The First Term in the Expansion of Plethysm of Schur Functions

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Abstract
Plethysm of two Schur functions can be expressed as a sum of Schur functions with nonnegative integer coefficients. Recently, a new algorithm has been developed to compute the coefficients individually. Knowing the first non-zero term (the largest term under the reverse lexicographic ordering of partitions) in the expansion will significantly speed up calculations. In this paper, we show that by using combinatorial properties of nested inverse Kostka numbers, we are able to obtain some general results regarding the first term. In a lot of cases, the first term can be easily found, and the coefficient turns out to be unity. We also give some results on the last term.

Résumé
Le pléthysme de deux fonctions de Schur peut s'exprimer comme somme de fonct ions de Schur à coefficients entiers. Récemment, un nouvel algorithme a été développé pour calculer chaque coefficient. Le fait de connaitre le premier terme non nul du développement (premier au sens de l'ordre lexicographique inverse des partitions) permettrait d'accélerer de façon significative les calculs. Dans cette communication nous montrons que l'utilisation des inverses emboités des nombres de Kostka, nous permet d'obtenir des résultats généraux sur ce premier terme. Dans de nombreux cas, il est calculable et s'avère être égal à l'unité. Nous donnons également des résulats sur le dernier terme.

1 Introduction

Throughout this paper, by *partition*, we mean a non-decreasing sequence of non-negative integers. The common notation $\ell_\lambda, |\lambda|$, and λ' are used for the *length, weight,* and *conjugate* of a partition λ, respectively, and λ/μ denotes the *skew* partition. We often use $\alpha^{(i)}$ with $i = 1, 2, \cdots, k$ to denote a sequence of k partitions. For two partitions λ and μ of the same length ℓ (add some initial zeros if necessary) we say μ is *lexicographically smaller* than μ and write $\mu <_{\text{LC}} \lambda$ if the first non-zero difference $\lambda_{\ell-i} - \mu_{\ell-i}$ for $i = 0, 1, \cdots \ell - 1$ is positive. Given two *Schur functions* $s_\lambda(x)$ and $s_\mu(x)$ where $x = (x_1, x_2, \cdots)$, the *plethysm* $s_\lambda[s_\mu(x)]$ can be expanded as $s_\lambda[s_\mu(x)] = \sum_\gamma c_\gamma s_\gamma(x)$ where γ is a partition of weight $|\lambda| + |\mu|$ and c_γ is a nonnegative integer [5]. For simplicity, we will omit x and just write s_λ for $s_\lambda(x)$. Define the *first term* in the expansion to be a partition γ_{FT} such that $c_{\gamma_{\mathrm{FT}}} \neq 0$ and $\gamma <_{\mathrm{LC}} \gamma_{\mathrm{FT}}$ for all γ with $c_\gamma \neq 0$. The *last term* γ_{LT} is similarly defined and satisfies $\gamma_{\mathrm{LT}} <_{\mathrm{LC}} \gamma$. For instance, in the expansion $s_{(1,2)}[s_{(2)}] = s_{(1,5)} + s_{(2,4)} + s_{(1,2,3)}$ the first term is $\gamma_{\mathrm{FT}} = (1,5)$, and the last term is $\gamma_{\mathrm{LT}} = (1,2,3)$.

Plethysm was introduced by Littlewood [4] more than half a century ago in connection with the representation theory of matrix groups. ($s_\lambda[s_\mu] = \{\mu\} \otimes \{\lambda\}$ in Littlewood's notation.) Although there exist several algorithms [2] for the expansion, actual computation is still a formidable task for large weights due to the fact that the number of partitions γ increases dramatically as the weights $|\lambda|$ and $|\mu|$ increase. Recently, a new algorithm has been developed [7] which computes the coefficients c_λ one at a time. Unlike the traditional method, it does not explicitly involve product of Schur functions and therefore, does not require searching/sorting which demands large computer memory space. In using this new algorithm for the expansion, it will save us time if we can predict the zero coefficients in advance. In particular, it is useful to know the first and the last term in the expansion since only the terms between them may have non-zero coefficients.

The starting point of this paper is the formula for c_γ which was obtained [7] by using orthogonality and transition matrices of symmetric functions. Let $< F, s_\gamma >$ denote coefficient of s_γ in the expansion of a symmetric function F. The coefficient c_γ of s_γ in the expansion of $s_\lambda[s_\mu]$ is

$$< s_\lambda[s_\mu], s_\gamma >= \sum_{\sigma \vdash |\lambda|} \frac{\chi_\sigma^\lambda}{z_\sigma} \sum_{\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell_\sigma)}} \prod_{i=1}^{\ell_\sigma} K_{\mu, \alpha^{(i)}} N_{\gamma; \sigma_1 \alpha^{(1)}, \sigma_2 \alpha^{(2)}, \cdots, \sigma_{\ell_\sigma} \alpha^{(\ell_\sigma)}}, \tag{1}$$

where χ_σ^λ is the *character* of the symmetric group, z_σ is the number of permutations of cycle type σ given by $z_\sigma = \prod_{i \geq 1} i^{n_i(\sigma)} n_i(\sigma)!$ with $n_i(\sigma)$ being the number of parts of σ equal to i, $K_{\mu,\alpha^{(i)}}$ is the *Kostka number*, $\alpha^{(i)}$ is a partition of weight $|\mu|$ for $1 < i \leq \ell_\sigma$, and $N_{\gamma; \sigma_1 \alpha^{(1)}, \sigma_2 \alpha^{(2)}, \cdots, \sigma_\ell \alpha^{(\ell_\sigma)}}$ is the *nested inverse Kostka number* of shape γ, type $\sigma_1 \alpha^{(1)}, \sigma_2 \alpha^{(2)}, \cdots, \sigma_{\ell_\sigma} \alpha^{(\ell_\sigma)}$. Our main results are obtained by using the combinatorial properties of the *nested inverse Kostka numbers* which we will briefly review in Section 2. We also need the following two results. The first one is the conjugate relation of plethysm [5]:

$$< s_\lambda[s_\mu], s_\gamma >= \begin{cases} < s_\lambda[s_{\mu'}], s_{\gamma'} > & \text{if } |\mu| \text{ is even,} \\ < s_{\lambda'}[s_{\mu'}], s_{\gamma'} > & \text{if } |\mu| \text{ is odd.} \end{cases} \tag{2}$$

The second result concerns with the sum of $\frac{\chi_\sigma^\lambda}{z_\sigma}$ where z_σ is as defined earlier. It is known that Schur function $s_\lambda(x)$ can be expressed in terms of the power sum symmetric function $p_\alpha(x)$ via $s_\lambda(x) = \sum_{\sigma\vdash n} \frac{\chi_\sigma^\lambda}{z_\sigma} p_\sigma(x)$. If we put $x = (1, 0, 0, \cdots,)$, then $p_\sigma(x) = 1$ for any σ and $s_\lambda(x) = 1$ if $\ell_\lambda = 1$, $s_\lambda(x) = 0$ if $\ell_\lambda > 1$. It follows that

$$\sum_{\sigma\vdash n} \frac{\chi_\sigma^\lambda}{z_\sigma} = \begin{cases} 1 & \text{if} \quad \ell_\lambda = 1 \\ 0 & \text{if} \quad \ell_\lambda > 1. \end{cases} \tag{3}$$

In the special case when $\lambda = (1^n)$, we have $\chi_\sigma^{(1^n)} = (-1)^{|\sigma|-\ell_\sigma} = \epsilon_\sigma$ and

$$\sum_{\sigma\vdash n} \frac{\epsilon_\sigma}{z_\sigma} = 0 \quad \text{for } n > 1. \tag{4}$$

The arrangement of the paper is as follows. In Section 2, we review properties of nested inverse Koska numbers. In Section 3 and 4, we give results on the first and last term. In Section 5, we list some open problems.

2 Properties of the Nested Inverse Kostka numbers

The *nested inverse Kostka number* $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ of shape γ, type $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$ is a generalization of the *inverse Kostka number* $K_{\alpha,\gamma}^{(-1)}$ and is defined by

$$N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}} = \sum_{\gamma^{(1)},\gamma^{(2)},\ldots,\gamma^{(\ell)}} \prod_{i=1}^{\ell} K^{-1}_{\alpha^{(i)},\gamma^{(i-1)}/\gamma^{(i)}} \tag{5}$$

where the summation is over all sequences of nested partitions $\gamma = \gamma^{(0)} \supset \gamma^{(1)} \supset \gamma^{(2)} \supset \cdots \supset \gamma^{(\ell)} = \emptyset$ satisfying $|\gamma^{(i-1)}/\gamma^{(i)}| = |\alpha^{(i)}|$ for $1 \leq i \leq \ell$. To compute $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ combinatorially, we divide the shape γ into ℓ segments and fill each segment $\gamma^{(i-1)}/\gamma^{(i)}$ with *special rim hooks of type* $\alpha^{(i)}$. Briefly speaking, a rim hook (or boundary strip) of $\gamma^{(i-1)}/\gamma^{(i)}$ is called *special* if it starts from the North-Western corner cell of the Ferrers diagram of $\gamma^{(i-1)}/\gamma^{(i)}$. The *type* of the rim hooks used in the filling is the partition obtained when we order the hook lengths in non-decreasing order. Since the filling of each segment $\gamma^{(i-1)}/\gamma^{(i)}$ with special rim hooks of lengths $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_{\ell_{\alpha^{(i)}}}^{(i)}$ is called a *special rim hook tabloid* of shape $\gamma^{(i-1)}/\gamma^{(i)}$ and type $\alpha^{(i)}$ (cf. [3]), we will call the complete filling H of γ, which contains ℓ segments, a *nested special rim hook tabloid* of shape γ, type $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$. The *row-sign* $\omega_r(H)$ of the filling H is defined as

$$\omega_r(H) = \prod_{h_i \in H} \omega_r(h_i) \quad \text{with} \quad \omega_r(h_i) = (-1)^{r(h_i)-1}, \tag{6}$$

where $r(h_i)$ is the number of rows overed by the hook h_i. Finally,

$$N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}} = \sum_H \omega_r(H), \tag{7}$$

summed over all possible fillings H of the given shape and type.

Originally [7], nested inverse Kostka number $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ is obtained as the coefficient of s_γ in the expansion of product of ℓ copies of monomial symmetric functions $m_{\alpha^{(1)}}m_{\alpha^{(2)}}\cdots m_{\alpha^{(\ell)}}$, where $\alpha^{(i)}$ for $i = 1, 2, \cdots, \ell$ are partitions. Because the product is commutative, $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ is invariant under permutations of partitions $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$. It is easy to see either algebraically or combinatorially, that $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ is related to character of the symmetric group, the Kostka number and the inverse Kostka number as follows.

(a) when each $\alpha^{(i)}$ has only one part, say $\alpha^{(i)} = (\alpha_i)$ for $1 \le i \le \ell$,

$$N_{\gamma;(\alpha_1),(\alpha_2),(\alpha_\ell)} = \chi^\gamma_\alpha, \tag{8}$$

(b) when $\alpha^{(i)} = (1^{\alpha_i})$ for $1 \le i \le \ell$,

$$N_{\gamma;(1^{\alpha_1}),(1^{\alpha_2}),\cdots,(1^{\alpha_\ell})} = K_{\gamma',\alpha}, \tag{9}$$

(c) when $\alpha^{(1)} = \alpha$, and $\alpha^{(i)} = \emptyset$ for $2 \le i \le \ell$,

$$N_{\gamma;\alpha^{(1)}} = K^{-1}_{\alpha,\gamma} \tag{10}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell)$. Nested inverse Kostka number $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell)}}$ can be computed by successively subtracting partitions $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$, from the bubble sequence $B_\gamma = (0 + \gamma_1, 1 + \gamma_2, \cdots, \ell - 1 + \gamma_\ell)$ in certain ways (cf.[7]). The algorithm has been implemented on computer.

3 Results on the First Terms

Theorem 3.1 *If* $< s_\lambda[s_\mu], s_\gamma > \neq 0$, *then* γ *satisfies*

$$\gamma \le_{\text{LC}} |\lambda|\mu, \tag{11}$$

and

$$\gamma' \le_{\text{LC}} |\lambda|\mu'. \tag{12}$$

Proof. Suppose $< s_\lambda[s_\mu], s_\gamma > \neq 0$. By (1), there exist some partitions σ and $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell_\sigma)}$ such that $N_{\gamma;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\cdots,\sigma_{\ell_\sigma}\alpha^{(\ell_\sigma)}} \neq 0$ and $K_{\mu,\alpha^{(i)}} \neq 0$ for $1 \leq i \leq \ell_\sigma$. It is a known property of the Kostka number [5] that

$$\text{if} \quad K_{\mu,\alpha^{(i)}} \neq 0, \quad \text{then,} \quad \alpha^{(i)} \leq_{\mathrm{LC}} \mu. \tag{13}$$

Let $ord(\gamma^{(i-1)}/\gamma^{(i)})$ denote the partition obtained by rearranging the rows of the Ferrers diagram of $\gamma^{(i-1)}/\gamma^{(i)}$ in non-decreasing order. For instance, $ord(1446/125) = (1123)$. It follows from the combinatorial definition of the inverse Kostka number that

$$\text{if} \quad K^{-1}_{\theta^{(i)},\gamma^{(i-1)}/\gamma^{(i)}} \neq 0, \quad \text{then,} \quad ord(\gamma^{(i-1)}/\gamma^{(i)}) \leq_{\mathrm{LC}} \theta^{(i)}. \tag{14}$$

since for a fixed partition $\theta^{(i)}$, the lexicographically largest possible shape $ord(\gamma^{(i-1)}/\gamma^{(i)})$ is obtained when we lay all special rim hooks of lengths $\theta_1^{(i)}, \theta_2^{(i)}, \cdots, \theta_{\ell_\theta}^{(i)}$ one below another horizontally. Define the *sum* $\lambda + \mu$ of two partitions of the same length as the vector sum. If they are of different lengths, add some initial zeros. For instance, $(1,2) + (1,1,1,4) = (0,0,1,2)+(1,1,1,4) = (1,1,2,6)$. It is easy to see that the linear ordering $\leq_{\mathrm{LC}}$ is preserved under addition and scalar multiplication. That is, if $\lambda \leq_{\mathrm{LC}} \mu$, $\sigma \leq_{\mathrm{LC}} \tau$, and $k > 0$, then

$$\lambda + \sigma \leq_{\mathrm{LC}} \mu + \tau, \quad \text{and} \quad k\lambda \leq_{\mathrm{LC}} k\mu. \tag{15}$$

Using (15) together with (14) and (13) we see that if $N_{\gamma;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\cdots,\sigma_{\ell_\sigma}\alpha^{(\ell_\sigma)}} \neq 0$ and $K_{\mu,\alpha^{(i)}} \neq 0$, then

$$\begin{aligned}
\gamma &\leq_{\mathrm{LC}} \sum_{i=1}^{\ell_\sigma} ord(\gamma^{(i-1)}/\gamma^{(i)}) \quad \text{since } \gamma^{(i-1)}/\gamma^{(i)} \leq_{\mathrm{LC}} ord(\gamma^{(i-1)}/\gamma^{(i)}) \\
&\leq_{\mathrm{LC}} \sum_{i=1}^{\ell_\sigma} \sigma_i\alpha^{(i)} \quad \text{by (14), with } \theta^{(i)} = \sigma_i\alpha^{(i)} \\
&\leq_{\mathrm{LC}} \sum_{i=1}^{\ell_\sigma} \sigma_i\mu \quad \text{by (13)} \\
&= |\sigma|\mu \\
&= |\lambda|\mu, \quad \text{since } |\sigma| = |\lambda|,
\end{aligned}$$

and the proof of (11) is completed. Now, by combining the above result with the conjugate relation (2) and using the fact that $|\lambda| = |\lambda'|$, we have $\gamma' \leq_{\mathrm{LC}} |\lambda|\mu'$. □

Example 3.2 Consider the non-zero terms s_γ in $s_{(1,2)}[s_{(1,3,4)}]$. Here $\lambda = (1,2)$, $|\lambda| = 3, \mu = (1,3,4)$ and $\mu' = (1,2,2,3)$. By Theorem 3.1, γ is a partition of 24 satisfying

$\gamma \leq_{\text{LC}} 3(1,3,4) = (3,9,12)$ and $\gamma' \leq 3(1,2,2,3) = (3,6,6,9)$. Let $P[n,A]$ denote the number of partitions γ of n satisfying condition A. We find by recursive method and generating functions that $P[24, \gamma \leq_{\text{LC}} (3,9,12)$ and $\gamma' \leq_{\text{LC}} (3,6,6,9)] = 837$. So, only 837 terms out of the $P[24] = 1575$ partitions of 24 may have non-zero coefficients. Actual calculations produce only 743 terms with non-zero coefficients.

Looking at Theorem (3.1), one might expect γ_{FT} to be $|\lambda|\mu$. However, this is only true if λ has one part. For convenience, let $s_\lambda[s_\mu]_{\text{FT}}$ denote the first term γ_{FT} in the plethysm $s_\lambda[s_\mu]$.

Theorem 3.3 *The coefficient of $s_{|\lambda|\mu}$ in the plethysm $s_\lambda[s_\mu]$ is*

$$< s_\lambda[s_\mu], s_{|\lambda|\mu} >= \begin{cases} 1 & \text{if} \quad \ell_\lambda = 1 \\ 0 & \text{if} \quad \ell_\lambda > 1, \end{cases} \tag{16}$$

and

$$s_\lambda[s_\mu]_{\text{FT}} \begin{cases} = |\lambda|\mu & \text{if} \quad \ell_\lambda = 1 \\ <_{\text{LC}} |\lambda|\mu & \text{if} \quad \ell_\lambda > 1. \end{cases} \tag{17}$$

Proof. It follows from the proof of Theorem 3.1 that $N_{|\lambda|\mu;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\cdots,\sigma_{\ell_\sigma}\alpha^{(\ell_\sigma)}} \neq 0$ iff $\alpha^{(i)} = \mu$ for $i = 1,2,\cdots,\ell_\sigma$. This is because each segment $\gamma^{(i-1)}/\gamma^{(i)}$ of $|\lambda|\mu$ can be filled only by using the longest hooks allowed, namely, hooks of type $\sigma_i\mu$ by (13), and all hooks must be laid horizontally one below another. Since the row-sign of a horizontal hook is, by (6), +1 and we have just one filling H of $|\lambda|\mu$ possible for any σ, it follows

$$N_{|\lambda|\mu;\sigma_1\mu,\sigma_2\mu,\cdots,\sigma_{\ell_\sigma}\mu} = \omega_r(H) = 1 \quad \text{for any } \sigma. \tag{18}$$

We also know that $K_{\mu,\mu} = 1$. Thus, it follows from (1) that

$$\begin{aligned} < s_\lambda[s_\mu], s_{|\lambda|\mu} > &= \sum_{\sigma \vdash |\lambda|} \frac{\chi_\sigma^\lambda}{z_\sigma} \prod_{i=1}^{\ell_\sigma} K_{\mu,\mu} N_{|\lambda|\mu;\sigma_1\mu,\sigma_2\mu,\cdots,\sigma_{\ell_\sigma}\mu} \\ &= \sum_{\sigma \vdash |\lambda|} \frac{\chi_\sigma^\lambda}{z_\sigma} \\ &= \begin{cases} 1 \text{ if } \ell_\lambda = 1 \\ 0 \text{ if } \ell_\lambda > 1 \quad \text{by (3).} \end{cases} \end{aligned}$$

(17) follows immediately from Theorem 3.1 and (16). □

As a special case of Theorem 3.3, we have $s_n[s_m]_{\text{FT}} = (nm)$ with coefficient of the first term being unity. This result can be proved (cf. [6]) using elementary representation theory of the symmetric group. For $\ell_\lambda > 1$, we have the following theorem.

Theorem 3.4 *Suppose $\ell_\lambda > 1$. The the first term $\gamma_{\rm FT}$ in $s_\lambda[s_\mu]$ is*

$$\gamma_{\rm FT} = \tau_{\rm FT} \cup |\lambda|\mu^d, \tag{19}$$

where $\tau_{\rm FT}$ is the first term in $s_\lambda[s_{(\mu_1)}]$, and $\mu^d = (\mu_2, \mu_3, \cdots, \mu_\ell)$. Further, the coefficient of $s_{\gamma_{\rm FT}}$ satisfies

$$< s_\lambda[s_\mu], s_{\gamma_{\rm FT}} >=< s_\lambda[s_{(\mu_1)}], s_{\tau_{\rm FT}} > . \tag{20}$$

By this theorem, $\gamma_{\rm FT}$ is obtained simply by multiplying each, except the first, of the rows of μ by a factor of $|\lambda|$ and add $\tau_{\rm FT}$ on the top.

Proof: By Theorem 3.3, $\gamma_{\rm FT} <_{\rm LC} |\lambda|\mu$ since $\ell_\lambda > 1$. Let us assume the form of the first term to be

$$\gamma_{\rm FT} = \tau \cup |\lambda|\mu^d = (\tau, |\lambda|\mu_2, |\lambda|\mu_3, \cdots, |\lambda|\mu_{\ell_\mu}), \tag{21}$$

where τ is some partition of weight $|\lambda|\mu_1$. When we use (1) to compute the coefficient, in order for $N_{\gamma_{\rm FT};\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\cdots,\sigma_{\ell_\sigma}\alpha^{(\ell_\sigma)}} \neq 0$, we have to fill each segment $\gamma^{(i-1)}/\gamma^{(i)}$ of $\gamma_{\rm FT}$ with special rim hooks of type $\sigma_i\alpha^{(i)}$, with $\alpha^{(i)} \leq_{\rm LC} \mu$ by (13). It is crucial to note that in this case, each segment $\gamma^{(i-1)}/\gamma^{(i)}$ decomposes into two parts, one is contained in τ, the other in $|\lambda|\mu^d$, and the two parts are filled disjointly. Exactly like the filling of $|\lambda|\mu$ discussed in the proof of Theorem 3.3 , the $|\lambda|\mu^d$ part of $\gamma^{(i-1)}/\gamma^{(i)}$ is filled uniquely using the longest hooks possible, i.e. hooks of type $\sigma_i\mu^d$, and all hooks must be laid horizontally one below another, as shown in Figure 1. The τ part of $\gamma^{(i-1)}/\gamma^{(i)}$ (if is is not empty) can be filled with special rim hooks of type $\sigma_i\beta^{(i)}$ for some partition $\beta^{(i)} \vdash \mu_1$. It is impossible for any special rim hook to start from the τ part and reach the $|\lambda|\mu^d$ part since it is not long enough. (More precisely, the longer hooks have to be used to fill $|\lambda|\mu^d$ part of the segment $\gamma^{(i-1)}/\gamma^{(i)}$ and only shorter hooks are available to fill the τ part of the segment.)

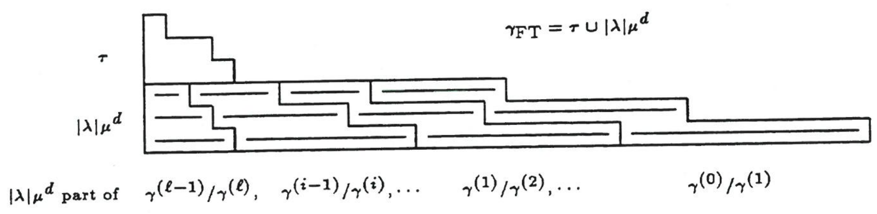

Figure 1.

Thus, we have for $1 \leq i \leq \ell_\sigma$,

$$\alpha^{(i)} = \beta^{(i)} \cup \mu^d \tag{22}$$

where $\beta^{(i)}$ is some partition of μ_1. By (21) and (22),

$$\begin{aligned}
& N_{\gamma_{\mathrm{FT}};\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)}\cdots,\sigma_{\ell_\sigma}\alpha^{(\ell_\sigma)}} && (23)\\
= & N_{\tau\cup|\lambda|\mu^d;\sigma_1(\beta^{(1)}\cup\mu^d),\sigma_2(\beta^{(2)}\cup\mu^d),\cdots,\sigma_{\ell_\sigma}(\beta^{(\ell_\sigma)}\cup\mu^d)} && (24)\\
= & N_{\tau;\sigma_1\beta^{(1)},\sigma_2\beta^{(2)},\cdots,\sigma_{\ell_\sigma}\beta^{(\ell_\sigma)}} N_{|\lambda|\mu^d;\sigma_1\mu^d,\sigma_2\mu^d,\cdots,\sigma_{\ell_\sigma}\mu^d} && (25)\\
= & N_{\tau;\sigma_1\beta^{(1)},\sigma_2\beta^{(2)},\cdots,\sigma_{\ell_\sigma}\beta^{(\ell_\sigma)}}. && (26)
\end{aligned}$$

where $N_{|\lambda|\mu^d;\sigma_1\mu^d,\sigma_2\mu^d,\cdots,\sigma_{\ell_\sigma}\mu^d} = 1$ by (18). We also have

$$\begin{aligned}
K_{\mu,\alpha^{(i)}} &= K_{\mu,\beta^{(i)}\cup\mu^d} \quad \text{by (22)} && (27)\\
&= K_{\mu_1,\beta^{(i)}} K_{\mu^d,\mu^d} && (28)\\
&= K_{\mu_1,\beta^{(i)}} && (29)
\end{aligned}$$

since $K_{\mu^d,\mu^d} = 1$. So, it follows from (1) (26) and (29) that

$$\begin{aligned}
< s_\lambda[s_\mu], s_{\gamma_{\mathrm{FT}}} > &= \sum_{\sigma\vdash|\lambda|} \frac{\chi^\lambda_\sigma}{z_\sigma} \sum_{\beta^{(1)},\beta^{(2)},\cdots,\beta^{\ell_\sigma}\vdash\mu_1} \prod_{i=1}^{\ell_\sigma} K_{\mu_1,\beta^{(i)}} N_{\tau;\sigma_1\beta^{(1)},\sigma_2\beta^{(2)},\cdots,\sigma_{\ell_\sigma}\beta^{(\ell_\sigma)}}\\
&= < s_\lambda[s_{(\mu_1)}], s_\tau >
\end{aligned}$$

and γ_{FT} is the first term in $s_\lambda[s_\mu]$ iff τ is the first term λ in $s_\lambda[s_{(\mu_1)}]$. □

In Section 5, we will give a conjecture for $s_\lambda[s_{\mu_1}]$ which is yet to be proved. However, in the special case when $\mu_1 = (1)$ we know that $s_\lambda[s_{(1)}]_{\mathrm{FT}} = \lambda$, since $s_\lambda[s_{(1)}] = s_\lambda$ which contains one term only. Thus, when $\mu_1 = 1$, $s_\lambda[s_\mu]_{\mathrm{FT}}$ can be found by Theorem 3.4.

Corollary 3.5 *When* $\mu = (1, \mu^d) = (1, \mu_2, \mu_3, \cdots, \mu_\ell)$,

$$s_\lambda[s_{(1,\mu^d)}]_{\mathrm{FT}} = \lambda \cup |\lambda|\mu^d \tag{30}$$

with coefficient

$$< s_\lambda[s_{(1,\mu^d)}], s_{\tau\cup|\lambda|\mu^d} >= \begin{cases} 1 & \textit{if } \tau = \lambda \\ 0 & \textit{otherwise.} \end{cases} \tag{31}$$

Example 3.6 Consider the first term in $s_{(2,3)}[s_{(1,3,4)}]$. Here $\lambda = (2,3), |\lambda| = 5, \mu = (1,3,4)$, and $\mu^d = (3,4)$. So, by Corollary 3.5, we have

$$s_{(2,3)}[s_{(1,3,4)}]_{\mathrm{FT}} = (2,3) \cup 5(3,4) = (2,3) \cup (15,20) = (2,3,15,20)$$

and

$$< s_{(2,3)}[s_{(1,3,4)}], s_{(\tau,15,20)} >= \begin{cases} 1 \text{ if } \tau = (2,3) \\ 0 \text{ otherwise.} \end{cases} \tag{32}$$

Combining the above result with (12), we see that any term γ which does not satisfy both $\gamma \leq_{\text{LC}} (2,3,15,20)$ and $\gamma' \leq_{\text{LC}} (1^5, 3^5, 4^5)$ will have zero coefficient in the expansion of $s_{(2,3)}[s_{(1,3,4)}]$. For instance, the coefficient of $s_{(4,16,20)}$ in the expansion is zero since $(4,16,20) >_{\text{LC}} (2,3,15,20)$.

As special cases of Corollary (3.5), we have

$$s_\lambda[s_{(1^m)}]_{\text{FT}} = \lambda \cup (|\lambda|^{m-1}) \tag{33}$$

$$s_{(1^n)}[s_{(1^m)}]_{\text{FT}} = (1^n, n^{m-1}) \tag{34}$$

$$s_n[s_{(1^m)}]_{\text{FT}} = (n^m) \tag{35}$$

and the coefficient of the first term in each case is unity. Further, we have

$$< s_\lambda[s_{(1^m)}], s_{\tau \cup (|\lambda|^{m-1})} >= \begin{cases} 1 & \text{if } \tau = \lambda \\ 0 & \text{otherwise,} \end{cases} \tag{36}$$

$$< s_{(1^n)}[s_{(1^m)}], s_{(\tau, n^{m-1})} >= \begin{cases} 1 & \text{if } \tau = (1^n) \\ 0 & \text{otherwise,} \end{cases} \tag{37}$$

and

$$< s_n[s_{(1^m)}], s_{\tau \cup (n^{m-1})} >= \begin{cases} 1 & \text{if } \tau = (n) \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

4 Some Results on the Last Terms

The last term γ_{LT} of the plethysm $s_\lambda[s_\mu]$ is not, in general, equal to the conjugate γ'_{FT} of the first term γ_{FT} of the plethysm. This is due to the fact that although it is true that in most cases

$$\text{if} \quad \kappa \leq_{\text{LC}} \gamma, \quad \text{then} \quad \gamma' \leq_{\text{LC}} \kappa', \tag{39}$$

the above relation is not always true. For instance, $(3^2) \leq_{\text{LC}} (1^2, 4)$, but the conjugates $(2^3) \leq_{\text{LC}} (1^3, 3)$. We note, however, that if partition κ can not be fit inside the $(\ell_\gamma - 1)$ by $(\ell_{\gamma'} - 1)$ rectangular box, for instance, if $|\kappa| > (\ell_\gamma - 1)(\ell_{\gamma'} - 1)$, then (39) will be true. We have the following proposition.

Proposition 4.1 *Let γ and κ be partitions of the same weight and $\kappa \leq_{\text{LC}} \gamma$. If $|\gamma| > (\ell_\gamma - 1)(\ell_{\gamma'} - 1)$, then, $\gamma' \leq_{\text{LC}} \kappa'$.*

Let γ be the first term in $s_\lambda[s_{\mu'}]$ if $|\mu|$ is even, the first term in $s_{\lambda'}[s_{\mu'}]$ if $|\mu|$ is odd. If γ satisfies condition in Proposition 4.1, then the last term γ_{LT} in $s_\lambda[s_\mu]$ can indeed be found by taking the conjugate of γ. The following theorem follows directly from the conjugate relation (2), Theorem 3.4 and Proposition 4.1.

Theorem 4.2 *Let τ be the first term in $s_\lambda[s_{(\mu'_1)}]$ if $|\mu|$ is even, the first term in $s_{\lambda'}[s_{(\mu'_1)}]$ if $|\mu|$ is odd, where μ'_1 is the first part in the conjugate partition μ' of μ. Further, let $\gamma = \tau \cup |\lambda|\mu'^d$, where $\mu'^d = (\mu'_2, \mu'_3, \cdots, \mu'_\ell)$. If $|\gamma| > (\ell_\gamma - 1)(\ell_{\gamma'} - 1)$, then the last term $s_\lambda[s_\mu]_{\text{LT}}$ in $< s_\lambda[s_\mu], s_\gamma >$ is γ'. That is,*

$$s_\lambda[s_\mu]_{\text{LT}} = (\tau \cup |\lambda|\mu'^d)' = \tau' + (|\lambda|\mu'^d)' \tag{40}$$

and the coefficient

$$< s_\lambda[s_\mu], s_{\gamma_{\text{LT}}} >= \begin{cases} < s_\lambda[s_{(\mu'_1)}], s_\tau > & \text{if } |\mu| \text{ is even} \\ < s_{\lambda'}[s_{(\mu'_1)}], s_\tau > & \text{if } |\mu| \text{ is odd.} \end{cases} \tag{41}$$

Note that γ_{LT} here is obtained simply by multiplying each, except the last, of the columns of μ by a factor of $|\lambda|$ and append τ' on the right.

Example 4.3 Consider the last term γ_{LT} in $s_{(1,2)}[s_{(3,4)}]$. Here $\lambda = (1,2)$, $\mu = (3,4)$, $\lambda' = (1,2)$, $\mu' = (1,2,2,2)$, $\mu'_1 = 1$, and $\mu'^d = (2,2,2)$. By the Theorem (4.2),

$$\tau = s_{(1,2)}[s_{(1)}]_{\text{FT}} = (1,2)$$

and

$$\gamma = (1,2) \cup 3(2,2,2) = (1,2) \cup (6,6,6) = (1,2,6,6,6).$$

Since $|\gamma| = |\lambda||\mu| = 21 > (\ell_\gamma - 1)(\ell_{\gamma'} - 1) = (5-1)(6-1) = 20$, we have

$$\gamma_{\text{LT}} = \gamma' = (3^4, 4, 5).$$

Theorem 4.4 *Let $s_\lambda^{even} = s_\lambda$ with $|\lambda|$ being an even number, and $s_\lambda^{odd} = s_\lambda$ with $|\lambda|$ being a odd number, respectively. Then,*

$$s_{(n)}[s^{even}_{(m^k)}]_{\text{LT}} = (m^{kn}) \tag{42}$$

$$s_{(1^n)}[s^{odd}_{(m^k)}]_{\rm LT} = (m^{kn}) \tag{43}$$

with coefficient of the last term in both cases being unity.

Before giving the proof we remark that for the opposite parity, we can show by (1) and (4) that

$$< s_{(n)}[s^{odd}_{(m^k)}], s_{(m^{kn})} > \; = \; 0 \tag{44}$$

$$< s_{(1^n)}[s^{even}_{(m^k)}], s_{(m^{kn})} > \; = \; 0 \tag{45}$$

and the last terms are more difficult to find (see [1] for a conjecture).

Proof. For both (42) and (43), we have $|\lambda| = n, \mu = (m^k), \mu' = (k^m), \mu'_1 = k$ and $\mu'^d = (k^{m-1})$. By Theorem (4.2) we have

$$\tau = s_n[s_k]_{\rm FT} = (kn),$$

and

$$\gamma = (kn) \cup n(k^{m-1}) = ((kn)^m).$$

Clearly, the weight $|\gamma| = kmn$ is bigger than $(\ell_\gamma - 1)(\ell_{\gamma'} - 1) = (m-1)(kn-1)$ and thus by Theorem 4.2, $\gamma_{\rm LT} = \gamma' = (m^{kn})$ with coefficient of $\gamma_{\rm LT}$ being equal to $< s_{(n)}[s_{(k)}], s_{(kn)} >= 1$. □

5 Open Problems

In [1] Agaoka has given several conjectures regarding the first and last term. Most conjectures on the first term have been proved in this paper, except one which we will state below.

Conjecture 5.1 *Let λ be a partition of length ℓ.*

$$s_\lambda[s_{(m)}]_{\rm FT} = (\lambda_1, \lambda_2, \cdots, \lambda_{\ell-1}, |\lambda|m - (\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell-1})) \tag{46}$$

and when $\lambda = (1^n)$,

$$s_{(1^n)}[s_{(m)}]_{\rm FT} = (1^{n-1}, mn - n + 1) \tag{47}$$

with coefficient of the first term being unity.

We know that (46) is true for $\ell_\lambda = 1$ or $m = 1$. To prove (47), we have by Theorem 3.1 that $\gamma \leq_{\mathrm{LC}} (mn)$ and $\gamma' \leq_{\mathrm{LC}} (n^m)$. So, γ_{FT} has at most n rows. It has been proved [6] that for $1 \leq a \leq n$,

$$< s_{(1^n)}[s_{(m)}], s_{(1^{a-1}, mn-a+1)} >= \begin{cases} 1 & \text{if } a = n \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

which can also be proved using the method of this paper by establishing a one-one mapping between the nested inverse Kostka number of a hook shape and a single-row shape. To complete the the proof of (47), it remains to show that $< s_{(1^n)}[s_{(m)}], s_{(\tau, mn-n+1)} >= 0$ if τ is not a single column. The difficulty lies in the fact that in this case, the filling of $\gamma = (\tau, mn - n + 1)$ is not unique, and there are a lot of cancellations.

The coefficient of the last term is generally not unity and finding the last term is a much more difficult problem. In [1] explicit form of γ_{FT} in some special cases have been conjectured.

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