A Macdonald vertex operator and standard tableaux statistics for the two-column (q, t)-Kostka coefficients

Mike Zabrocki
Centre de Recherche Mathématiques, Université de Montréal/LaCIM, Université du Québec à Montréal
email: zabrocki@math.uqam.ca

Abstract

On donnera une formule pour l'opérateur H_2^{qt} avec la propriété que $H_2^{qt} H_{(2^a1^b)}[X; q, t] = H_{(2^{a+1}1^b)}[X; q, t]$ avec les fonctions symétriques $H_\mu[X; q, t] = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda[X]$. L'opérateur donne une méthode pour créer les tableaux standards de longueur $n + 2$ à partir des tableaux standards de longueur n et les statistiques $a_\mu(T)$ et $b_\mu(T)$ tel que $H_{(2^a1^b)}[X; q, t] = \sum_T t^{a_{(2^a1^b)}(T)} q^{b_{(2^a1^b)}(T)} s_{\lambda(T)}[X]$ oú le somme est sur les tableaux standards.

We present a formula for a symmetric function operator with the property that $H_2^{qt} H_{(2^a1^b)}[X; q, t] = H_{(2^{a+1}1^b)}[X; q, t]$ with the symmetric functions $H_\mu[X; q, t] = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda[X]$. The operator gives a method for building the standard tableaux of size $n + 2$ from the tableaux of size n and statistics $a_{(2^a1^b)}(T)$ and $b_{(2^a1^b)}(T)$ such that $H_{(2^a1^b)}[X; q, t] = \sum_T t^{a_{(2^a1^b)}(T)} q^{b_{(2^a1^b)}(T)} s_{\lambda(T)}[X]$ where the sum is over all T standard tableaux.

1 The Vertex Operator

The Macdonald basis for the symmetric functions generalizes many other bases by specializing the values of t and q. The symmetric function basis $\{P_\mu[X; q, t]\}_\mu$ is defined by the following two conditions

$$a) \quad P_\lambda = s_\lambda + \sum_{\mu < \lambda} s_\mu c_{\mu\lambda}(q, t)$$

$$b)\ \langle P_\lambda, P_\mu\rangle_{qt} = 0 \quad for \quad \lambda \neq \mu$$

where $\langle,\rangle_{qt}$ denotes the scalar product of symmetric functions defined on the power symmetric functions by $\langle p_\lambda, p_\mu\rangle_{qt} = \delta_{\lambda\mu} z_\lambda \prod_{1\le i\le l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$ (z_λ is the size of the stabilizer of the permutations of cycle structure λ and $\delta_{xy} = 1$ if $x = y$ and 0 otherwise). We also introduce the scalar product $\langle,\rangle_t$ defined by $\langle p_\lambda, p_\mu\rangle_t = \delta_{\lambda\mu} z_\lambda \prod_{1\le i\le l(\lambda)} \frac{1}{1-t^{\lambda_i}}$.

The integral form of the basis is defined by setting $J_\mu[X;q,t] = P_\mu[X;q,t] \prod_{s\in\mu}(1 - q^{a_\mu(s)} t^{l_\mu(s)+1})$ ($s \in \mu$ means run over all cells s in μ and $a_\mu(s)$ and $l_\mu(s)$ are the arm and leg of s in μ respectively). The $\{J_\mu[X;q,t]\}_\mu$ are used to define the (q,t)-Kostka coefficients as

$$K_{\lambda\mu}(q,t) = \langle J_\mu[X;q,t], s_\lambda[X]\rangle_t$$

These coefficients are known to be polynomials and conjectured to have non-negative integer coefficients. It is known that $K_{\lambda\mu}(1,1) = f_\lambda =$ the number of standard tableaux of shape λ and so it is conjectured that these coefficients (q,t) count these standard tableaux.

We will also refer to the basis $H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X]$ that is of interest in this paper as Macdonald symmetric functions. They have the specializations that $H_\mu[X;0,t] = H_\mu[X;t]$ (the Hall-Littlewood basis of symmetric functions defined below), $H_\mu[X;0,0] = s_\mu[X]$, $H_\mu[X;0,1] = h_\mu[X]$, and the property that $H_\mu[X;q,t] = q^{n(\mu')} t^{n(\mu)} \omega H_\mu[X;1/q,1/t]$ where $n(\mu) = \sum_i (i-1)\mu_i$ and $H_\mu[X;q,t] = \omega H_{\mu'}[X;t,q]$.

The Hall-Littlewood symmetric functions $H_\mu[X;t]$ can be defined by the following formula.

$$H_\mu[X;t] = \prod_{i\ge 0, 1\le j\le k} \frac{1}{1 - z_j x_i} \prod_{1\le i\le j\le k} \frac{1 - z_j/z_i}{1 - t z_j/z_i}\Bigg|_{Z^\mu}$$

where μ is a partition with k parts and $\Big|_{Z^\mu}$ represents taking the coefficient of the monomial $z_1^{\mu_1} z_2^{\mu_2} \cdots z_k^{\mu_k}$.

These symmetric functions are not the same, but are related to the symmetric functions referred to as Hall-Littlewood polynomials in [13] p. 208. The Hall-Littlewood functions are related to the Schur symmetric functions by letting $t \to 0$ and to the homogeneous symmetric functions by letting $t \to 1$.

For each of the homogeneous, Schur, and Hall-Littlewood symmetric functions there are vertex operators with the property that for $m \ge \mu_1$ $h_m h_\mu[X] = h_{(m,\mu)}[X]$, $S_m s_\mu[X] = s_{(m,\mu)}[X]$, and $H_m^t H_\mu[X;t] = H_{(m,\mu)}[X;t]$ where (m,μ) represents the partition $(m, \mu_1, \mu_2, \ldots, \mu_k)$. These are each given by the following formulas:

$$i)\quad h_m = h_m[X] \qquad ii)\quad S_m = \sum_{i\ge 0} (-1)^i h_{m+i}[X] e_i^\perp \qquad iii)\quad H_m^t = \sum_{j\ge 0} t^j S_{m+j} h_j^\perp$$

where $f^\perp$ denotes the adjoint to multiplication for a symmetric function f with respect to the standard inner product. Therefore $\langle f^\perp g, h\rangle = \langle g, fh\rangle$. Note that $h_k^\perp$ and $e_k^\perp$ act on the Schur function basis with the formulas

$$e_k^\perp s_\mu = \sum_{\mu/\lambda \in \mathcal{V}_k} s_\lambda \qquad\qquad h_k^\perp s_\mu = \sum_{\mu/\lambda \in \mathcal{H}_k} s_\lambda$$

where the first sum is over all λ partitions of $|\mu| - k$ that differ from μ by a vertical strip and the second sum is over all partitions that differ from μ by removing a horizontal k strip.

The vertex operator of formula for the Hall-Littlewood polynomials is due to Jing ([5], [3]). The Schur function vertex operator is due to Bernstein [13] (p. 96). The action of each of these operators on the Schur basis is known ([15]). It is hopeful that a similar vertex operator, H_m^{qt}, can be found for the $H_m[X; q, t]$ symmetric functions and the action on the Schur basis can be expressed easily (although in general it may not have any relation to the formulas for the other vertex operators).

In the case that $m = 2$, an operator that adds a row of size 2 to the $H_\mu[X; q, t]$ can be expressed in terms of the Hall-Littlewood vertex operator and the first theorem we present is

Theorem 1.1 *Let*

$$H_2^{qt} = H_2^t + q\omega H_2^{\frac{1}{t}}\omega R^t$$

where R^t *is an operator with the property that* $R^t f = t^{deg(f)} f$ *for a homogeneous symmetric function* f. *This operator has the property that* $H_2^{qt} H_{(2^a 1^b)}[X; q, t] = H_{(2^{a+1}1^b)}[X; q, t]$.

This theorem follows from a formula by John Stembridge [12] that gives an expression for the Macdonald polynomial indexed by a shape with two columns in terms of Hall-Littlewood polynomials. Susanna Fischel [2] has already used the Stembridge result to find statistics on rigged configurations that are known to be isomorphic to standard tableaux. It would be better to have these statistics directly for standard tableau since the bijection between standard tableau and rigged configurations is not trivial ([7], [8], [4]).

For the remainder of this paper the symbol $H_2^{\substack{\boxed{2}\\ \boxed{1}}}$ will represent the expression $\omega H_2^{\frac{1}{t}}\omega R^t$ and the symbol $H_2^{\boxed{1}\boxed{2}}$ will represent the operator H_2^t so that $H_2^{qt} = H_2^{\boxed{1}\boxed{2}} + qH_2^{\substack{\boxed{2}\\ \boxed{1}}}$.

For integers $n \geq 0$, define

$$(a; t)_n = (1 - a)(1 - at)\cdots(1 - at^{n-1})$$

In Theorem 1.1 of [12], an expansion of the 2-column Macdonald polynomials in terms of the Hall-Littlewood polynomials is given as

$$H_{(2^a1^b)}[X;q,t] = \sum_{i=0}^{a} q^{a-i}(qt^{a+b};t^{-1})_i \frac{(t^a;t^{-1})_i}{(t^i;t^{-1})_i} H_{(2^i1^{b+2a-2i})}[X;t]$$

This result, along with the following lemmas and algebraic manipulation, is used to prove that the H_2^{qt} operator has the vertex operator property.

Lemma 1.2

$$H_2^{\substack{\boxed{2}\\ \boxed{1}}} H_2^{\boxed{1}\boxed{2}} = t H_2^{\boxed{1}\boxed{2}} H_2^{\substack{\boxed{2}\\ \boxed{1}}}$$

Lemma 1.3

$$H_2^{\substack{\boxed{2}\\ \boxed{1}}} H_{(2^a1^b)}[X;t] = t^a H_{(2^a1^{b+2})}[X;t] - t^{a+b+1} H_{(2^{a+1}1^b)}[X;t]$$

One result that follows from 1.1 is that the $H_\mu[X;q,t]$ when $\mu = (2^a1^b)$ has an unusual breakdown into 'atoms' as in the following formula.

Corollary 1.4

$$H_{(2^a1^b)}[X;q,t] = \sum_{s \in \left\{\boxed{1}\boxed{2},\, \substack{\boxed{2}\\ \boxed{1}}\right\}^a} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] q^{\sum_i co(s_i)}$$

where $co(\boxed{1}\boxed{2}) = 0$ *and* $co\left(\substack{\boxed{2}\\ \boxed{1}}\right) = 1$.

It is interesting that the symmetric functions $H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t]$ have coefficients that are polynomials in t with non-negative integer coefficients when expanded in the Schur basis (Schur positive). This will be the main result of the next section and we will refer to these as the atoms of the symmetric functions $H_{(2^a1^b)}[X;q,t]$.

Because of the relation from Lemma 1.2, for $\sum_i co(s_i) = k$ we have that

$$H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = t^x H_2^{\boxed{1}\boxed{2}} \cdots H_2^{\boxed{1}\boxed{2}} H_2^{\substack{\boxed{2}\\ \boxed{1}}} \cdots H_2^{\substack{\boxed{2}\\ \boxed{1}}} H_{(1^b)}[X;t] \qquad (1.1)$$

for some $x \geq 0$ where the $H_2^{\boxed{1}\boxed{2}}$ occurs $a-k$ times and $H_2^{\substack{\boxed{2}\\ \boxed{1}}}$ occurs k times. Therefore all atoms with exactly k occurrences of $H_2^{\substack{\boxed{2}\\ \boxed{1}}}$ are equal up to a factor of t. In fact we may derive the following identity.

Corollary 1.5

$$H_{(2^a1^b)}[X;q,t] = \sum_{i=0}^{a} \begin{bmatrix} a \\ i \end{bmatrix}_t (H_2^{\boxed{1}\boxed{2}})^{a-i} (H_2^{\begin{array}{c}\boxed{2}\\ \boxed{1}\end{array}})^i H_{(1^b)}[X;t]q^i$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \frac{(t^n;t^{-1})_k}{(t^k;t^{-1})_k}$$

This corollary can be used to recover the atoms of $H_{(2^a1^b)}[X;q,t]$ from the symmetric function, for it says that the coefficient of q^k will be a t-binomial coefficient multiplied by the atom indexed by $\left(\boxed{1}\boxed{2}^{a-k}, \begin{array}{c}\boxed{2}\\ \boxed{1}\end{array}^k\right)$.

2 Statistics on standard tableaux

One advantage of having the formula H_m^{qt} is that it provides fast methods for computing (q,t)-Kostka coefficients, but our main purpose for finding the vertex operator H_m^{qt} and its action on the Schur function basis is to use it to discover statistics $a_\mu(T)$ and $b_\mu(T)$ on standard tableau so that $K_{\lambda\mu}(q,t) = \sum_{T \in ST^\lambda} q^{a_\mu(T)} t^{b_\mu(T)}$. If these statistics exist, then the family of symmetric functions $\{H_\mu[X;q,t]\}_\mu$ can be thought of as generating functions for the standard tableaux in the sense that $H_\mu[X;q,t] = \sum_{T \in ST^{|\mu|}} q^{a_\mu(T)} t^{b_\mu(T)} s_{\lambda(T)}[X]$.

The vertex operator property has the combinatorial interpretation that H_m^{qt} changes the generating function for the standard tableaux of size n to the generating function for the standard tableaux of size $n+m$. Knowing the action of H_m^{qt} on the Schur function basis gives a description of how the shape of the tableau changes when a block of size m is added.

In the case of $m = 2$, the action of H_2^t (and $\omega H_2^{\frac{1}{t}} \omega R^t$ and hence H_2^{qt}) on the Schur function basis is well understood. The operator H_2^{qt} can be interpreted as instructions for building the standard tableaux of size $n+2$ from the standard tableaux of size n. A tableaux operator can be defined and used to build tableaux of larger content from smaller and state explicitly how cancellation of any negative terms in the expression $H_2^{qt} H_{(2^a1^b)}[X;q,t] = H_{(2^{a+1}1^b)}[X;q,t]$ occurs. This operator suggests that the standard tableaux are divided into subclasses of tableaux and that each subclass is represented by the atoms $H_{(2^a1^b)}[X;q,t]$.

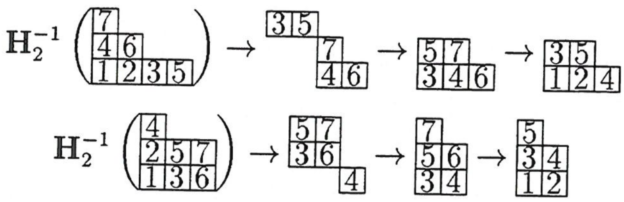
Let T be a standard tableau of size larger than 1. Either the cell labeled by 2 lies to the right of 1 or directly above. If 2 lies to the right then let $V(T) = \boxed{1}\boxed{2}$. If the 2 lies above then let $V(T) = \begin{array}{c}\boxed{2}\\ \boxed{1}\end{array}$.

Define an operator $\mathbf{H}_2^{-1}$ on the standard tableaux of size n and maps this space to the standard tableaux of size $n-2$ by the following procedure:

1. If $V(T) = \boxed{1}\boxed{2}$ then let R_1 be the first row of T and $\tilde{T}$ be T with the first row removed. Column insert the cells of R_1 that are not 1 or 2 into $\tilde{T}$ from largest to smallest and decrease each label by 2 in this new tableau. The result will be $\mathbf{H}_2^{-1}T$.

2. If $V(T) = \substack{\boxed{2}\\ \boxed{1}}$ then let C_1 be the first column of T and let $\tilde{T}$ be T with the first column removed. Row insert the cells of C_1 that are not 1 or 2 into $\tilde{T}$ from largest to smallest and decrease by 2 each of the labels of the cells in this new tableau. The result will be $\mathbf{H}_2^{-1}T$.

Example 2.1 *There are two cases in this procedure but the action is the same in both cases. The cells are separated into left and right of* $\substack{\boxed{2}\\ \boxed{1}}$ *or* $\boxed{1}\boxed{2}$, *the two groups switch places and the* 1 *and* 2 *are removed.*



This operator will be used to define the *type* of a standard tableau.

Let $\mu = (2^a 1^b)$. Let T be a standard tableau of size $2a+b$. The $\mu-type$ will represent the orientation of the "building blocks" of the standard tableau. It will be represented by the symbol $type_\mu(T)$ and be defined as the tuple of standard tableaux of size 1 or 2 with the following properties:

- If $a = 0$ and $\mu = (1^b)$ then $type_\mu(T) = (\boxed{1}^b)$.
- If $a = 1$ then $type_{(21^b)}(T) = (V(T), \boxed{1}^b)$.
- If $a > 1$ then $type_{(2^a1^b)}(T) = (V(T), type_{(2^{a-1}1^b)}(\mathbf{H}_2^{-1}T))$.

This type is used to define the statistics on standard tableaux. Let $T \in ST^{2a+b}$ and let μ be a partition with two columns with $\mu = (2^a 1^b)$. We will let the statistic $b_\mu(T)$ on standard tableaux be the number of occurrences of $\substack{\boxed{2}\\ \boxed{1}}$ in the $type_\mu(T)$. Let the statistic $a_\mu(T)$ be defined recursively with a base case of $a = 0$ so that $a_{(1^b)}(T) = c(T)$. For $a > 0$ let $a_\mu(T) = a_{\mu^r}(\mathbf{H}_2^{-1}T) + (\lambda(T)_1 - 2)$ if $type_\mu(T)_1 = \boxed{1}\boxed{2}$ and $a_\mu(T) = a_{\mu^r}(\mathbf{H}_2^{-1}T) + |\lambda(T)^c|$ if $type_\mu(T)_1 = \substack{\boxed{2}\\ \boxed{1}}$ where $\lambda(T)^c$ is the shape of the tableau T with the first column removed.

The a_μ statistic is related to the charge statistic of a standard tableau that was introduced by Lascoux and Schützenberger. An index is given to each letter in the word.

The index 0 is assigned to 1. If the letter i has index k then the index of the letter $i+1$ is k if $i+1$ lies to the left of i and the index is $k+1$ if $i+1$ lies to the right of i. The charge of the word is defined to be the sum of the indices.

Example 2.2 *Let $w = 638152479$ then the index for each letter is given by* $\substack{638152479 \\ 213021234}$ *and the charge of w is $2+1+3+0+2+1+2+3+4 = 18$.*

The charge and the $a_\mu(T)$ statistic are related by the following proposition.

Proposition 2.3 *Let $n = 2a+b$. The statistic $a_\mu(T)$ where $\mu = (2^a1^b)$ satisfies the formula*

$$a_\mu(T) = c(T) - \sum_{i=1}^{a}((n+1) - 2i)\chi(type_\mu(T)_i = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array})$$

where we use the notation $\chi(TRUE) = 1$ and $\chi(FALSE) = 0$.

Example 2.4 *Let $\mu = (2^41)$ then the standard tableau*

$$T = \begin{array}{|c|c|c|}\hline 7 \\ \cline{1-1} 4 & 6 \\ \cline{1-3} 3 & 5 & 9 \\ \hline 1 & 2 & 8 \\ \hline\end{array} \qquad \mathbf{H}_2^{-1}T = \begin{array}{|c|c|c|}\hline 6 \\ \cline{1-1} 5 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 3 & 7 \\ \hline\end{array} \qquad \mathbf{H}_2^{-1}\mathbf{H}_2^{-1}T = \begin{array}{|c|c|}\hline 5 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

$$\mathbf{H}_2^{-1}\mathbf{H}_2^{-1}\mathbf{H}_2^{-1}T = \begin{array}{|c|c|}\hline 2 \\ \cline{1-2} 1 & 3 \\ \hline\end{array} \qquad \mathbf{H}_2^{-1}\mathbf{H}_2^{-1}\mathbf{H}_2^{-1}\mathbf{H}_2^{-1}T = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

Calculate $type_{(2^41)}(T) = (\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array})$, $a_{(2^41)}(T) = 7$, $b_{(2^41)}(T) = 3$ and $c(T) = 15$.

It develops that the atoms of 1.4 are actually generating functions for the class of standard tableaux with fixed $\mu - type$.

Theorem 2.5 *Let $\mu = (2^a1^b)$. The symmetric functions $H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_{(1^b)}[X;t]$ for $s \in \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right\}^a$ are generating functions for the standard tableaux of $\mu - type = (s, \begin{array}{|c|}\hline 1 \\ \hline\end{array}^b)$ in the sense that*

$$H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_{(1^b)}[X;t] = \sum_{\substack{T \in ST^{2a+b} \\ type_\mu(T) = (s, \begin{array}{|c|}\hline 1 \\ \hline\end{array}^b)}} t^{a_\mu(T)} s_{\lambda(T)}[X]$$

This gives the next corollary that follows from this theorem and Corollary 1.4

Corollary 2.6 *Let $\mu = (2^a1^b)$. The $H_\mu[X;q,t]$ are generating functions for the standard tableaux in the sense that*

$$H_\mu[X;q,t] = \sum_{T \in ST^{2a+b}} t^{a_\mu(T)} q^{b_\mu(T)} s_{\lambda(T)}[X]$$

The atoms of the Macdonald polynomials and the $\mu - type$ of the standard tableaux suggest that the tableaux naturally fall into standard tableaux classes. For a sequence $s \in \left\{\boxed{1}\boxed{2}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right\}^a \times \{\boxed{1}\}^b$ set $STC^s = \{T \in ST^{2a+b} | type_{(2^a1^b)}(T) = s\}$.

This breakdown of the standard tableaux into classes is very beautiful when one sees where the tableau that correspond to the atoms lie in the standard tableaux when they are ranked by the charge. The figure at the end of this paper are the posets of the standard tableaux of size 6 when they are ranked by the charge. The standard tableau classes are grouped together in this poset and shaded so that each class is separated. The horizontal position of each tableau is slightly related to cyclage. Many of the properties of the Macdonald polynomials can be observed in these diagrams and expansions for $H_{(2^a1^b)}[X;q,t]$ in terms of Schur functions can be immediately written down.

These classes have the property that $STC^{(s,\boxed{1}\boxed{2},\boxed{1}^b)} \cup STC^{\left(s,\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array},\boxed{1}^b\right)} = STC^{(s,\boxed{1}^{b+2})}$ simply by definition of the type. There is also a relation between the a_μ and b_μ statistics over this set of tableaux.

Proposition 2.7 *If $type_{(2^{a+1}1^b)}(T)_{a+1} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ then $a_{(2^{a+1}1^b)}(T) = a_{(2^a1^{b+2})}(T)$ and $b_{(2^{a+1}1^b)}(T) = b_{(2^{a+1}1^b)}(T) + 1$.*

If $type_{(2^{a+1}1^b)}(T)_{a+1} = \boxed{1}\boxed{2}$ then $a_{(2^{a+1}1^b)}(T) = a_{(2^a1^{b+2})}(T) + (b+1)$ and $b_{(2^{a+1}1^b)}(T) = b_{(2^{a+1}1^b)}(T)$.

This relationship is consistent with observations made by Lynne Butler [1] about adjacent rows of the q,t-Kostka matrix. Comparing $K_{\lambda(2^a1^{b+2})}(q,t)$ to $K_{\lambda(2^{a+1}2^b)}(q,t)$, one notices that every term either changes by a factor of q or a factor of t^{b+1}.

Taking the transpose (flipping the shape and entries of the diagram about the $x = y$ line) of a standard tableau T will be represented by the operator ωT. It has the property that if the $type_\mu(T) = s$ then the $type_\mu(\omega T) = (\omega s_1, \omega s_2, \ldots, \omega s_k)$. This gives a simple method for computing the a_μ and b_μ statistics of ωT from the a_μ and b_μ statistics of T.

Proposition 2.8 *$a_{(2^a1^b)}(\omega T) = n((2^a1^b)) - a_{(2^a1^b)}(T)$ and $b_{(2^a1^b)}(\omega T) = n((a+b,a)) - b_{(2^a1^b)}(T)$.*

There exists outstanding conjectures about the number of standard tableaux that fall in a catabolism type when ranked by charge. Since the standard tableaux classes that we have defined here are generalizations for the catabolism type, it seems likely that the same conjectures will hold true for these classes. Again we let $s \in \left\{\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right\}^a \times \{\boxed{1}\}^b$ then define the symbol $A_s^i = \#\{T | T \in STC^s, a_{(2^a1^b)}(T) = i\}$.

Conjecture 2.9 *The sequence $A_s^* = (A_s^0, A_s^1, A_s^2, \dots)$ is a unimodal sequence (that is, it increases and then decreases).*

Example 2.10

$$\mu = (2^3) \qquad s = (\boxed{1}\boxed{2}, \boxed{1}\boxed{2}, \boxed{1}\boxed{2}) \qquad A_s^* = (1,1,2,3,2,1,1)$$
$$s = \left(\boxed{1}\boxed{2}, \boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right) \qquad A_s^* = (1,2,3,2,1)$$
$$s = \left(\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right) \qquad A_s^* = (1,2,3,2,1)$$
$$s = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right) \qquad A_s^* = (1,1,2,3,2,1,1)$$
$$\mu = (2^21^2) \qquad s = (\boxed{1}\boxed{2}, \boxed{1}\boxed{2}, \boxed{1}^2) \qquad A_s^* = (1,3,4,4,4,2,1,1)$$
$$s = \left(\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^2\right) \qquad A_s^* = (1,2,4,4,4,2,1)$$
$$s = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^2\right) \qquad A_s^* = (1,1,2,4,4,4,3,1)$$
$$\mu = (2^11^4) \qquad s = (\boxed{1}\boxed{2}, \boxed{1}^4) \qquad A_s^* = (1,2,4,5,7,6,5,4,2,1,1)$$
$$s = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^4\right) \qquad A_s^* = (1,1,2,4,5,6,7,5,4,2,1)$$
$$\mu = (1^6) \qquad s = (\boxed{1}^6) \qquad A_s^* = (1,1,2,4,5,7,9,9,9,9,7,5,4,2,1,1)$$

We list here the A_s^* sequences for only the classes $\left(\boxed{1}\boxed{2}^l, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}^{a-l}, \boxed{1}^b\right)$ since the other classes are isomorphic to these. By the observations from Proposition 2.7 we know that for $s \in \left\{\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right\}^a$ we have that the sequence $A^*_{\left(s, \boxed{1}^{b+2}\right)}$ can be calculated from the sequences $A^*_{\left(s, \boxed{1}\boxed{2}, \boxed{1}^b\right)}$ and $A^*_{\left(s, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^b\right)}$ since $A^i_{\left(s, \boxed{1}^{b+2}\right)} = A^{i-b-1}_{\left(s, \boxed{1}\boxed{2}, \boxed{1}^b\right)} + A^i_{\left(s, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^b\right)}$.

In [16], the two main results of were the following vertex operator formulas

Theorem 2.11 *The operator*

$$\begin{aligned} H_3^{qt} &= H_3^t + (e_1[X]H_2^t - H_3^t)q + (e_1[X]\bar{H}_2^t - \bar{H}_3^t)q^2 + \bar{H}_3^t q^3 \\ &= (1-q)H_3^t + qe_1[X]H_2^{qt} + q^2(q-1)\bar{H}_3^t \end{aligned}$$

has the property that $H_3^{qt} H_{(2^a1^b)}[X;q,t] = H_{(32^a1^b)}[X;q,t]$.

Theorem 2.12 *The operator*

$$\begin{aligned} H_4^{qt} = H_4^t &+ (h_1[X]H_3^t - H_4^t)q + (h_2[X]H_2^t - H_4^t)q^2 + (h_3[X]H_1^t - H_4^t)q^3 \\ &+ (e_3[X]\bar{H}_1^t - \bar{H}_4^t)q^3 + (e_2[X]\bar{H}_2^t - \bar{H}_4^t)q^4 + (e_1[X]\bar{H}_3^t - \bar{H}_4^t)q^5 + \bar{H}_4^t q^6 \end{aligned}$$

has the property that $H_4^{qt} H_{(1^k)}[X;q,t] = H_{(41^k)}[X;q,t]$.

where $\bar{H}_m^t = \omega H_m^{\frac{1}{t}} \omega R^t$. These operators provide fast methods of calculating the Macdonald polynomials through $n = 7$. At $n = 8$, there are 4 partitions, $(4,2,2)$, $(3,3,1,1)$, $(4,2,1,1)$, and $(3,3,2)$, that are not covered by these formulas and at $n = 9$, there are 10.

Since these operators are also expressed in terms of the Hall-Littlewood vertex operators they can be used to extend the standard tableaux statistics [17] to the cases when $\mu = (32^a1^b)$ or $\mu = (41^a)$. These same methods might be used to prove some more special cases of vertex operators and standard tableaux statistics but it seems that the general case will not be so closely tied to the Hall-Littlewood symmetric functions.

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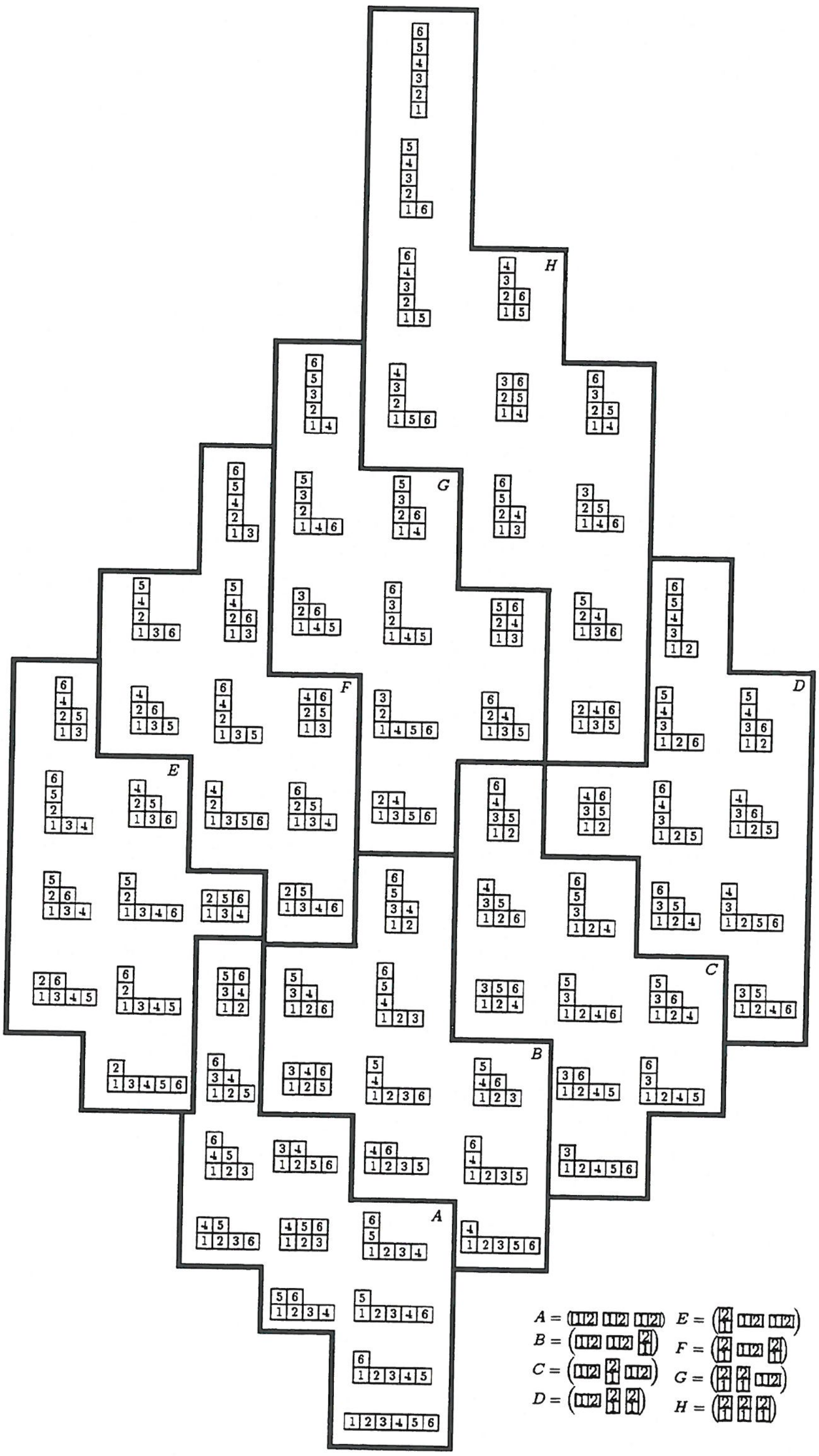

Atoms of the cyclage poset for $n = 6$