Bell permutations and Stirling numbers interpolation

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Abstract. We present a family of number sequences which interpolates between the sequences $B(n)$, of Bell numbers, and $n!$. It is defined in terms of permutations with forbidden patterns. The introduction, as a parameter, of the number k of left–to–right minima yields an interpolation between Stirling numbers of the second kind $S(n, k)$ and of the first kind (signless) $c(n, k)$. Moreover, q-counting the restricted permutations by inversions gives an interpolation between the usual q-analogues of these numbers.

Résumé. Nous présentons une famille de suites de nombres qui interpole entre la suite $B(n)$ des nombres de Bell et la suite $n!$. Cette famille est définie en termes de permutations à motifs interdits. L'introduction comme paramètre du nombre d'éléments saillants minimums de gauche a droite donne une interpolation plus fine entre les nombres de Stirling de deuxième espèce $S(n, k)$ et de première espèce (sans signe) $c(n, k)$. De plus, un q-comptage des permutations selon leurs inversions donne une interpolation entre les q-analogues habituels de ces nombres.

1 Introduction

The study of Stirling numbers and their q–analogues dates back a long time; in the last twenty years mathematicians were interested in models giving combinatorial interpretations of classical relations involving the q–analogues of Stirling numbers. In 1961, Gould [11] gives his expression in terms of symmetric functions; a combinatorial treatment of q–Stirling numbers of second kind, involving finite dimensional vector spaces over a field $\mathcal{K}_q$ of cardinality q appears in [16, 17, 18]; Garsia and Remmel [9] introduce particular rook placements in Ferrers boards. Later, Leroux [13] introduces 0–1 tableaux to prove the conjecture of Butler [6] concerning the q–log concavity for q–Stirling numbers and De Médicis and Leroux [14, 15] study and generalize q–Stirling numbers of both kinds, using this interpretation.

On the other hand the study of permutations with forbidden subsequences made meaningful progresses in the last thirty years: the n–th Catalan number is the common value for the number of permutations with a single forbidden subsequence of length three [21]; some results for permutations avoiding a single forbidden subsequence of length four can be found in [4, 5, 10]. Concerning permutations avoiding a single subsequence of length greater than four, Regev [19] obtained an interesting result, that is: the number of permutations of length n avoiding the pattern $1 \ldots (k+1)$ is asymptotically equal to $c(k-1)^{2n} n^{(2k-k^2)/2}$, where c is a constant. Pell, Fibonacci, Motzkin and Schröder numbers are sequences which count permutations avoiding more than one forbidden subsequence. We refer to [12] for an exhaustive survey on the results about permutations with forbidden subsequences.

In this paper we put these two research areas together and give combinatorial interpretations of q–analogues of Stirling numbers of both kinds in terms of permutations with forbidden subsequences. From another point of view it can be seen as a continuation of the two previous works [2, 3], here, the interpolation is between Bell numbers and factorials, and, moreover, between $S(n, k)$ and $c(n, k)$ and their q–analogues.

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2 Notations and Definitions

In this Section we recall the concepts of generating tree for a set of succession rules and of permutations with forbidden subsequences. Moreover, we generalize some classical definitions about permutations.

The concept of generating tree was introduced in [7] for the study of Baxter permutations and later applied to the study of various permutations with forbidden subsequences by different authors. Generating trees and succession rules can be used in combinatorics to deduce enumerative results about various combinatorial objects [1].

A *generating tree* is a rooted, labeled tree in which the size and labels of the set of children of each node x are determined solely from the label of x. Thus, any particular generating tree can be specified by a set of succession rules, that is, a recursive definition, consisting of

1. the basis, the label of the root,
2. the inductive step, a set of succession rules that yields a multiset of labeled children which depends solely on the label of the parent.

A succession rule can be used to describe the growth of the objects to which it is related and also to obtain the number sequence counting the objects themselves. The introduction of a parameter, say j, in a succession rule allows us to obtain a denumerable family of number sequences. In [2] the introduction of such a parameter into the classical succession rule for the Motzkin numbers allowed the authors to define number sequences such that the n–th number of each of them is lying between the n–th Motzkin and Catalan numbers. Moreover, the permutations enumerated by each number sequence are identified: they are permutations with two forbidden subsequences; the first, of length three, is fixed and the second has a length which increases with j. In [3] the introduction of the parameter j in the classical succession rule for the Catalan numbers defines number sequences such that the n–th term interpolates between the n–th Catalan number and $n!$. The objects that each sequence counts are permutations with $j!$ forbidden subsequences of length $(j+2)$.

A permutation $\pi = \pi(1)\,\pi(2)\ldots.\pi(n)$ on $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to $[n]$. Let S_n be the set of permutations on $[n]$. A permutation $\pi \in S_n$ *contains a subsequence of type* $\tau \in S_k$ iff a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that $\pi(i_1)\,\pi(i_2)\ldots\pi(i_k)$ is ordered as τ. We denote the set of permutations of S_n *avoiding* subsequences of type τ by $S_n(\tau)$.

Example 2.1 The permutation 58132674 belongs to $S_8(4321)$ because none of its subsequences of length 4 are of type 4321. This permutation does not belong to $S_8(4132)$ because there exist some subsequences of type 4132 like, for example, $\pi(2)\pi(3)\pi(6)\pi(8) = 8164$.

A *barred* forbidden subsequence $\bar{\tau}$ on $[k]$ is a permutation of S_k having a bar over one of its elements. Let τ be a permutation on $[k]$ identical to $\bar{\tau}$ but unbarred and $\hat{\tau}$ be the permutation on $[k-1]$ made up of the $(k-1)$ unbarred elements of $\bar{\tau}$, rewritten to be a permutation on $[k-1]$. A permutation $\pi \in S_n$ *contains a type* $\bar{\tau}$ *subsequence* if π contains a type $\hat{\tau}$ subsequence that, in turn, is not a subsequence of a type τ subsequence. We denote the set of permutations of S_n *avoiding* type $\bar{\tau}$ subsequences by $S_n(\bar{\tau})$.

Example 2.2 If $\bar{\tau} = 4\bar{1}32$ then $\tau = 4132$ and $\hat{\tau} = 321$. The permutation $\pi = 58132674$ belongs to $S_8(\bar{\tau})$ because all its subsequences of type $\hat{\tau}$: $\pi(1)\,\pi(4)\,\pi(5) = 532$, $\pi(2)\,\pi(4)\,\pi(5) = 832$, $\pi(2)\,\pi(6)\,\pi(8) = 864$ and $\pi(2)\,\pi(7)\,\pi(8) = 874$ are subsequences of a sequence of type τ because: $\pi(1)\,\pi(3)\,\pi(4)\,\pi(5) = 5132$, $\pi(2)\,\pi(3)\,\pi(4)\,\pi(5) = 8132$, $\pi(2)\,(4)\,\pi(6)\,\pi(8) = 8364$ and $\pi(2)\,\pi(5)\,\pi(7)\,\pi(8) = 8274$ are of type τ.

If we have the set $\tau_1 \in S_{k_1}, \ldots, \tau_p \in S_{k_p}$ of barred or unbarred permutations, we denote the set $S_n(\tau_1) \cap \ldots \cap S_n(\tau_p)$ by $S_n(\tau_1, \ldots, \tau_p)$. We call the family $F = \{\tau_1, \ldots, \tau_p\}$ a *family of forbidden subsequences*, the set $S_n(F)$, a *family of permutations with forbidden subsequences* and $S(F) = \sum_{n \geq 1} S_n(F)$ a *class of permutations with forbidden subsequences*. For $\pi \in S_n$, we call *sites* the positions lying on the left of $\pi(i)$,

$1 \leq i \leq n$, and on the right of $\pi(n)$, so the site i is on the left of $\pi(i)$ and the site $(n+1)$ on the right of $\pi(n)$. The site i of $\pi \in S_n(F)$ is *active* if the insertion of $(n+1)$ into the position between $\pi(i-1)$ and $\pi(i)$ gives a permutation belonging to the set $S_{n+1}(F)$; otherwise it is said to be *inactive*.

Example 2.3 The permutation $\pi = 58132674 \in S_8(4\bar{1}32)$ has 4 active sites that is the sites: 3, 5, 8 and 9. Indeed the permutations: 589132674, 581392674, 581326794 and 581326749 belong to $S_9(4\bar{1}32)$, while the remaining sites are inactive, for example 581326974 has the subsequence 974 of type 321 but it is not a subsequence of a sequence of type 4132.

Let π be a permutation on $[n]$. The element $\pi(i)$, $1 \leq i \leq n$, is a *left–to–right minimum* if $\pi(i) < \pi(t)$, for all $t \in [i+1, n]$. This means that an index i_1, $i+1 \leq i_1 \leq n$, such that $\pi(i) > \pi(i_1)$ does not exist. We propose to generalize the concept of left–to–right minimum as follows: let π be a permutation on $[n]$; the element $\pi(i)$, $1 \leq i \leq n$, is a *j–th kind left–to–right minimum* if and only if a sequence of indices of length j: $i_1, \cdots, i_j$, $i+1 \leq i_1 < \cdots < i_j \leq n$, such that $\pi(i) > \pi(i_l)$, $1 \leq l \leq j$ does not exist. This implies that the j rightmost elements of π are trivialy j–th kind left–to–right minima. Of course a left–to–right minimum is the same as a first kind left–to–right minimum while each element of the permutation is an ∞–kind left–to–right minimum. Hence the number of ∞–kind left–to–right minima is the length of the permutation.

Example 2.4 The permutation $\pi = 58132674$ has:

- 3 *left–to–right minima*: $\pi(3) = 1$, $\pi(5) = 2$ and $\pi(8) = 4$;
- 6 *second kind left–to–right minima*: $\pi(3) = 1$, $\pi(4) = 3$ $\pi(5) = 2$, $\pi(6) = 6$, $\pi(7) = 7$ and $\pi(8) = 4$;
- 8 *∞–kind left–to–right minima.*

Let π be a permutation on $[n]$. An *inversion* is an ordered pair of indices: (s,t), $1 \leq s < t \leq n$, such that $\pi(s) > \pi(t)$. We say that the couple of indices (s,t), $1 \leq s < t \leq n$, such that $\pi(s) > \pi(t)$, is a *j–th kind inversion* if $\pi(t)$ is a j–th kind left–to–right minimum. Following this definition the classical concept of an inversion becomes an ∞–kind inversion, while the number of inversions with respect to the left–to–right minima are first kind inversions.

Example 2.5 The permutation $\pi = 58132674$ of Example 2.4 has:

- 9 *first kind inversions*: $(1,3)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,5)$,$(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$;
- 13 *second kind inversions*: $(1,3)$,$(1,4)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,4)$,$(2,5)$,$(2,6)$,$(2,7)$,$(2,8)$,$(4,5)$,$(6,8)$,$(7,8)$;
- 13 *∞–kind inversions*: $(1,3)$,$(1,4)$,$(1,5)$,$(1,8)$,$(2,3)$,$(2,4)$,$(2,5)$,$(2,6)$,$(2,7)$,$(2,8)$, $(4,5)$,$(6,8)$,$(7,8)$.

3 Bell permutations and set partitions

The Stirling numbers of the second kind, denoted by $S(n,k)$, for $n \geq k \geq 0$, count the ways of partitioning a set of n objects into k nonempty subsets, called blocks. The number of partitions of an n–element set is given by the sum over k, $0 \leq k \leq n$, of $S(n,k)$; this defines the n–th Bell number, denoted by B_n [20]. For example, there are 7 ways of partitioning a 4–element set into two blocks: $\{1,2,3\}$ $\{4\}$; $\{1,2,4\}$ $\{3\}$; $\{1,3,4\}$ $\{2\}$; $\{1,2\}$ $\{3,4\}$; $\{1,3\}$ $\{2,4\}$; $\{1,4\}$ $\{2,3\}$; $\{1\}$ $\{2,3,4\}$, and the total number of partitions is $B_4 = \sum_{k=0}^{4} S(4,k) = 0+1+7+6+1 = 15$. Note that $S(0,0) = B(0) = 1$.

The standard representation of a given set partition consists in using the increasing order within each block and, in listing the blocks according to the increasing order of their minimum elements. We consider a new representation of the partition by moving the minimum element from the first to the last position in each block and then erasing the curly braces. The sequence of elements thus obtained is a permutation such that its (first kind) left–to–right minima are exactly the minimum elements of the blocks in the partition.

Example 3.1 Let us consider the following partition of an 8–element set into three blocks: $\{1,5,8\}$ $\{2,3\}$ $\{4,6,7\}$. The new representation described above is the permutation: $5\ 8\ \underline{1}\ 3\ \underline{2}\ 6\ 7\ \underline{4}$ which has exactly three (underlined) left–to–right minima.

Proposition 3.1 *Permutations in $S_n(4\bar{1}32)$ are counted by the n–th Bell number* (this is the reason why we call them Bell permutations)*, and $S(n,k)$ counts the permutations in $S_n(4\bar{1}32)$ with k left–to–right minima.*

Proof. Following the previous discussion, we observe that the permutation π obtained from a partition of an n–element set contains a subsequence of type $\hat{\tau} = 321$ if and only if it is a subsequence of any sequence of type $\tau = 4132$. In other words, three indices i_1, i_2, i_3, $i_1 < i_2 < i_3$, such that $\pi(i_1) > \pi(i_2) > \pi(i_3)$ can be found in π if and only if it exists an index j, $i_1 < j < i_2 < i_3$, such that $\pi(i_1)\pi(j)\pi(i_2)\pi(i_3)$ is of type 4132. Such a condition is described by the forbidden subsequence $4\bar{1}32$. Let $\pi(i_1), \cdots, \pi(i_k)$ be the k left–to–right minima of π, then $\pi(i_l)$, $1 \leq l \leq k$, is the first element of the l^{th} block in the corresponding partition; while the elements between $\pi(i_{l-1})$ and $\pi(i_l)$ are all the elements belonging to the l^{th} block of the partition. Permutations in $S_n(4\bar{1}32)$ with k left–to–tight minima are counted by the Stirling numbers. So, $S_n(4\bar{1}32)$ is enumerated by the Bell numbers. $\square$

The first construction we take into consideration for the class $S(4\bar{1}32)$ is a recursive construction which allows to obtain $S_{n+1}(4\bar{1}32)$, starting with $S_n(4\bar{1}32)$. It uses the concept of *active site* of a permutation (see Fig. 1: the active sites are represented by "_").

Proposition 3.2 *Let $\pi \in S_n(4\bar{1}32)$ be a permutation with $k \geq 2$ active sites, that is the sites i_1, i_2, ..., i_{k-2}, n and $(n+1)$. Then the number of active sites is still k in the permutation obtained by inserting $(n+1)$ into any active site different from the rightmost one; the permutation obtained from π by inserting $(n+1)$ into the site $(n+1)$ has $k+1$ active sites: i_1, i_2, ..., i_{k-2}, n, $(n+1)$ and $(n+2)$.*

Proof. Let $i_1 < i_2 < \cdots < i_{k-2} < n$ be the indices of the $k-1$ left–to–right minima of π, namely $\pi(i_1), \pi(i_2), \cdots, \pi(i_{k-2}), \pi(n)$. The active sites of π are the sites on the immediate left of each left–to–right minimum and on the right of the last element, that is, active sites of π are i_1, i_2, ..., i_{k-2}, n and $(n+1)$. Indeed, the insertion of $(n+1)$ into the site $(n+1)$ does not cause any occurrence of the forbidden subsequence 321; by inserting $(n+1)$ into the site l, $l = i_1, \cdots, i_{k-2}, n$ we can obtain the forbidden subsequences 321 if and only if there exist two indices t_1, t_2 such that $l < t_1 < t_2$ and $(n+1) > \pi(t_1) > \pi(t_2)$, but in this case $(n+1)\pi(l)\pi(t_1)\pi(t_2)$ is of type 4132. Each other site is inactive: if a site lies on the left of $\pi(i)$ that is not a left–to–right minimum, then there exists $i_1 > i :\ \pi(i) > \pi(i_1)$, and the insertion of $(n+1)$ on the left of $\pi(i)$ gives $(n+1)\pi(i)\pi(i_1)$, that is a decreasing sequence of length three, with $(n+1)$ and $\pi(i)$ adjacent elements and we get a forbidden subsequence 321. Observe that the insertion of $(n+1)$ into the site $(n+1)$ increases the number of left–to–right minima of π while each other insertion does not change this number in the permutation. $\square$

If we classify the permutations of $S_n(4\bar{1}32)$, $n \geq 1$, according to their number of active sites then we can synthetically describe the obtained recursive construction by the succession rule:

$$\begin{cases} basis: & (2) \\ inductive\ \ step: & (k) \rightarrow (k)^{k-1}(k+1), \end{cases} \tag{3.1}$$

since $S_1(4\bar{1}32) = \{1\}$ has two active sites.

The expansion of this succession rule gives the generating tree of Fig. 1. Consequently if $p_{n,k} = |\{\pi \in S_n(4\bar{1}32) :\ \pi \text{ has } k \text{ active sites}\}|$ then

$$\begin{cases} p_{1,2} & = & 1, \\ p_{n+1,k} & = & p_{n,k-1} + (k-1)p_{n,k}, \quad 2 \leq k \leq (n+2), \end{cases} \tag{3.2}$$

which is the recursive relation of the Stirling numbers of the second kind [8] (replace $p_{n,k}$ by $S(n,k-1)$).

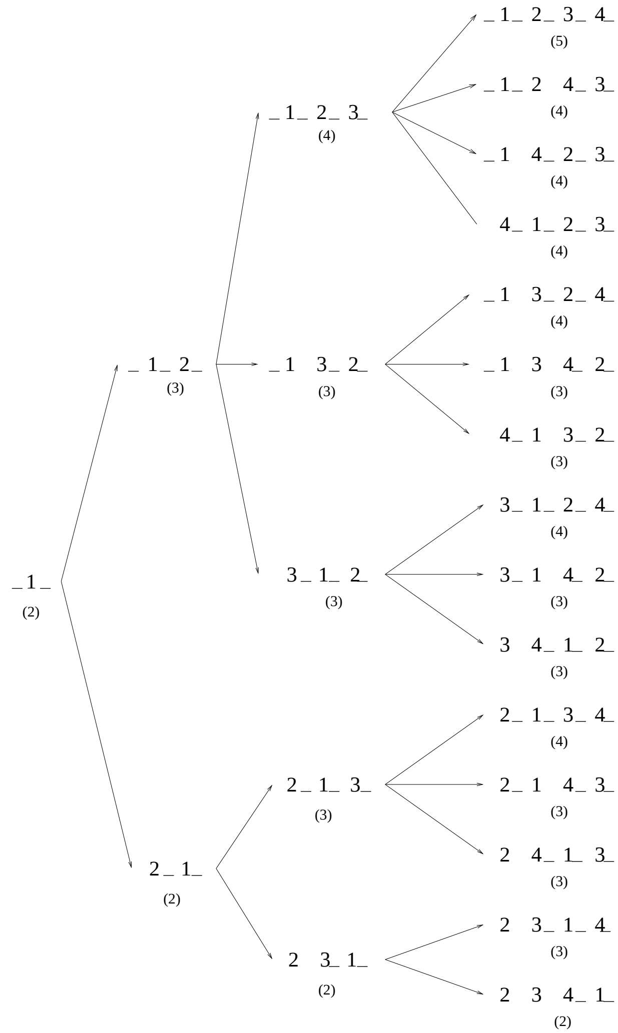

Figure 1: The generating tree for $4\bar{1}32$–avoiding permutations.

Let us note that the active sites of a permutation belonging to $S(4\bar{1}32)$ are the sites on the immediate left of each left–to–right minimum and the one on the right of the last element. Therefore the number of active sites in a permutation is the number of its left–to–right minima plus one.

The second approach we propose in order to generate $S(4\bar{1}32)$ permutations, is to construct $S_{n+1}(4\bar{1}32)$ starting from $S_1(4\bar{1}32), S_2(4\bar{1}32), \cdots, S_n(4\bar{1}32)$. The permutations in $S_{n+1}(4\bar{1}32)$ with k left–to–right minima can be obtained in the following way. For each value m such that $0 \leq m \leq n$:

- extract a subset of m elements from the set $\{2, \cdots, n+1\}$,
- construct the permutations in $S_m(4\bar{1}32)$ with $(k-1)$ left–to–right minima,
- add the element 1 on its left,
- place on the left of 1 the remaining $(n-m)$ elements in an increasing order.

The increasing order is required to avoid the forbidden subsequence 321 that would be obtained if there were two elements $\pi(t_1)$, $\pi(t_2)$ such that $\pi(t_1) > \pi(t_2)$ and $t_1 < t_2 \leq n-m$. This means that:

$$p_{n+1,k+1} = \sum_{m=0}^{n} \binom{n}{m} p_{m,k}, \qquad k \leq m.$$

As $p_{n,k} = S(n, k-1)$ we obtain a combinatorial interpretation of the well known relation involving the second kind signless Stirling numbers [8] by means of Bell permutations.

4 Generalized Bell permutations

In this Section we introduce a parameter j in the succession rule (3.1) giving the Bell numbers. Each value of j yields a number sequence such that the n–th term lies between B_n and $n!$. We are interested in characterizing the permutations enumerated by each number sequence.

Let us carefully examine the succession rule (3.1): the "exponents" of the terms on the right hand side of the inductive step are $k-1$ for the label (k) and 1 for the label $(k+1)$. We can make these "exponents" depend on a parameter j, thus giving the "exponent" $k-j$ to the label (k) and j to the label $(k+1)$; moreover if $k \leq j$ then only the label $(k+1)$ is obtained exactly k times. The exact form of the succession rule we obtain is

$$\begin{cases} basis: & (2) \\ inductive\ \ step: & (k) \rightarrow (k+1)^k, \qquad\qquad k \leq j \\ inductive\ \ step: & (k) \rightarrow (k)^{k-j}(k+1)^j, \quad k > j. \end{cases} \tag{4.3}$$

It is easy to verify that if $j=1$, then the succession rule (4.3) reduces to (3.1).

We recall that the "exponent" of a label in a succession rule means the number of times the label must be repeated. For example, the "exponent" of the label $(k+1)$ in the first inductive step is k because k is less or equal to j. Also the number of terms on the right hand side of the inductive step in a succession rule must be exactly k. The idea is to perform (4.3) on permutations and try to characterize the class we obtain. The first step is to give an interpretation of (4.3) in terms of active sites in a permutation; we have to decide how the active sites are modified when a new element is added into a permutation with a fixed number of active sites. The second step is to describe the resulting permutations in terms of forbidden subsequences. We refer to the first active site as the leftmost active site in the permutation and so on, and we make the following choices:

- if a new element is inserted in the l^{th} active site, $1 \leq l \leq k-j$, then the site on the left of the inserted element is inactive and the number of active sites do not change in the new permutation,
- if a new element is inserted in the l^{th} active site, $k \geq l \geq k-j+1$, then the site on the left of the inserted element is also active and the number of active sites grows by one.

In other words, the permutation obtained from π of length n with $k-1$ left–to–right minima of j–th kind, by inserting $(n+1)$ into its j rightmost active sites has its number of active sites increased by one; while the permutation obtained by inserting $(n+1)$ into the remaining active sites has an unchanged number of active sites.

We now show that the permutations we obtain avoid the subsequences $(j+2)(j+1)\sigma$ where $\sigma \in S_j$ and the elements corresponding to $(j+2)$ and $(j+1)$ are consecutive. In terms of permutations with forbidden subsequences such a condition is given by the union of j sets of permutations with forbidden subsequences: $\bigcup_{i=1}^{j} S\left(\bar{F}_i^j\right)$ where $\bar{F}_i^j$ is a set of barred subsequences $\bar{\tau} = (j+3)\bar{i}(j+2)\sigma_i$ with σ_i a permutation on the set $\{(j+1), \cdots, (i+1), (i-1), \cdots, 1\}$; so $\left|\bar{F}_i^j\right| = j!$ and $|\bar{\tau}| = j+3$.

Example 4.1 Let $j=2$ then $1 \leq i \leq 2$. The set $\bar{F}_2^2$ obtained for $i=2$ is $\{5\bar{2}431, 5\bar{2}413\}$.

Let us note that in the union i can assume all values between 1 and j. This means that we are not interested in the value of the element lying between $(j+3)$ and $(j+2)$, but at least one element must exist between $(j+3)$ and $(j+2)$. Such a condition avoids subpatterns of two adjacent decreasing elements having at least j smaller elements on their right. Moreover, i cannot be equal to $(j+1)$ because the subsequence $(j+3)(j+1)\sigma$ (σ being a permutation of length j) is of the forbidden type. Let S^j be the class of permutations defined by $S^j = \bigcup_{n \geq 1} \bigcup_{i=1}^{j} S_n\left(\bar{F}_i^j\right)$. We show that the class S^j has a recursive construction described by (4.3).

Proposition 4.1 *Let* $\pi \in \bigcup_{i=1}^{j} S_n\left(\bar{F}_i^j\right)$, $j \geq 1$, *be a permutation with* $k \geq 2$ *active sites:* $i_1, \ldots, i_{k-j}$, $(n-(j-2)), \ldots, (n+1)$. *Then the number of active sites does not change in the permutation obtained by inserting* $(n+1)$ *into the site* i_t, $t = 1, \cdots, k-j$; *the permutation obtained from* π *by inserting* $(n+1)$ *into the site* $(n+1-t)$, $0 \leq t \leq j-1$, *has* $k+1$ *active sites:* $i_1, \ldots, i_{k-j}$, $(n-(j-2)), \ldots, (n+1), (n+2)$.

Proof. The j rightmost sites of π that is $(n-(j-2)), \ldots, (n+1)$ are always active, if they exist, because the insertion of $(n+1)$ into the site $(n+1-t)$, $0 \leq t \leq j-1$, cannot create any occurrence of any forbidden subsequences. Thus, the element $(n+1)$ has exactly t elements on its right so it is the first and largest element of a sequence of length $(t+1) \leq j$ and any unbarred forbidden subsequence has length $(j+2)$. The site i_1 is still active if and only if a sequence of indices $i_2, \cdots, i_{j+1}$, $i_1 + 1 \leq i_2 < \cdots < i_{j+1} \leq n$, such that $\pi(i_1) > \pi(i_l)$, $2 \leq l \leq j+1$, does not exist, meaning that an active site must lie on the left of a j–th kind left–to–right minima.

Let the k active sites of a permutation π be $i_1, \ldots, i_{k-j}$, $(n-(j-2)), \ldots, (n+1)$. Observe that the site i_{k-j} is the site $(n-j+1)$, but it behaves as the sites i_t, $1 \leq t \leq k-j-1$. The active sites of the permutation obtained from π by inserting $(n+1)$ into the site $(n+1-t)$, $0 \leq t \leq j-1$, are: $i_1, \ldots, i_{k-j}$, $(n-(j-2)), \ldots, (n+1), (n+2)$. The site $(n+2-t)$, $0 \leq t \leq j$, is trivialy active; the remaining active sites are those that were active in the original permutation as the new inserted element $(n+1)$ plays no role in the creation of any forbidden subsequence. The sites that in π were inactive are always inactive because $(n+1)$ cannot play the role of the barred element in a forbidden subsequence.

By inserting $(n+1)$ into the site i_t, $1 \leq t \leq k-j$, the active sites in the new permutation are: $i_1, \ldots, i_{t-1}, (i_t+1), \ldots, (i_{k-j}+1), (n-(j-3)), \ldots, (n+2)$. The site on the left of $(n+1)$ is inactive because we would have $(n+2)(n+1)\sigma$, $|\sigma| = j$, which is forbidden. The sites that were active in π are always active because if they do not create any forbidden subsequences in π, then they do not create any problem in the new permutation and the inactive sites in π are still inactive in the new permutation. $\square$

5 Bicolored set partitions and permutations

In Section 3 we illustrated the case $j = 1$, that is we showed that $4\bar{1}32$–avoiding permutations are counted by the Bell numbers and gave a bijection with set partitions. For $j = 2$ we show that the number of $(5\bar{1}432, 5\bar{1}423)$ or $(5\bar{2}431, 5\bar{2}413)$–avoiding permutations are the values of Bell polynomials whose $(n-1)$–th term is defined by $\sum_{k \geq 0} 2^k S(n-1, k)$ ([22], sequence M1662). These numbers count bicolored set partitions (that is to say each block can be red or black) and there is a bijection between these two classes of structures. This correspondence can be easily obtained by applying the succession rules

$$\begin{cases} basis: \quad (2) \\ inductive \;\; step: \quad (2) \rightarrow (3)(3), \\ inductive \;\; step: \quad (k) \rightarrow (k)^{k-2}(k+1)^2, \quad k > 2, \end{cases} \tag{5.4}$$

to the bicolored set partitions, obtaining a constructive bijection. In bicolored set partitions the label k represents the number of blocks plus two. Given an n–element set bicolored partition with $k-2$ blocks, labeled by (k), we can add on its right the block $\{(n+1)\}$ that can be red or black and in this case the number of blocks becomes $k-1$, so the label of these new partitions is $(k+1)$; or we can insert $(n+1)$ into any of the blocks of the partition, the color remaining the same. This bijection is represented in Fig. 2, where the red blocks are those with the underlined elements. In an n–element bicolored set partition with k blocks, let i be a number belonging to the m^{th} block, $1 \leq m \leq k$, which is different from the minimum element of the block. We then define the *weighted inversions* related to i as: the number of blocks on the right of its own block such that their minimum element is smaller than i, plus 2. The total weighted inversions of a partition is given by the sum, over each i satisfying the above condition, of its weighted inversions.

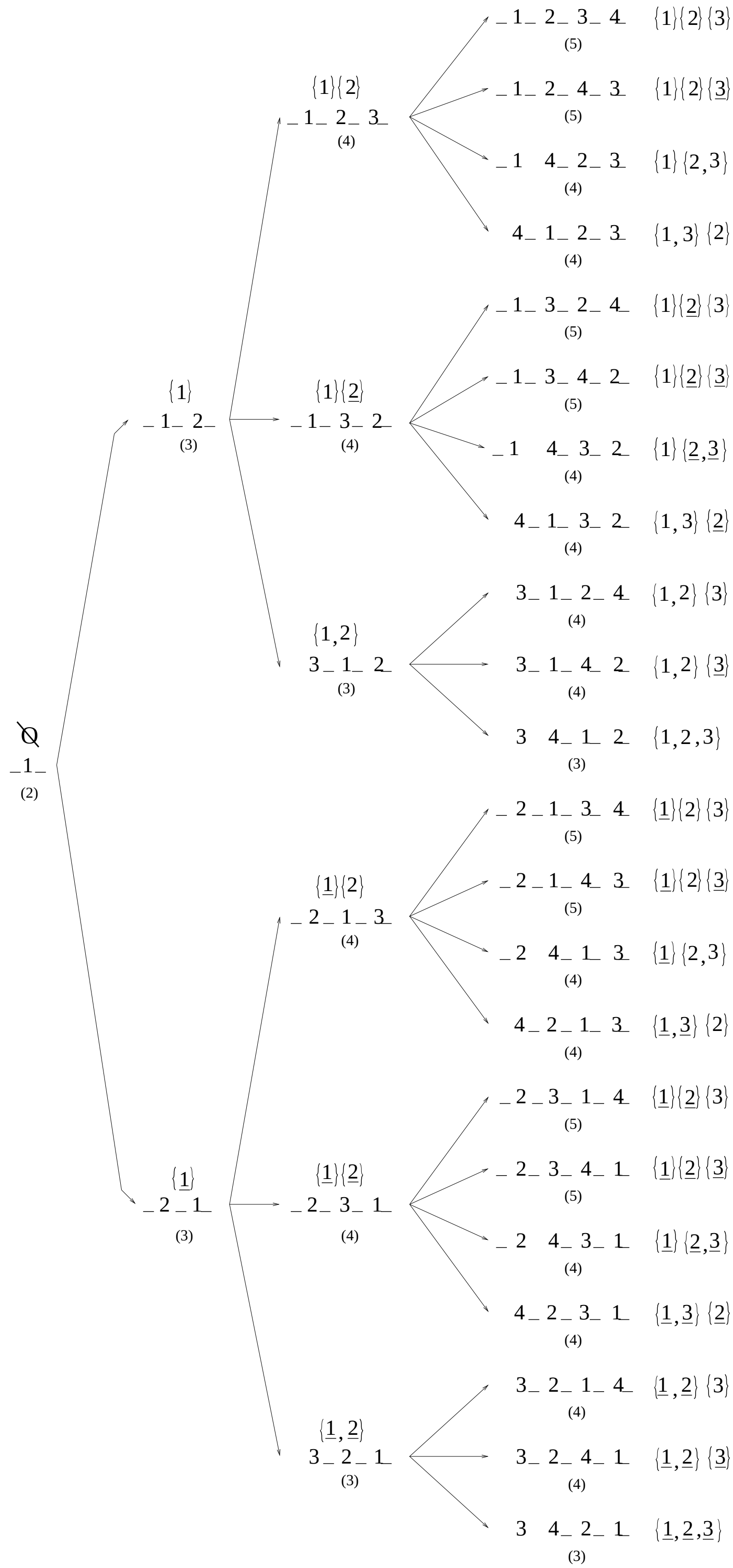

Figure 2: The first four levels of the generating tree for permutations in $S^2 = \bigcup_{n\geq 1} (S_n(5\bar{1}432, 5\bar{1}423) \bigcup S_n(5\bar{2}431, 5\bar{2}413))$ and the constructive bijection with the bicolored set partitions.

We have the following parameter correspondences:

Bicolored set partitions	**S^2-permutations**
cardinality of the partitioned set	length of the permutations+1
number of black blocks	number of left–to–right minima-1
number of blocks	number of second kind left–to–right minima-1
number of red blocks + number of weighted inversions	number of second–kind inversions

We do not know of a direct bijection between these two classes of structures.

If $j = \infty$, then we obtain all permutations and $n!$ appears; for each other value of $j \geq 3$ we obtain sequences of numbers such the n–th term of each of them is between B_n and $n!$ (see Fig. 3). These sequences do not appear in the Sloane–Plouffe book [22]: "The Encyclopedia of Integer Sequences", and verify the following property: the $(j+2)$–th number of the $(j+1)$–th sequence is obtained from the $(j+2)$–th number of the (j)–th sequence by adding $j!$.

Index	Family of permutations	Numbers
j = 1	$S_n(4\bar{1}32)$	1 2 5 15 52 203 877 4140 21147 115975 678570 . . .
		+1!
j = 2	$S_n(5\bar{1}432, 5\bar{1}423) \cup S_n(5\bar{2}431, 5\bar{2}413)$	1 2 6 22 94 454 2430 14214 89918 610182 4412798 . . .
		+2!
j = 3	$S_n(6\bar{1}5432, 6\bar{1}5423, 6\bar{1}5342, 6\bar{1}5324, 6\bar{1}5243, 6\bar{1}5234) \cup$ $S_n(6\bar{2}5431, 6\bar{2}5413, 6\bar{2}5341, 6\bar{2}5314, 6\bar{2}5143, 6\bar{2}5134) \cup$ $S_n(6\bar{3}5421, 6\bar{3}5412, 6\bar{3}5241, 6\bar{3}5214, 6\bar{3}5142, 6\bar{3}5124)$	1 2 6 24 114 618 3732 24702 177126 1363740 11195286 . . .
		+3!
j = 4	⋮	1 2 6 24 120 696 4536 32568 254136 2133816 19130040 . . .
		+4!
j = 5	⋮	1 2 6 24 120 720 4920 37320 309120 2763720 26440920 . . .
.	⋮	⋮
.	⋮	
$j = \infty$	S_n	1 2 6 24 120 720 5040 40320 362880 3628800 39916800 . . .

Figure 3: Table of permutations.

6 Enumerative results for S^j-permutations

For each j, we are interested in the enumeration of the permutations in S^j according to their length, the number of left–to–right minima and the number of j–th kind inversions. The reason we introduce this parameter is to give a combinatorial interpretation of the q–analogue that we obtain in a natural way from (4.3) by giving a "weight" to the label on the right–hand side of each inductive step in (4.3). More precisely the i–th child of a label (k) q–counts for $k - i$; the result of this "weight assignment procedure" is expressed in Proposition 6.1.

Let $\pi \in S^j$ and π^l be the permutation obtained from π by inserting the next element into the l^{th} active site, from left to right. We denote the length of π by $\mathrm{n}(\pi)$, the number of its left–to–right minima by $\mathrm{rm}(\pi)$ and the number of its j–th kind inversions by $\mathrm{inv}_j(\pi)$.

From (4.3) we deduce that the sites in a permutation $\pi \in S^j$ with length $k - 1 \leq j$ are all active, so π is the father of k permutations obtained by inserting the element k into its first, second, ..., k^{th} sites. The parameters change as follows in the new permutation:

$$\mathrm{n}(\pi^l) = \mathrm{n}(\pi) + 1; \quad \begin{cases} \mathrm{rm}(\pi^l) = \mathrm{rm}(\pi), & 1 \leq l \leq k-1, \\ \mathrm{rm}(\pi^k) = \mathrm{rm}(\pi) + 1, & l = k; \end{cases} \quad \mathrm{inv}_j(\pi^l) = \mathrm{inv}_j(\pi) + k - l;$$

and the number of active sites becomes $k + 1$.

A permutation π with $k > j$ active sites is the father of k permutations obtained by inserting the next element into each of its active sites: $i_1, i_2, \ldots, i_{k-j}, (n(\pi)-(j-2)), \ldots, (n(\pi)+1)$. Again the parameters in the new permutations change as follows:

- for the leftmost $(k-j)$ active sites:
  $\mathrm{n}(\pi^l) = \mathrm{n}(\pi)+1$; $\mathrm{rm}(\pi^l) = \mathrm{rm}(\pi)$; $\mathrm{inv}_j(\pi^l) = \mathrm{inv}_j(\pi)+k-l$;
  and the number of active sites is unchanged;

- for the remaining active sites:
  $$\mathrm{n}(\pi^l) = \mathrm{n}(\pi)+1; \quad \begin{cases} \mathrm{rm}(\pi^l) = \mathrm{rm}(\pi), & k-j+1 \leq l \leq k-1, \\ \mathrm{rm}(\pi^k) = \mathrm{rm}(\pi)+1, & l = k; \end{cases} \quad \mathrm{inv}_j(\pi^l) = \mathrm{inv}_j(\pi)+k-l;$$
  and the number of active sites increases by one unit.

Let $a_k^j(x,y,q)$ be the generating function of S^j–permutations with k active sites according to their length (x), the number of left–to–right minima (y) and the number of j–th kind inversions (q). The above considerations on the parameter modifications yield the following recursive relations for $a_k^j(x,y,q)$:

$$\begin{cases} a_2^j(x,y,q) &= xy, \\ a_k^j(x,y,q) &= xya_{k-1}^j(x,y,q) + xq[k-2]_q a_{k-1}^j(x,y,q), & 3 \leq k \leq j, \\ a_k^j(x,y,q) &= xya_{k-1}^j(x,y,q) + xq[j-1]_q a_{k-1}^j(x,y,q) + xq^j[k-j]_q a_k^j(x,y,q), & k \geq j+1; \end{cases} \tag{6.5}$$

where $[i]_q$ denotes the classical q–analogue of i that is $[i]_q = 1+\cdots+q^{i-1} = \frac{q^i-1}{q-1}$.
Solving the recursions, we obtain the following:

Proposition 6.1 *The generating function $a_k^j(x,y,q)$ for S^j–permutations verify:*

$$\begin{aligned} a_k^j(x,y,q) &= x^{k-1}\prod_{i=0}^{k-2}(y+q[i]_q), & 2 \leq k \leq j; \\ a_k^j(x,y,q) &= x^{k-1}(y+q[j-1]_q)^{k-j}\,\frac{\prod_{i=0}^{j-2}(y+q[i]_q)}{\prod_{i=1}^{k-j}(1-xq^j[i]_q)}, & k \geq j+1. \end{aligned}$$

The coefficient $[x^ny^m]\,a_k^j(x,y,q)$ gives a polynomial in q–counting the S^j–permutations with length n, having m left–to–right minima and k active sites, according to their number of j–th kind inversions. Let $c_q[h,i]$ and $S_q[h,i]$ be the classical q–analogues of the (signless) Stirling numbers of the first and second kind respectively, as defined in [14, 15]. These polynomials are characterized by:

$$\sum_{i=0}^{h} c_q[h,i]z^{h-i}y^i = \prod_{i=0}^{h-1}(y+[i]_q z), \tag{6.6}$$

$$\sum_{i\geq h} S_q[i,h]z^{i-h} = \prod_{i=1}^{h}\frac{1}{1-z[i]_q}. \tag{6.7}$$

Corollary 6.2 *Let $a_{n,m}^{(k,j)}(q) = [x^ny^m]\,a_k^j(x,y,q)$, $m \leq k-1$; then we have:*

$$a_{n,m}^{(k,j)}(q) = \delta_{n,k-1}c_q[k-1,m]q^{k-1-m}, \quad 2 \leq k \leq j; \tag{6.8}$$

$$a_{n,m}^{(k,j)}(q) = S_q[n+1-j,k-j]q^{j(n+1-k)+(k-m-1)}([j-1]_q)^{k-j-m}\sum_{i=0}^{j-1}\binom{k-j}{m-i}c_q[j-1,i]([j-1]_q)^i, \quad k \geq j+1; \tag{6.9}$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. The value of $a^{(k,j)}_{n,m}(q)$ in the case of $2 \le k \le j$ is an immediate consequence of (6.6). In the case of $k \ge j+1$, by applying (6.6) and (6.7) we can write:
$a^j_k(x,y,q) =$
$$=x^{k-1}\left(\sum_{t=0}^{k-j}\binom{k-j}{t}q^t\left([j-1]_q\right)^t y^{k-j-t}\right)\left(\sum_{i=0}^{j-1}c_q[j-1,i]q^{j-1-i}y^i\right)\left(\sum_{i\ge 0}S_q[i+k-j,k-j]\left(xq^j\right)^i\right),$$
and the second equality can then be easily proved. □

Let us now examine the polynomials $a^{(k,j)}_{n,m}(q)$ for some particular values of the parameter j.

- If $j=1$, then equation (6.9) should be used and the result is different from 0 if and only if the exponent of $([j-1]_q)=([0]_q)$ is zero, that is $k=m+1$. Once n and m are fixed the only possibility is:
$$a^{(m+1,1)}_{n,m}(q) = S_q[n,m]q^{n-m}.$$
This confirms the results of Section 3 for the number of left–to–right minima in Bell permutations of length n. Moreover it shows that the classical q–analogue of the Stirling numbers of the second kind, $S_q[n,m]$, multiplied by q^{n-m}, count restricted permutations according to first kind inversions.
- If $j=2$ then equations (6.8) and (6.9) give:
$$\begin{cases} a^{(2,2)}_{1,1}(q) &= 1,\\ a^{(k,2)}_{n,m}(q) &= \binom{k-2}{m-1}S_q[n-1,k-2]q^{2n+1-k-m}, \quad k\ge 3.\end{cases}$$
By summing over k and m we obtain the polynomials for the permutations with forbidden subsequences $(5\bar{1}432, 5\bar{1}423)$ or $(5\bar{2}431, 5\bar{2}413)$ of length n according to the number of their second kind inversions:
$$\sum_{k\ge 2}\sum_{1\le m\le n} a^{(k,2)}_{n,m}(q) = \sum_{k=0}^{n-1} S_q[n-1,k]q^{2(n-1-k)}(1+q)^k, \quad n\ge 2. \tag{6.10}$$
This expression reduces to a value of the $(n-1)$–th Bell polynomials: $\sum_{k\ge 0} 2^k S(n-1,k)$ for $q=1$ as said in Section 5, so (6.10) defines a q–analogue for these numbers.
- If $j=\infty$ then equation (6.8) gives:
$$a^{(n+1,\infty)}_{n,m}(q) = c_q[n,m]q^{n-m}, \qquad n\ge 1.$$
This means that the classical q–analogue of the first kind signless Stirling numbers, $c_q[n,m]$, multiplied by q^{n-m} correspond to q–counting the inversions in the permutations. A direct combinatorial explanation can be given.

The meaning of "Stirling numbers interpolation" lies in the observation that the permutations of length n having m left–to–right minima are counted by the second kind Stirling numbers for $j=1$ and by the first kind Stirling numbers for $j=\infty$. In the intermediate cases this number, $p^{(j)}_{n,m}$, is such that $S(n,m)\le p^{(j)}_{n,m}\le c(n,m)$, $c(n,m)$ denoting the first kind signless Stirling numbers, and it verifies the recursive relation:
$$p^{(j)}_{n,m} = p^{(j)}_{n-1,m-1} + \sum_{k=2}^{n}(k-1)a^{(k,j)}_{n-1,m}(1), \tag{6.11}$$
where:
$$a^{(k,j)}_{n-1,m}(1) = \begin{cases} c(n-1,m), & \text{for} \quad 2\le k=n\le j,\\ S(n-j,k-j)(j-1)^{k-j-m}\sum_{i=0}^{j-1}\binom{k-j}{m-i}c(j-1,i))(j-1)^i, & \text{for} \quad k\ge j+1.\end{cases}$$
Note that the sum in equation (6.11) reduces to a single term if $j=1$ (namely, the term for $k=m+1$) and if $j=\infty$ (namely, the term for $k=n$). These two cases yield classical recurrence relations for the Stirling numbers of the second kind, $S(n,m)$, and unsigned first kind, $c(n,m)$, respectively.

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