $1 \otimes \gamma_j = \vartheta(\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j)$ (cf. (3.3)). This product can be expressed in the following form:

$$1 \otimes w^{\kappa,\phi} = \left(\frac{1}{\sqrt{2}}\right)^{k'-l} \times \sum (\text{a product of the } \vartheta(\tau_p)) \times \vartheta(\sigma^{\kappa,\phi})$$

where $\sigma^{\kappa,\phi} = g_1 g_2 \cdots g_l$, $g_i = \sigma_{a+1}\sigma_{a+2}\dots\sigma_{a+\nu_i-1}$ $(a = \sum_{j=1}^{i-1} \kappa_j)$. Then the $2^{k'-l}$ terms in the summation are conjugate to $\vartheta(\sigma^{\kappa,\phi})$ in $\vartheta((\mathcal{B}_k)^\times)$. Therefore we have

$$\begin{aligned}\mathrm{Ch}[W']\left((1 \otimes w^{\kappa,\phi}) \otimes E\right) &= 2^{k'-l}(\sqrt{2})^{-(k'-l)}\,\mathrm{Ch}[W']\left(\vartheta(\sigma^{\kappa,\phi}) \otimes E\right) \\ &= 2^{\frac{k'+l}{2}} p_\kappa(x_1,\dots,x_{n_0},y_1,\dots,y_{n_1}).\end{aligned}$$

Put $l' = l(\nu)$. Similarly we have

$$\begin{aligned}\mathrm{Ch}[W'']\left((1 \otimes w^{\phi,\nu}) \otimes E\right) &= 2^{k-k'-l'}(\sqrt{2})^{-(k-k'-l')}\,\mathrm{Ch}[W''](\vartheta(\sigma'^{\phi,\nu}) \otimes E) \\ &= 2^{\frac{k-k'+l'}{2}} p_\nu(x_1,\dots,x_{n_0},-y_1,\dots,-y_{n_1})\end{aligned}$$

where $\sigma'^{\phi,\nu} = g'_1 g'_2 \cdots g'_{l'}$, $g'_i = \sigma_{b+1}\sigma_{b+2}\dots\sigma_{b+\nu_i-1}\tau'_{b+\nu_i}$ $(b = \sum_{j=1}^{i-1} \nu_j)$. By (4.12) and (4.13), we have

$$\begin{aligned}&\mathrm{Ch}[W^\varepsilon]\left((1 \otimes w^{\kappa,\nu}) \otimes E\right) \\ &= \begin{cases} 2^{\frac{l+l'}{2}} p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots) p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots) & \text{if } k \text{ is even,} \\ 2^{\frac{l+l'+1}{2}} p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots) p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots) & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

By (4.6), Theorem B.1 and (B.1), we have

$$\begin{aligned}&\mathrm{Ch}[V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}](w^{\kappa,\nu} \otimes E) \\ &= \begin{cases} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)}{2}} Q_\lambda(x_1,\dots,x_{n_0}) Q_\mu(y_1,\dots,y_{n_1}) & \text{if } k \text{ is even,} \\ \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)+1}{2}} Q_\lambda(x_1,\dots,x_{n_0}) Q_\mu(y_1,\dots,y_{n_1}) & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

Since these hold for all n_0 and n_1, the result follows. □

Let us mention a relationship between this formula and Stembridge's formula (cf. Theorem E). Define an algebra endomorphism ι of $\Omega_x \otimes \Omega_y$ by $\iota(f \otimes 1) = f(x,y) = f(x_1, x_2, \ldots, y_1, y_2, \ldots)$ and $\iota(1 \otimes g) = g(x,-y) = g(x_1, x_2, \ldots, -y_1, -y_2, \ldots)$. Note that $\{Q_\lambda(x,y)Q_\mu(x,-y) \mid (\lambda,\mu) \in DP^2\}$ is a basis of $\Omega_x \otimes \Omega_y$ (cf. [10, Th. 7.1, Lem. 7.5]). It follows that ι is an automorphism, since $\iota(Q_\lambda(x)Q_\mu(y) = Q_\lambda(x,y)Q_\mu(x,-y)$. Moreover, since $\iota(p_r(x,y)) = 2p_r(x)$ and $\iota(p_r(x,-y)) = 2p_r(y)$ for any odd r, it follows that the image of (4.11) under ι coincides with Stembridge's formula.

Appendix

A. Symmetric functions. Let Λ_x denote the ring of the symmetric functions in the variables $x = \{x_1, x_2, \ldots\}$ with coefficients in $\mathbb{C}$; namely our Λ_x is the scalar extension of the Λ_x in [6], which is $\mathbb{Z}$-algebra, to $\mathbb{C}$. Let Ω_x denote the subring of Λ_x generated by the power sums of odd degrees, namely the $p_r(x)$, $r = 1, 3, 5, \ldots$. Then $\{p_\lambda(x) \mid \lambda \in OP\}$ is a basis of Ω_x, where $p_\lambda = \prod_{i \geq 1} p_{\lambda_i}$. For $\lambda \in DP$, let $Q_\lambda(x) \in \Lambda_x$ denote Schur's Q-function indexed by λ (cf. [7], [9, §6]). Then $\{Q_\lambda(x) \mid \lambda \in DP\}$ is also a basis of Ω_x.

B. Semisimple superalgebras. A $\mathbb{Z}_2$-graded algebra A, which is called a **superalgebra** in this paper, is called **simple** if it does not have nontrivial $\mathbb{Z}_2$-graded two-sided ideals. If A is a simple superalgebra, then it is either isomorphic to $M(m,n)$ (denoted by $M(m|n)$ in [2]) for some m and n, or isomorphic to $Q(n)$ for some n (see [2], [13, §1] for the definitions of simple superalgebras $M(m,n)$, $Q(n)$).

Let V be an **A-module**, namely a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ together with a representation $\rho\colon A \longrightarrow \mathrm{End}(V)$ satisfying $\rho(A_\alpha)V_\beta \subset V_{\alpha+\beta}$ $(\alpha, \beta \in \mathbb{Z}_2)$. By an A-submodule of V we mean a $\mathbb{Z}_2$-graded $\rho(A)$-stable subspace of V. We say that V is **simple** if it does not have nontrivial A-submodules.

Let V and W be two A-modules. Let $\mathrm{Hom}_A^\alpha(V,W)$ $(\alpha \in \mathbb{Z}_2)$ denote the subspace of $\mathrm{Hom}^\alpha(V,W) = \{f \in \mathrm{Hom}(V,W)\,;\, f(V_\beta) \subset W_{\alpha+\beta}\}$ consisting of all elements $f \in \mathrm{Hom}^\alpha(V,W)$ such that $f(av) = (-1)^{\alpha\cdot\beta}af(v)$ for all $a \in A_\beta$ $(\beta \in \mathbb{Z}_2)$ and $v \in V$. Put $\mathrm{Hom}_A^{\cdot}(V,W) = \mathrm{Hom}_A^0(V,W) \oplus \mathrm{Hom}_A^1(V,W)$ and put $\mathrm{End}_A^{\cdot}(V) = \mathrm{Hom}_A^{\cdot}(V,V)$. We call $\mathrm{End}_A^{\cdot}(V)$ the **supercentralizer** of A in $\mathrm{End}(V)$. Two A-modules V and W are called **isomorphic** if there exists an invertible linear map $f \in \mathrm{Hom}_A^{\cdot}(V,W)$. If this is the case, we write $V \cong_A W$ (or simply write $V \cong W$). If V and W are simple A-modules, then $V \cong W$ if and only if there exists an invertible element in $\mathrm{Hom}_A^0(V,W)$ or $\mathrm{Hom}_A^1(V,W)$. Note that, in [11] we distinguished between V and the shift of V which is defined to be the same vector space as V with the switched grading. In this paper, however, we identify V and the shift of V.

If V is a simple A-module, then $\mathrm{End}_A^{\cdot}(V)$ is isomorphic to either $M(1,0) \cong \mathbb{C}$ or $Q(1) \cong C_1$ (cf. [1, Prop. 2.17], [2, Prop. 2.5, Cor. 2.6]). In the former (resp. latter) case, we say that V is of **type** M (resp. of **type** Q). This gives the following theorem (see [1], [2], [11, §1] for the definition of the "supertensor product" of the superalgebras or modules).

Theorem B.1. *Let $C = A \hat{\otimes} B$ be the supertensor product of superalgebras A and B and let $V = U \otimes W$ be the supertensor product of a simple A-module U and a* *simple B-module W.*

(a) *If U, W are of type M, then V is a simple C-module of type M.*

(b) *If one of U and W is of type M and the other is of type Q, then V is a simple C-module of type Q.*

(c) *If U and W are of type Q, then V is a sum of two copies of a simple C-module X of type M: $V = X \oplus X$.*

Moreover, the above construction gives all simple $A \dot{\otimes} B$-modules.

Using the above U, W, V and X, define an $A \dot{\otimes} B$-module $U \dot{\circ} W$ by

$$(B.1) \qquad U \dot{\circ} W = \begin{cases} V & \text{if } U \text{ or } W \text{ is of type } M, \\ X & \text{if } U \text{ and } W \text{ are of type } Q. \end{cases}$$

Let $\operatorname{Irr} A$ denote the set of all isomorphism classes of simple A-modules for any superalgebra A.

Corollary B.2. *We have a bijection*

$$\dot{\circ}\colon \operatorname{Irr} A \times \operatorname{Irr} B \ni (U, W) \overset{\sim}{\mapsto} U \dot{\circ} W \in \operatorname{Irr} A \dot{\otimes} B.$$

C. Sergeev's duality. We review Sergeev's duality relation between $\mathcal{B}'_k$ and $\mathcal{U}_n$ on the space W in Theorem 4.1. Define a map $d\colon DP_k \to \mathbb{Z}_2$ by $d(\lambda) = 0$ (resp. $d(\lambda) = 1$) if $l(\lambda)$ is even (resp. $l(\lambda)$ is odd).

Theorem C. [8] (1) *The two superalgebras $\Psi(\mathcal{B}_k)$ and $\mathcal{U}_n$ act on W as mutual supercentralizers of each other:*

$$(C.1) \qquad \operatorname{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W) = \Psi(\mathcal{B}_k), \quad \operatorname{End}^{\cdot}_{\Psi(\mathcal{B}_k)}(W) = \Theta(\mathcal{U}_n).$$

(2) *The simple $\mathcal{B}_k$-module W_λ ($\lambda \in DP_k$) occurs in W if and only if $l(\lambda)) \leq n$. Then we have*

$$(C.2) \qquad W \cong_{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\lambda \in DP_k,\ l(\lambda) \leq n} W_\lambda \dot{\circ} U_\lambda$$

where U_λ denotes a simple $\mathcal{U}_n$-module corresponding to W_λ in W in the sense of DSCT.

(3) *The character values of* $\operatorname{Ch}[U_\lambda]$ *are given as follows:*

$$(C.3) \qquad \operatorname{Ch}[U_\lambda](x_1, x_2, \ldots, x_n) = 2^{\frac{d(\lambda) - l(\lambda)}{2}} Q_\lambda(x_1, x_2, \ldots, x_n).$$

D. A duality of $\mathcal{A}_k$ and $\mathfrak{q}(n)$. We established a duality relation between $\mathcal{A}_k$ and $\mathcal{U}_n$ on the space W^ε in Theorem 4.2.

Theorem D. [11, Th. 4.1] *The $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-submodule W^ε of W is decomposed as a multiplicity-free sum of simple $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-modules as follows :*

(D.1) $$W^\varepsilon \cong_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\lambda \in DP_k} V_\lambda \dot{\circ} U_\lambda.$$

(1) *Assume that k is even. Then the simple $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-modules $V_\lambda \dot{\circ} U_\lambda$ in W^ε are of type M. Furthermore we have*

(D.2) $$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n).$$

(2) *Assume that k is odd. Then the simple $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$-modules $V_\lambda \dot{\circ} U_\lambda$ in W^ε are of type Q. Furthermore we have*

(D.3) $$\mathrm{End}^{\cdot}_{\Theta(U_n)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Theta(\mathcal{U}_n).$$

E. Stembridge's character formula for $\mathcal{B}'_k$. We review Stembridge's formula for the character values of simple $\mathcal{B}'_k$-modules , in a form adapted to the simple modules in the $\mathbb{Z}_2$-graded sense.

Theorem E. (cf. [10, Lem. 7.5]) *We have*

$$2^{\frac{3(l(\kappa)+l(\nu))}{2}} p_\kappa(x) p_\nu(y) = \sum_{(\lambda,\mu)\in(DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu})\, 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x,y) Q_\mu(x,-y)$$

for all $(\kappa,\nu) \in (OP^2)_k$, where $Q_\lambda(x,y) = Q_\lambda(x_1,x_2,\ldots,y_1,y_2,\ldots)$ and $Q_\mu(x,-y)$ $= Q_\mu(x_1,x_2,\ldots,-y_1,-y_2,\ldots)$.

References

1. T. Józefiak, *Semisimple superalgebras*, Some Current Trends in Algebra, Proceedings of the Varna Conference 1986, Lecture Notes in Math., No. 1352, 1988, pp. 96–113.
2. T. Józefiak, *Characters of projective representations of symmetric grou- ps*, Expo. Math. **7** (1989), 193–247.
3. T. Józefiak, *Schur Q-functions and applications*, Proceedings of the Hyderabad Conference on Algebraic groups, Manoj Prakashan, 1989, pp. 205–224.
4. T. Józefiak, *A class of projective representations of hyperoctahedral groups and Schur Q-functions*, Topics in Algebra, Banach Center Publications, Vol. 26, Part 2, PWN-Polish Scientific Publishers, Warsaw, 1990, pp. 317–326.
5. V. G. Kac, *Lie superalgebras*, Adv. in Math. **26** (1977), 8–96.
6. I. G. Macdonald, *Symmetric functions and Hall polynomials, 2nd ed.*, Clarendon Press, Oxford, 1998.
7. I. Schur, *Über die Darrstellung der symmetrischen und der alternierenhen Gruppe durch gebrohene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.
8. A. N. Sergeev, *Tensor algebra of the identity representation as a module over Lie superalgebras GL(n, m) and Q(n)*, Math. USSR Sbornik **51** (1985), no. No. 2, 419-425.
9. J. R. Stembridge, *Shifted Tableaux and the Projective Representations of Symmetric groups*, Adv. in Math. **74** (1989), 87–134.
10. J. R. Stembridge, *The Projective Representations of the Hyperoctahedral Group*, J. Algebra **145** (1992), 396–453.
11. M. Yamaguchi, *A duality of the twisted group algebra of the symmetric group and a Lie superalgebra*, submitted to J. Algebra (to see the subbmited paper, use http:// xxx.lanl.gov/ abs/ math.RT9811090).